BORDERED FLOER HOMOLOGY AND EXISTENCE OF INCOMPRESSIBLE TORI IN HOMOLOGY SPHERES

EAMAN EFTEKHARY

ABSTRACT. Let Y be a homology sphere which contains an incompressible torus. We show that Y can not be a L-space, i.e. the rank of $\widehat{\mathrm{HF}}(Y)$ is greater than 1. In fact, if the homology sphere Y is an irreducible L-space then Y is either S^3, or the Poicaré sphere $\Sigma(2,3,5)$, or it is hyperbolic.

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Key words and phrases. Floer homology, splicing, incompressible torus.

1. Introduction

1.1. **Background and main results.** Heegaard Floer theory, defined by Ozsváth and Szabó [OS1], has been powerful in extracting topological properties of three-manifolds. Surprisingly, in rare cases homology spheres have the Heegaard Floer homology of S^3. The Poincaré sphere $\Sigma(2,3,5)$ is an example of an irreducible homology sphere with $\widehat{\mathrm{HF}}(\Sigma(2,3,5)) = \widehat{\mathrm{HF}}(S^3) = \mathbb{Z}$. It is thus not true in general, that Heegaard Floer homology is capable to distinguish S^3 from other homology spheres. However, a conjecture of Ozsváth and Szabó predicts that $\Sigma(2,3,5)$ is the only non-trivial example of an irreducible homology sphere with trivial Heegaard Floer homology. In this paper, we address the case of a 3-manifold which contains an incompressible torus. Throughout the paper, we let $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

Theorem 1.1. *If a homology sphere Y contains an incompressible torus*
$$\widehat{\mathrm{HF}}(Y;\mathbb{F}) \neq \mathbb{F} = \widehat{\mathrm{HF}}(S^3;\mathbb{F}).$$

By Thurston's geometrization conjecture/Perelman's theorem ([Thu, Per], also [MT1, MT2]), Theorem 1.1 reduces Ozsváth-Szabó conjecture to the homology spheres which are either Seifert fibered or hyperbolic. It is shown that $\Sigma(2,3,5)$ and S^3 are the only Seifert fibered homology spheres with trivial Heegaard Floer homology [Rus], [Ef5]. Ozsváth-Szabó conjecture is thus reduced to the following.

Conjecture 1.2. *If the homology sphere Y is hyperbolic then $\widehat{\mathrm{HF}}(Y;\mathbb{F}) \neq \mathbb{F}$.*

If the homology sphere Y includes an incompressible torus, it is obtained by splicing the complements of a pair of non-trivial knots K_1 and K_2 in the homology spheres Y_1 and Y_2, respectively. In this case, we write $Y = Y(K_1, K_2)$. Theorem 1.1 may then be re-stated as the following.

Theorem 1.3. *If K_1 and K_2 are non-trivial, $\widehat{\mathrm{HF}}(Y(K_1,K_2);\mathbb{F}) \neq \mathbb{F}$.*

When both Y_1 and Y_2 are L-spaces, Theorem 1.3 is the main result of [HL]. When $Y_1 = S^3$ and K_1 is the trefoil, Theorem 1.3 is Corollary 1.3 from [Ef4].

The reduced Khovanov homology of a knot $K \subset S^3$ is related to the Heegaard Floer homology of the double cover of S^3 branched over K [OS5]. Ozsváth-Szabó Conjecture 1.2 thus implies that the reduced Khovanov homology (and thus Khovanov homology) detects the unknot; a theorem of Kronheimer and Mrowka [KM]. The result of this paper reproves a few special cases of the aforementioned theorem. A knot $K \subset S^3$ is π-*hyperbolic* if $S^3 - K$ admits a Riemannian metric with constant negative curvature which becomes singular folding with angle π around K.

Corollary 1.4. *Suppose that $K \subset S^3$ is either a prime satellite knot or it is not π-hyperbolic. Then the rank of the reduced Khovanov homology $\widetilde{\mathrm{Kh}}(K)$ is greater than 1.*

1.2. **Bordered Floer homology for a knot complement.** Let K be a knot inside the homology sphere Y and $Y(K)$ denote the bordered manifold determined from the knot complement $Y - \mathrm{nd}(K)$ by parametrizing its boundary using a meridian and a zero-framed longitude for K. The proof of Theorem 1.1 rests heavily on a construction of the bordered Floer module $\widehat{\mathrm{CFD}}(Y(K))$ using the knot Floer complex $\mathrm{CFK}^{\bullet}(Y,K)$, which we will now describe. Consider a doubly pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u, v)$ for K and let $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ denote the totally real tori in the symmetric product $\mathrm{Sym}^g(\Sigma)$ which correspond to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively. The markings u and v determine the map

$$\mathfrak{s} = \mathfrak{s}_{u,v} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \underline{\mathrm{Spin}^c}(Y, K)$$

where $\mathfrak{s}(\mathbf{x})$ denotes the relative Spin^c class assigned to $\mathbf{x}$ in the sense of [Ni], which is defined by assigning a nowhere vanishing vector field on $Y - \mathrm{nd}(K)$ to $\mathbf{x}$ which is tangent to the boundary. Multiplying the vector fields by -1 gives an involution map J on $\underline{\mathrm{Spin}^c}(Y, K)$, and the map $\mathfrak{s} \mapsto c_1(\mathfrak{s}) = \mathfrak{s} - J(\mathfrak{s}) \in \mathrm{H}^2(Y, K; \mathbb{Z})$ gives an identification of $\underline{\mathrm{Spin}^c}(Y, K)$ with $\mathbb{Z}$, which will be implicit in this paper. Let

$$C = \big\langle [\mathbf{x}, i, j] \;\big|\; \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta,\ \mathfrak{s}(\mathbf{x}) - i + j = 0 \big\rangle_{\mathbb{Z}}$$

denote the $\mathbb{Z} \oplus \mathbb{Z}$ filtered chain complex associated with K. Following [OS4] we may consider the sub-modules

$$C\{i = a, j = b\}, C\{i = a, j \leq b\} \quad \text{and} \quad C\{i \leq a, j = b\} \qquad a, b \in \mathbb{Z} \cup \{\infty\}$$

with the induced structure as a chain complex. Set $C\{i = a\} = C\{i = a, j \leq \infty\}$ and $C\{j = b\} = C\{i \leq \infty, j = b\}$. For every relative Spin^c class $\mathfrak{s} \in \mathbb{Z}$ define

$$i_n^{\mathfrak{s}} = i_n^{\mathfrak{s}}(K) : C\{i \leq \mathfrak{s}, j = 0\} \oplus C\{i = 0, j \leq n - \mathfrak{s} - 1\} \longrightarrow C\{j = 0\}$$
$$i_n^{\mathfrak{s}}\left([\mathbf{x}, i, 0], [\mathbf{y}, 0, j]\right) := [\mathbf{x}, i, 0] + \Xi[\mathbf{y}, 0, j],$$

where $\Xi : C\{i = 0\} \to C\{j = 0\}$ is the chain homotopy equivalence corresponding to the Heegaard moves which change $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u)$ to $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; v)$. Let $Y_n(K)$ denote the three-manifold obtained from Y by n-surgery on K and let K_n denote the corresponding knot inside $Y_n(K)$, determined by the aforementioned surgery.

Proposition 1.5. *The homology of the mapping cone $M(i_n^{\mathfrak{s}})$ gives*

$$\mathbb{H}_n(K, \mathfrak{s}) = \widehat{\mathrm{HFK}}(Y_n(K), K_n, \mathfrak{s}).$$

Note that $M(i_0^{\mathfrak{s}})$ is a sub-complex of both $M(i_1^{\mathfrak{s}})$ and $M(i_1^{\mathfrak{s}+1})$. We denote the embedding maps by $F_\infty^{\mathfrak{s}} = F_\infty^{\mathfrak{s}}(K)$ and $\overline{F}_\infty^{\mathfrak{s}+1} = \overline{F}_\infty^{\mathfrak{s}+1}(K)$, respectively. The quotient of $M(i_1^{\mathfrak{s}})$ by $F_\infty^{\mathfrak{s}}(M(i_0^{\mathfrak{s}}))$ is isomorphic to $\widehat{\mathrm{CFK}}(K, \mathfrak{s}) \simeq C\{i = 0, j = -\mathfrak{s}\}$. Denote the quotient map by $F_0^{\mathfrak{s}} = F_0^{\mathfrak{s}}(K)$. We thus obtain a short exact sequence

$$0 \longrightarrow M\left(i_0^{\mathfrak{s}}\right) \xrightarrow{F_\infty^{\mathfrak{s}}} M\left(i_1^{\mathfrak{s}}\right) \xrightarrow{F_0^{\mathfrak{s}}} \widehat{\mathrm{CFK}}(K, \mathfrak{s}) \longrightarrow 0$$

Similarly, the quotient map $\overline{F}_0^{\mathfrak{s}} = \overline{F}_0^{\mathfrak{s}}(K)$ from $M(i_1^{\mathfrak{s}})$ to $M(i_1^{\mathfrak{s}})/\mathrm{Im}(\overline{F}_\infty^{\mathfrak{s}})$ sits in

$$0 \longrightarrow M\left(i_0^{\mathfrak{s}-1}\right) \xrightarrow{\overline{F}_\infty^{\mathfrak{s}}} M\left(i_1^{\mathfrak{s}}\right) \xrightarrow{\overline{F}_0^{\mathfrak{s}}} \widehat{\mathrm{CFK}}(K, \mathfrak{s}) \longrightarrow 0.$$

Let $C_\bullet(K) = \bigoplus_{\mathfrak{s} \in \mathbb{Z}} C_\bullet(K, \mathfrak{s})$, where $C_\bullet(K, \mathfrak{s}) = M(i_\bullet^{\mathfrak{s}})$ for $\bullet = 0, 1$ and $C_\infty(K, \mathfrak{s}) = C\{i = \mathfrak{s}, j = 0\}$. Denote the differential of $C_\bullet(K)$ by $d_\bullet$ for $\bullet \in \{0, 1, \infty\}$. Set $M(K) = C_0(K) \oplus C_1(K)$ and $L(K) = C_1(K) \oplus C_\infty(K)$. The maps $F_\bullet = F_\bullet(K)$, obtained by putting all $F_\bullet^{\mathfrak{s}}$ together, will be called the *bypass homomorphisms*.

A differential graded algebra $\mathcal{A}(T^2, 0)$ is associated with the torus boundary of $Y - \mathrm{nd}(K)$, which will be denoted by $-T^2$. The bordered Floer module $\widehat{\mathrm{CFD}}(Y(K))$ is then a module over $\mathcal{A}(T^2, 0)$. Following the notation of Subsection 4.2 from [LOT2], $\mathcal{A}(T^2, 0)$ is generated, as a module over $\mathbb{F}$, by the idempotents $\imath_0$ and $\imath_1$, and the chords $\rho_1, \rho_2, \rho_3, \rho_{12} = \rho_1\rho_2, \rho_{23} = \rho_2\rho_3$ and $\rho_{123} = \rho_1\rho_2\rho_3$.

Theorem 1.6. *The bordered Floer complex $\widehat{\mathrm{CFD}}(Y(K))$ is quasi-isomorphic to the left module over $\mathcal{A}(T^2,0)$, which is generated by $\imath_0.L(K)$ and $\imath_1.M(K)$ and is equipped with the differential $\partial\colon \widehat{\mathrm{CFD}}(Y(K)) \to \widehat{\mathrm{CFD}}(Y(K))$ defined by*

$$\partial\binom{\mathbf{x}}{\mathbf{y}} = \begin{cases} \begin{pmatrix} d_0(\mathbf{x}) \\ \overline{F}_\infty(\mathbf{x}) + d_1(\mathbf{y}) \end{pmatrix} + \begin{pmatrix} \rho_1 F_\infty(\mathbf{x}) \\ \rho_3\overline{F}_0(\mathbf{y}) + \rho_{123}\overline{F}_0(F_\infty(\mathbf{x})) \end{pmatrix} & \textit{if } \binom{\mathbf{x}}{\mathbf{y}} \in M(K) \\ \\ \begin{pmatrix} d_1(\mathbf{x}) \\ F_0(\mathbf{x}) + d_\infty(\mathbf{y}) \end{pmatrix} + \rho_2.\binom{0}{\mathbf{x}} & \textit{if } \binom{\mathbf{x}}{\mathbf{y}} \in L(K) \end{cases} \tag{1}$$

This theorem should be compared with Theorem 11.26 from [LOT1], which addresses the case where $Y = S^3$.

2. Surgery on null-homologous knots

2.1. A triangle of chain maps. By a Heegaard n-tuple we mean the data

$$(\Sigma, \boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_n; u_1, ..., u_r)$$

where Σ is a Riemann surface of genus g, each $\boldsymbol{\alpha}_i$ is g-tuples of disjoint simple closed curves for $i = 1, ..., n$ and u_j are markings in $\Sigma - \sqcup_{i=1}^n \boldsymbol{\alpha}_i$. Let $\mathbb{T}_i \subset \mathrm{Sym}^g(\Sigma)$ denote the torus associated with $\boldsymbol{\alpha}_i$. Choose $\mathbf{x}_i \in \mathbb{T}_i \cap \mathbb{T}_{i+1}$ for $i = 1, ..., n-1$ and $\mathbf{x}_n \in \mathbb{T}_1 \cap \mathbb{T}_n$. Let $\pi_2(\mathbf{x}_1, ..., \mathbf{x}_n)$ denote the set of homotopy classes of n-gons connecting $\mathbf{x}_1, ..., \mathbf{x}_n$ and $\pi_2^j(\mathbf{x}_1, ..., \mathbf{x}_n; u_1, ..., u_r) \subset \pi_2(\mathbf{x}_1, ..., \mathbf{x}_n)$ denote the subset of classes with Maslov index j with zero intersection number with the codimension-2 sub-varieties $L_{u_1}, ..., L_{u_r}$ of $\mathrm{Sym}^g(\Sigma)$ corresponding to the markings $u_1, ..., u_r$.

Let $K \subset Y$ be a knot inside a homology sphere Y. Consider a Heegaard diagram

$$H = (\Sigma, \boldsymbol{\alpha} = \{\alpha_1, ..., \alpha_g\}, \boldsymbol{\beta} = \{\beta_1, ..., \beta_g\})$$

for Y so that $(\Sigma, \boldsymbol{\alpha}, \widehat{\boldsymbol{\beta}} = \boldsymbol{\beta} - \{\beta_g\})$ is a Heegaard diagram for $Y - \mathrm{nd}(K)$. Suppose that $\lambda \subset \Sigma - \widehat{\boldsymbol{\beta}}$ represents a zero-framed longitude for K. Let λ_n be a small perturbation of the juxtaposition $\lambda + n\beta_g$ and β_i^n denote a small Hamiltonian isotope of β_i for $i = 1, ..., g-1$. The Heegaard diagram

$$H_n = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n = \{\beta_1^n, ..., \beta_{g-1}^n, \lambda_n\}, p_n)$$

gives a diagram for $(Y_n(K), K_n)$, where p_n is a marking at the intersection of λ_n and β_g which distinguishes λ_n from other curves in $\boldsymbol{\beta}_n$. With the integers $m > n \geq 0$ fixed, we assume that λ_n and λ_{n+m} intersect each other in m transverse points, and that for an intersection point q of these latter curves the points q, p_n, p_{m+n} are the vertices of a triangle Δ, which is one of the connected components in $\Sigma - (\boldsymbol{\alpha} \cup \boldsymbol{\beta} \cup \{\lambda_n, \lambda_{n+m}\})$. From the 4 quadrants which have q as a corner two of them belong to the neighbors of Δ. Place a pair of markings u and v in these two quadrants, and use them as the punctures in the following discussion.

The complex associated with the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; u, v)$ is denoted by $\widehat{\mathrm{CF}}(Y)$, the complex associated with $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n; u, v)$ and a given relative Spin^c class $\mathfrak{s}$ is denoted by $\widehat{\mathrm{CFK}}(K_n, \mathfrak{s})$ and the complex associated with $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_{n+m}; u, v)$ and the classes $\mathfrak{s}$ and $\mathfrak{s}+m$ is denoted by $C_{n,m}(\mathfrak{s}) = \widehat{\mathrm{CFK}}(K_{m+n}, \mathfrak{s}) \oplus \widehat{\mathrm{CFK}}(K_{m+n}, \mathfrak{s}+m)$.

Let Θ_f denote the top generator associated with $(\Sigma, \boldsymbol{\beta}_{n+m}, \boldsymbol{\beta}; u, v)$. Consider the holomorphic triangle map $f^{\mathfrak{s}} : C_{n,m}(\mathfrak{s}) \to \widehat{\mathrm{CF}}(Y)$ which is defined by

$$f^{\mathfrak{s}}(\mathbf{x}) := \sum_{\mathbf{z}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\Delta\in\pi_2^0(\mathbf{x},\Theta_f,\mathbf{z};u,v)} \#\big(\widehat{\mathcal{M}}(\Delta)\big).\mathbf{z}. \tag{2}$$

The diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_n, \boldsymbol{\beta}_{n+m}; u, v)$ determines a cobordism from $Y_n(K) \coprod L$ to $Y_{n+m}(K)$, where $L = L(m,1)\#(\#^{g-1}S^1 \times S^2)$. The intersection point q determines a canonical Spin^c class $\mathfrak{s}_q \in \mathrm{Spin}^c(L)$ in the sense of Definition 3.2 of [OS4]. Let Θ_g denote the top generator of $\widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\beta}_n, \boldsymbol{\beta}_{n+m}; u, v)$ which corresponds to $\mathfrak{s}_q$, or equivalently to the intersection point q. Define $g^{\mathfrak{s}} : \widehat{\mathrm{CFK}}(K_n, \mathfrak{s}) \to \widehat{\mathrm{CFK}}(K_{n+m})$ by

$$g^{\mathfrak{s}}(\mathbf{x}) := \sum_{\mathbf{z}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_{n+m}}} \sum_{\Delta\in\pi_2^0(\mathbf{x},\Theta_g,\mathbf{z};u,v)} \#\big(\widehat{\mathcal{M}}(\Delta)\big).\mathbf{z}.$$

If $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}$ then $g^{\mathfrak{s}}(\mathbf{x}) \in C_{n,m}(\mathfrak{s})$ (see the discussion after Lemma 8.2 in [AE]). Finally, the top generator $\Theta_h \in \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta}_n; u, v)$ and the triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}_n; u, v)$ determine the map $h^{\mathfrak{s}} : \widehat{\mathrm{CF}}(Y) \to \widehat{\mathrm{CFK}}(K_n, \mathfrak{s})$ defined by

$$h^{\mathfrak{s}}(\mathbf{x}) := \sum_{\substack{\mathbf{z}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_n}\\ \mathfrak{s}(\mathbf{z})=\mathfrak{s}}} \sum_{\Delta\in\pi_2^0(\mathbf{x},\Theta_h,\mathbf{z};u,v)} \#\big(\widehat{\mathcal{M}}(\Delta)\big).\mathbf{z}.$$

We thus arrive at the following the triangle of chain maps

$$\begin{array}{ccc} \widehat{\mathrm{CF}}(Y) & \xrightarrow{\quad h^{\mathfrak{s}} = h^{\mathfrak{s}}_n \quad} & \widehat{\mathrm{CFK}}(K_n, \mathfrak{s}) \\ {}^{f^{\mathfrak{s}} = f^{\mathfrak{s}}_n}\nwarrow & & \swarrow{}^{g^{\mathfrak{s}} = g^{\mathfrak{s}}_n} \\ & C_{n,m}(\mathfrak{s}) = \widehat{\mathrm{CFK}}(K_{m+n}, \mathfrak{s}) \oplus \widehat{\mathrm{CFK}}(K_{m+n}, \mathfrak{s}+m) & \end{array} \tag{3}$$

2.2. **Exactness of triangle.** Let $M(f^{\mathfrak{s}}_n)$ denote the mapping cone of $f^{\mathfrak{s}} = f^{\mathfrak{s}}_n$.

Theorem 2.1. *For $m \gg 1$ there is a map $H^{\mathfrak{s}}_{h_n} : \widehat{\mathrm{CFK}}(K_n, \mathfrak{s}) \to \widehat{\mathrm{CF}}(Y)$ which satisfies $d \circ H^{\mathfrak{s}}_{h_n} + H^{\mathfrak{s}}_{h_n} \circ d = f^{\mathfrak{s}}_n \circ g^{\mathfrak{s}}_n$. Moreover, the map $\imath^{\mathfrak{s}}_n : \widehat{\mathrm{CFK}}(K_n, \mathfrak{s}) \to M(f^{\mathfrak{s}}_n)$, defined by $\imath^{\mathfrak{s}}_n(\mathbf{x}) := (g^{\mathfrak{s}}_n(\mathbf{x}), H^{\mathfrak{s}}_{h_n}(\mathbf{x}))$ for $\mathbf{x} \in \widehat{\mathrm{CFK}}(K_n, \mathfrak{s})$, is a quasi-isomorphism.*

Proof. The proof is almost identical to the proof used in Section 8 from [AE]. We outline the proof to set up the notation. Define $H^{\mathfrak{s}}_f : \widehat{\mathrm{CF}}(Y) \to C_{n,m}(\mathfrak{s})$ by

$$H^{\mathfrak{s}}_f(\mathbf{x}) := \sum_{\substack{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_{n+m}}\\ \mathfrak{s}(\mathbf{y})\equiv\mathfrak{s}\ (\mathrm{mod}\ m)}} \sum_{\Box\in\pi_2^{-1}(\mathbf{x},\Theta_h,\Theta_g,\mathbf{y};u,v)} \#\big(\mathcal{M}(\Box)\big).\mathbf{y}.$$

The condition $\mathfrak{s}(\mathbf{y}) \equiv \mathfrak{s} \pmod m$ implies $\mathfrak{s}(\mathbf{y}) \in \{\mathfrak{s}, \mathfrak{s}+m\}$ since m is large. Considering all possible boundary degenerations of the one-dimensional moduli space corresponding to $\Box \in \pi_2^0(\mathbf{x}, \Theta_h, \Theta_g, \mathbf{y}; u, v)$ we find $d \circ H^{\mathfrak{s}}_f + H^{\mathfrak{s}}_f \circ d = h^{\mathfrak{s}} \circ g^{\mathfrak{s}}$. For this, one should note that the contributing classes $\Delta \in \pi_2^0(\Theta_h, \Theta_g, \Theta; u, v)$ with $\Theta \in \mathbb{T}_{\beta_{n+m}} \cap \mathbb{T}_\beta$ come in canceling pairs, where the difference between the coefficients of every canceling pair at a marking s (placed on the left-hand-side of β_g and close to u) is always a multiple of m. Similarly, define $H_g^{\mathfrak{s}}: C_{n,m}(\mathfrak{s}) \to \widehat{\mathrm{CF}}(Y)$ by

$$H_g^{\mathfrak{s}}(\mathbf{x}) := \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_n}} \sum_{\substack{\square\in\pi_2^{-1}(\mathbf{x},\Theta_f,\Theta_h,\mathbf{y};u,v)\\ n_s(\square)\equiv 0 \pmod m}} \#\big(\mathcal{M}(\square)\big).\mathbf{y}.$$

Since the contributing holomorphic triangles for $(\Sigma,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta},\boldsymbol{\beta}_n;u,v)$ and the closed top generators Θ_f,Θ_h come in canceling pairs, $d\circ H_g^{\mathfrak{s}}+H_g^{\mathfrak{s}} = h^{\mathfrak{s}}\circ f^{\mathfrak{s}}$. Finally, define the homotopy map $H_h^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_n,\mathfrak{s})\to\widehat{\mathrm{CF}}(Y)$ by

$$H_h^{\mathfrak{s}}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta}} \sum_{\square\in\pi_2^{-1}(\mathbf{x},\Theta_g,\Theta_f,\mathbf{y};u,v)} \#\big(\mathcal{M}(\square)\big).\mathbf{y}.$$

Employ the same argument again to show $d\circ H_h^{\mathfrak{s}}+H_h^{\mathfrak{s}}\circ d = f^{\mathfrak{s}}\circ g^{\mathfrak{s}}$.

We next introduce the pentagon maps. Let $\boldsymbol{\beta}_n'$ denote a g-tuple of simple closed curves which are small Hamiltonian isotopes of the curves in $\boldsymbol{\beta}_n$. Choosing the Hamiltonian isotopy sufficiently small we may assume that the chain complex associated with $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_n';u,v)$ and the Spin^c class $\mathfrak{s}$ may be identified with $\widehat{\mathrm{CFK}}(K_n,\mathfrak{s})$. There is a top generator Θ_h' for $\widehat{\mathrm{CF}}(\boldsymbol{\beta},\boldsymbol{\beta}_n';u,v)$ which is in correspondence with Θ_h. Define $P_f^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_n,\mathfrak{s})\to\widehat{\mathrm{CFK}}(K_n,\mathfrak{s})$ by

$$P_f^{\mathfrak{s}}(\mathbf{x}) := \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_n}} \sum_{\substack{\pentagon\in\pi_2^{-2}(\mathbf{x},\Theta_g,\Theta_f,\Theta_h',\mathbf{y};u,v)\\ n_s(\pentagon)\equiv 0 \pmod m}} \#\big(\mathcal{M}(\pentagon)\big).\mathbf{y}.$$

Five types of the ten possible degenerations in the boundary of the 1-dimensional moduli space associated with a class $\pentagon\in\pi_2^{-1}(\mathbf{x},\Theta_g,\Theta_f,\Theta_h',\mathbf{y};u,v)$ with $n_s(\pentagon)$ a multiple of m, which correspond to a degeneration to a bigon and a pentagon, contribute to the coefficient of $\mathbf{y}$ in $(d\circ P_f^{\mathfrak{s}}+P_f^{\mathfrak{s}}\circ d)(\mathbf{x})$. The remaining five types correspond to the degenerations of $\pentagon$ into a square $\square$ and a triangle Δ. Note that

- There is a unique contributing class $\square\in\pi_2^{-1}(\Theta_g,\Theta_f,\Theta_h',\Theta;u,v)$ which corresponds to the quadruple $(\Sigma,\boldsymbol{\beta}_n,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta},\boldsymbol{\beta}_n';u,v)$. Moreover, $\Theta=\Theta_n$ is the top generator for the diagram $(\Sigma,\boldsymbol{\beta}_n,\boldsymbol{\beta}_n';u,v)$.
- The contributing triangle classes
$$\Delta\in\pi_2^0(\Theta_g,\Theta_f,\Theta;u,v) \quad\text{and}\quad \Delta'\in\pi_2^0(\Theta_f,\Theta_h',\Theta;u,v)$$
corresponding to the triples $(\Sigma,\boldsymbol{\beta}_n,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta};u,v)$ and $(\Sigma,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta},\boldsymbol{\beta}_n';u,v)$ come in canceling pairs.

These observations imply that $d\circ P_f^{\mathfrak{s}}+P_f^{\mathfrak{s}}\circ d+J_f^{\mathfrak{s}} = h^{\mathfrak{s}}\circ H_h^{\mathfrak{s}}+H_g^{\mathfrak{s}}\circ g^{\mathfrak{s}}$, where

$$J_f^{\mathfrak{s}}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_n'}} \sum_{\Delta\in\pi_2^0(\mathbf{x},\Theta_n,\mathbf{y};u,v)} \#\big(\mathcal{M}(\Delta)\big).\mathbf{y}.$$

Consider the 5-tuple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}_n,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta}';u,v,z)$, where $\boldsymbol{\beta}'=\{\beta_1',...,\beta_g'\}$ is a set of g simple closed curves which are obtained from $\boldsymbol{\beta}$ by a small Hamiltonian isotopy. Thus β_i and β_i' intersect each other is a pair of canceling intersection points. We assume that the small area bounded between the two curves β_g and β_g' is formed as a union of two bigons; a small bigon which is a subset of the connected component of $\Sigma^\circ=\Sigma-\boldsymbol{\alpha}-\boldsymbol{\beta}-\boldsymbol{\beta}_n-\boldsymbol{\beta}_{n+m}$ which contains the marking v and a long and thin bigon which is stretched along β_g. We assume that the marking z is chosen in the intersection of the second bigon with the connected component in Σ°

which corresponds to u. For small Hamiltonian perturbations, the chain complex $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}';u)$ may be identified with $\widehat{\mathrm{CF}}(Y)$. Define $P_g^{\mathfrak{s}}:\widehat{\mathrm{CF}}(Y)\to\widehat{\mathrm{CF}}(Y)$ by

$$P_g^{\mathfrak{s}}(\mathbf{x}) := \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'}} \sum_{\substack{\pentagon\in\pi_2^{-2}(\mathbf{x},\Theta_h,\Theta_g,\Theta'_f,\mathbf{y};u,v)\\ n_z(\pentagon)+\mathfrak{s}(\mathbf{x})\equiv\mathfrak{s}\pmod m}} \#\big(\mathcal{M}(\pentagon)\big).\mathbf{y}.$$

Five types of the ten possible degenerations in the boundary of the 1-dimensional moduli space associated with a pentagon class $\pentagon\in\pi_2^{-1}(\mathbf{x},\Theta_h,\Theta_g^q,\Theta'_f,\mathbf{y};u,v)$ contribute to the coefficient of $\mathbf{y}$ in $(d\circ P_g^{\mathfrak{s}}+P_g^{\mathfrak{s}}\circ d)(\mathbf{x})$. The remaining five types correspond to the degenerations of $\pentagon$ into a square and a triangle. The choice of the markings implies that two of these degeneration types contribute to the coefficient of $\mathbf{y}$ in $(f^{\mathfrak{s}}\circ H_f^{\mathfrak{s}}+H_h^{\mathfrak{s}}\circ h^{\mathfrak{s}})(\mathbf{x})$. There is a unique contributing square class, corresponding to $(\Sigma,\boldsymbol{\beta},\boldsymbol{\beta}_n,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta}';u,v)$ and the intersection points $\Theta_h,\Theta_g,\Theta'_f,\Theta_\infty$, where Θ_∞ denotes the top generator for $(\Sigma,\boldsymbol{\beta},\boldsymbol{\beta}';u,v)$. Moreover, the triangles which contribute in $\pi_2(\Theta_h,\Theta_g,\Theta_f)$ and $\pi_2(\Theta_g,\Theta'_f,\Theta'_h)$ come in canceling pairs. Thus $d\circ P_g^{\mathfrak{s}}+P_g^{\mathfrak{s}}\circ d+J_g^{\mathfrak{s}}=f^{\mathfrak{s}}\circ H_f^{\mathfrak{s}}+H_h^{\mathfrak{s}}\circ h^{\mathfrak{s}}$, where

$$J_g^{\mathfrak{s}}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'}} \sum_{\Delta\in\pi_2^0(\mathbf{x},\Theta_\infty,\mathbf{y};u,v)} \#\big(\mathcal{M}(\Delta)\big).\mathbf{y}.$$

Let $\boldsymbol{\beta}'_{n+m}$ denote a g-tuple of simple closed curves which are small Hamiltonian isotopes of the curves in $\boldsymbol{\beta}_{n+m}$. Again, we assume that the chain complex associated with $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}'_{n+m};u,v)$ and the Spin^c classes $\mathfrak{s},\mathfrak{s}+m$ is identified with $C_{n,m}(\mathfrak{s})$. There is a top generator Θ'_g for $(\boldsymbol{\beta}_n,\boldsymbol{\beta}'_{n+m})$ which is in correspondence with Θ_g. Define $P_h^{\mathfrak{s}}:C_{n,m}(\mathfrak{s})\to C_{n,m}(\mathfrak{s})$ by

$$P_h^{\mathfrak{s}}(\mathbf{x}) := \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'_{n+m}}} \sum_{\substack{\pentagon\in\pi_2^{-2}(\mathbf{x},\Theta_f,\Theta_h,\Theta'_g,\mathbf{y};u,v)\\ n_z(\pentagon)\equiv 0\pmod m}} \#\big(\mathcal{M}(\pentagon)\big).\mathbf{y}.$$

A similar argument implies that $d\circ P_h^{\mathfrak{s}}+P_h^{\mathfrak{s}}\circ d+J_h^{\mathfrak{s}}=g^{\mathfrak{s}}\circ H_g^{\mathfrak{s}}+H_f^{\mathfrak{s}}\circ f^{\mathfrak{s}}$, where

$$J_h^{\mathfrak{s}}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'_{n+m}}} \sum_{\Delta\in\pi_2^0(\mathbf{x},\Theta_{n+m},\mathbf{y};u,v)} \#\big(\mathcal{M}(\Delta)\big).\mathbf{y},$$

and Θ_{n+m} is the top generator of $(\Sigma,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta}'_{n+m};u,v)$. Since $J_f^{\mathfrak{s}},J_g^{\mathfrak{s}},J_h^{\mathfrak{s}}$ are quasi-isomorphisms, Lemma 3.3 from [AE] completes the proof. □

Choose the markings s and t on Σ so that for each one of the pairs (z,s) and (v,t) there is an arc connecting them on Σ which cuts β_g in a single transverse point and stays disjoint from all other curves in $\boldsymbol{\alpha}\cup\boldsymbol{\beta}\cup\boldsymbol{\beta}'\cup\boldsymbol{\beta}_n\cup\boldsymbol{\beta}_{n+m}$, see Figure 1. Let $\Xi:\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};s)\to\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u)$ denote the chain homotopy equivalence given by the Heegaard moves which change $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};s)$ to $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u)$.

Define $\Xi\circ\overline{f}^{\mathfrak{s}}:C_{n,m}(\mathfrak{s})\to\widehat{\mathrm{CF}}(Y)$ by setting

$$\overline{f}^{\mathfrak{s}}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\Delta\in\pi_2^0(\mathbf{x},\mathbf{y};s,t)} \#(\mathcal{M}(\Delta)).\mathbf{y},$$

Lemma 2.2. *The chain maps $f^{\mathfrak{s}}$ and $\Xi\circ\overline{f}^{\mathfrak{s}}$ are chain homotopic.*

Proof. Note that the aforementioned Heegaard moves consist of $2g-2$ handle slides (composed with isotopies) on $\boldsymbol{\beta}$, supported away from the markings s,t. Denote the corresponding g-tuples of curves by $\boldsymbol{\beta}^0=\boldsymbol{\beta},\boldsymbol{\beta}^1,...,\boldsymbol{\beta}^{2g-2}$, where $\widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}^{2g-2};s)$ may be identified with $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u)$. The triple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}^{i-1},\boldsymbol{\beta}^i;s)$ and the top generator Θ^i of the diagram $(\Sigma,\boldsymbol{\beta}^{i-1},\boldsymbol{\beta}^i;s,t)$ determine a chain map $\Xi^i:\widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}^{i-1};s)\to\widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}^i;s)$. The triple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m},\boldsymbol{\beta}^i;s,t)$ and the top generator Θ^i_f of $(\Sigma,\boldsymbol{\beta}_{n+m},\boldsymbol{\beta}^i;s,t)$ determine $f^i:\widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m};s,t)\to\widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}^i;s)$. Finally, the quadruple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m},\boldsymbol{\beta}^{i-1},\boldsymbol{\beta}^i;s,t)$ together with Θ^{i-1}_f and Θ^i, determines a homomorphism $H^i:\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m};s,t)\to\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}^i;s)$. Considering different boundary degenerations of the one-dimensional moduli space associated with a square class of index 0 we find

$$d\circ H^i+H^i\circ d=f^i+\Xi^i\circ f^{i-1},\qquad i=1,...,2g-2. \tag{4}$$

Let us define $\Xi=\Xi^{2g-2}\circ\dots\Xi^1$ and set

$$H=H^{2g-2}+\Xi^{2g-2}\circ H^{2g-3}+\Xi^{2g-2}\circ\Xi^{2g-3}H^{2g-4}+\dots+(\Xi^{2g-2}\circ\dots\Xi^2)\circ H^1.$$

Using (4), $d\circ H+H\circ d=f^{2g-2}+\Xi\circ f^0$. To complete the proof, note that $f^{\mathfrak{s}}$ and $\overline{f}^{\mathfrak{s}}$ are the restrictions of f^{2g-2} and f^0 to $C_{n,m}(\mathfrak{s})$, respectively. □

3. The homomorphisms in the surgery triangle

Consider the triply punctured Heegaard 5-tuple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_0,\boldsymbol{\beta}_1,\boldsymbol{\beta}_m,\boldsymbol{\beta};u,v,w)$, as before, and assume that the local picture around the curves $\lambda_0,\lambda_1,\lambda_m,\lambda_\infty$ is the one illustrated in Figure 1. The top generators $\Theta_{0,1}$, Θ_{g_1} and Θ_{f_1} of the Heegaard diagrams $(\Sigma,\boldsymbol{\beta}_0,\boldsymbol{\beta}_1;u,v,w)$, $(\Sigma,\boldsymbol{\beta}_1,\boldsymbol{\beta}_m;u,v,w)$ and $(\Sigma,\boldsymbol{\beta}_m,\boldsymbol{\beta};u,v,w)$ (respectively) determine the holomorphic pentagon map $P^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_0,\mathfrak{s})\to\widehat{\mathrm{CF}}(Y)$, where

$$P^{\mathfrak{s}}(\mathbf{x}):=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\pentagon\in\pi_2^{-2}(\mathbf{x},\Theta_{0,1},\Theta_{g_1},\Theta_{f_1},\mathbf{y};u,v,w)}\#\left(\mathcal{M}(\pentagon)\right).\mathbf{y}.$$

Every pentagon class $\pentagon\in\pi_2^{-1}(\mathbf{x},\Theta_{0,1},\Theta_{g_1},\Theta_{f_1},\mathbf{y};u,v,w)$ corresponds to a 1-dimensional moduli space with boundary. The boundary points are in correspondence with the degeneration of the domain of $\pentagon$ into two parts. Since the generators $\Theta_{0,1},\Theta_{g_1}$ and Θ_{f_1} are closed, the degenerations into a bigon and a pentagon correspond to the the coefficient of $\mathbf{y}$ in $(d\circ P^{\mathfrak{s}}+P^{\mathfrak{s}}\circ d)(\mathbf{x})$. The remaining degenerations are the degenerations $\pentagon=\square\star\Delta$ to a triangle Δ with Maslov index 0 and a square $\square$ with Maslov index -1 which miss u,v and w. The possibilities are

(1) $\square\in\pi_2(\mathbf{z},\Theta_{g_1},\Theta_{f_1},\mathbf{y})$ and $\Delta\in\pi_2(\mathbf{x},\Theta_{0,1},\mathbf{z})$,
(2) $\square\in\pi_2(\mathbf{x},\Theta_{0,1},\Theta_{g_1},\mathbf{z})$ and $\Delta\in\pi_2(\mathbf{z},\Theta_{f_1},\mathbf{y})$,
(3) $\square\in\pi_2(\mathbf{x},\Theta_{0,1},\Theta,\mathbf{y})$ and $\Delta\in\pi_2(\Theta_{g_1},\Theta_{f_1},\Theta)$,
(4) $\square\in\pi_2(\mathbf{x},\Theta,\Theta_{f_1},\mathbf{y})$ and $\Delta\in\pi_2(\Theta_{0,1},\Theta_{g_1},\Theta)$,
(5) $\square\in\pi_2(\Theta_{0,1},\Theta_{g_1},\Theta_{f_1},\Theta)$ and $\Delta\in\pi_2(\mathbf{x},\Theta,\mathbf{y})$.

Degenerations of type 1 correspond to the coefficient of $\mathbf{y}$ in $(H^{\mathfrak{s}}_{h_1}\circ\mathfrak{f}^{\mathfrak{s}}_\infty)(\mathbf{x})$, where the map induced by $\mathfrak{f}^{\mathfrak{s}}_\infty:\widehat{\mathrm{CFK}}(K_0,\mathfrak{s})\to\widehat{\mathrm{CFK}}(K_1,\mathfrak{s})$ in homology is the homomorphism $\mathfrak{f}^{\mathfrak{s}}_\infty:\mathbb{H}_0(K,\mathfrak{s})\to\mathbb{H}_1(K,\mathfrak{s})$, which appears in the splicing formula of [Ef4].

Degenerations of type 2 correspond to the coefficient of $\mathbf{y}$ in $f_1^{\mathfrak{s}} \circ H^{\mathfrak{s}}(\mathbf{x})$, where $H^{\mathfrak{s}} : \widehat{\mathrm{CFK}}(K_0, \mathfrak{s}) \to C_{0,m}(\mathfrak{s})$ is defined by

$$H^{\mathfrak{s}}(\mathbf{x}) = \sum_{\mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta_m}} \sum_{\square \in \pi_2^{-1}(\mathbf{x}, \Theta_{0,1}, \Theta_{g_1}, \mathbf{z}; u,v,w)} \#\left(\mathcal{M}(\square)\right).\mathbf{z}.$$

In a degeneration of type 3, the contributing triangle classes Δ come in canceling pairs. The total count of such degenerations is thus trivial. Furthermore, there are no holomorphic representatives for the square classes which appear in the boundary degenerations of type 5, i.e. we may assume that there are no such degenerations. In a degeneration of type 4, the moduli space corresponding to Δ is trivial unless $\Theta = \Theta_{g_0}$ and Δ corresponds to the union of small triangles connecting $\Theta_{0,1}, \Theta_{g_1}$ and Θ_{g_0}. In this latter case the contribution of such triangles is 1. The number of such boundary degenerations (modulo 2) is thus equal to

$$\sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\square \in \pi_2^{-1}(\mathbf{x}, \Theta_{g_0}, \Theta_{f_1}, \mathbf{y}; u,v,w)} \#\left(\mathcal{M}(\square)\right).\mathbf{y} = H_{h_0}^{\mathfrak{s}}(\mathbf{x}).$$

Summarizing the above observations we arrive at the following.

Lemma 3.1. *With the above notation fixed*

$$d \circ P^{\mathfrak{s}} + P^{\mathfrak{s}} \circ d + H_{h_0}^{\mathfrak{s}} = f_1^{\mathfrak{s}} \circ H^{\mathfrak{s}} + H_{h_1}^{\mathfrak{s}} \circ \mathfrak{f}_\infty^{\mathfrak{s}}. \tag{5}$$

Next, we analyse $H^{\mathfrak{s}}$ via degenerations of holomorphic squares. For a square class $\square \in \pi_2^0(\mathbf{x}, \Theta_{0,1}, \Theta_{g_1}, \mathbf{y}; u, v, w)$ the moduli space $\mathcal{M}(\square)$ is 1-dimensional, and has 6 types of boundary ends. Since $\Theta_{0,1}$ and Θ_{g_1} are closed, the 4 types of degenerations of the square class to a square and a bigon correspond to the coefficient of $\mathbf{y}$ in $(d \circ H^{\mathfrak{s}} + H^{\mathfrak{s}} \circ d)(\mathbf{x})$. The remaining boundary ends correspond to a degeneration of $\square$ to a pair of triangles. The degenerations $\square = \Delta' \star \Delta$ with $\Delta \in \pi_2(\mathbf{x}, \Theta_{0,1}, \mathbf{z})$ and $\Delta' \in \pi_2(\mathbf{z}, \Theta_{g_1}, \mathbf{y})$ correspond to the coefficient of $\mathbf{y}$ in $(g_1^{\mathfrak{s}} \circ \mathfrak{f}_\infty^{\mathfrak{s}})(\mathbf{x})$.

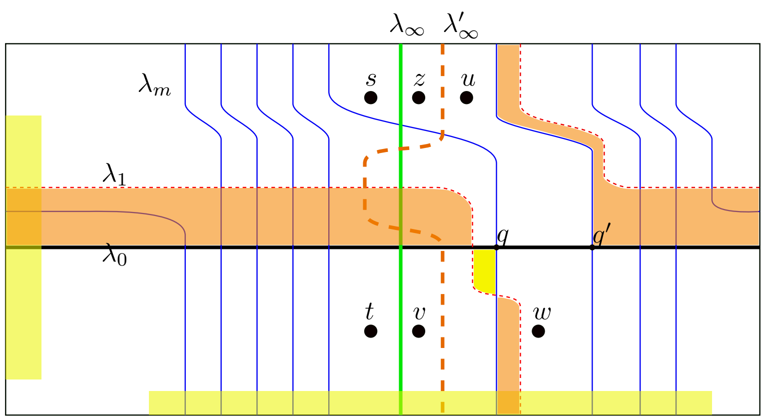

FIGURE 1. The arrangement of the curves on the Heegaard surface. Other curves and handles appear on the shaded yellow area.

The more tricky and interesting part is the contribution of the boundary degenerations of the form $\square = \Delta' \star \Delta$ with $\Delta' \in \pi_2^0(\Theta_{0,1}, \Theta_{g_1}, \Theta; u, v, w)$ and $\Delta \in \pi_2^0(\mathbf{x}, \Theta, \mathbf{y}; u, v, w)$. There are precisely two generators Θ with a corresponding

$\Delta' = \Delta_\Theta$ such that $\mathcal{M}(\Delta_\Theta)$ is non-empty. One of them corresponds to $\Theta = \Theta_{g_0}$ and the other one corresponds to Θ'_{g_0} which is obtained from Θ_{g_0} by changing q to the intersection point $q' \in \lambda_0 \cap \lambda_m$ which is next to q (see Figure 1). The total contribution of such boundary ends is thus given by

$$\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta_m}} \left(\sum_{\Delta\in\pi_2^0(\mathbf{x},\Theta_{g_0},\mathbf{y};u,v,w)} \#\left(\mathcal{M}(\Delta)\right) + \sum_{\Delta\in\pi_2^0(\mathbf{x},\Theta'_{g_0},\mathbf{y};u,v,w)} \#\left(\mathcal{M}(\Delta)\right) \right).\mathbf{y}.$$

Suppose that the decomposition of $g_0^\mathfrak{s}$ in $\widehat{\mathrm{CFK}}(K_m,\mathfrak{s})\oplus\widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m)$ is given by $g_0^\mathfrak{s}(\mathbf{x}) = (g_{0,1}^\mathfrak{s}(\mathbf{x}), g_{0,2}^\mathfrak{s}(\mathbf{x}))$. Then the above sum is equal to $g_{0,1}^\mathfrak{s}(\mathbf{x}) + G^\mathfrak{s}(\mathbf{x})$, where

$$G^\mathfrak{s} : \widehat{\mathrm{CFK}}(K_0,\mathfrak{s}) \longrightarrow \widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m-1)$$

is defined by the second sum above. In Section 4 we show that for sufficiently large m and an appropriate Heegaard 5-tuple we may assume that there is an embedding

$$J^\mathfrak{s} : \widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m) \longrightarrow \widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m-1)$$

such that $G^\mathfrak{s}(\mathbf{x}) = J^\mathfrak{s}(g_{0,2}^\mathfrak{s}(\mathbf{x}))$. Define $G_\infty^\mathfrak{s} : C_{0,m}(\mathfrak{s}) \to C_{0,m}(\mathfrak{s})$ by

$$G_\infty^\mathfrak{s}(\mathbf{x}_1,\mathbf{x}_2) := (\mathbf{x}_1, J^\mathfrak{s}(\mathbf{x}_2)) \quad \forall\, \mathbf{x}_1 \in \widehat{\mathrm{CFK}}(K_m,\mathfrak{s}),\ \forall\, \mathbf{x}_2 \in \widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m).$$

The above observations imply the following.

Lemma 3.2. *With the above notation fixed we have*

$$d\circ H^\mathfrak{s} + H^\mathfrak{s}\circ d = g_1^\mathfrak{s}\circ \mathfrak{f}_\infty^\mathfrak{s} - G_\infty^\mathfrak{s}\circ g_0^\mathfrak{s}. \tag{6}$$

Let us now consider the Heegaard 5-tuple $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_m, \boldsymbol{\beta}, \boldsymbol{\beta}'; u, v, z)$ where the curves in $\boldsymbol{\beta}'$ are small Hamiltonian isotopes of the corresponding curves in $\boldsymbol{\beta}$. Moreover, we assume that the intersection pattern between $\lambda_1, \lambda_m, \lambda_\infty = \beta_g$ and $\lambda'_\infty = \beta'_g$ and the location of u, v and z follows the pattern illustrated in Figure 1. We may assume that the chain complex associated with the punctured Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}'; u, v, z)$ and $\mathfrak{s}\in\mathbb{Z}$ is $\widehat{\mathrm{CFK}}(K,\mathfrak{s})$. Associated with $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta}'; u, v, z)$ there is a *top generator* which may be denoted by Θ'_∞. Unlike most of such situations Θ'_∞ is not closed and $d(\Theta'_\infty) = \Theta_\infty$ is the generator which is obtained from Θ'_∞ by changing the choice of intersection point in $\lambda_\infty \cap \lambda'_\infty$. By construction, Θ_∞ is closed. The diagram $\mathcal{H}$ defines a pentagon map $Q^\mathfrak{s} : \widehat{\mathrm{CFK}}(K_1,\mathfrak{s}) \to \widehat{\mathrm{CFK}}(K,\mathfrak{s})$:

$$Q^\mathfrak{s}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'},\mathfrak{s}(\mathbf{y})=\mathfrak{s}} \quad \sum_{\pentagon\in\pi_2^{-2}(\mathbf{x},\Theta_{g_1},\Theta_{f_1},\Theta_\infty,\mathbf{y};u,v,z)} \#\left(\mathcal{M}(\pentagon)\right).\mathbf{y}.$$

For $\pentagon \in \pi_2^{-1}(\mathbf{x}, \Theta_{g_1}, \Theta_{f_1}, \Theta_\infty, \mathbf{y}; u, v, z)$, the ends of the moduli space $\mathcal{M}(\pentagon)$ which correspond to the degenerations of the pentagon either to a bigon and a pentagon contribute to the coefficient of $\mathbf{y}$ in $(d\circ Q^\mathfrak{s} + Q^\mathfrak{s}\circ d)(\mathbf{x})$. Other ends correspond to the degeneration are of the form $\pentagon = \square \star \Delta$ of one of the following 5 types:

(1) $\square \in \pi_2(\mathbf{z}, \Theta_{f_1}, \Theta_\infty, \mathbf{y})$ and $\Delta \in \pi_2(\mathbf{x}, \Theta_{g_1}, \mathbf{z})$,
(2) $\square \in \pi_2(\mathbf{x}, \Theta_{g_1}, \Theta_{f_1}, \mathbf{z})$ and $\Delta \in \pi_2(\mathbf{z}, \Theta_\infty, \mathbf{y})$,
(3) $\square \in \pi_2(\mathbf{x}, \Theta_{g_1}, \Theta, \mathbf{y})$ and $\Delta \in \pi_2(\Theta_{f_1}, \Theta_\infty, \Theta)$,
(4) $\square \in \pi_2(\mathbf{x}, \Theta, \Theta_\infty, \mathbf{y})$ and $\Delta \in \pi_2(\Theta_{g_1}, \Theta_{f_1}, \Theta)$,
(5) $\square \in \pi_2(\Theta_{g_1}, \Theta_{f_1}, \Theta_\infty, \Theta)$ and $\Delta \in \pi_2(\mathbf{x}, \Theta, \mathbf{y})$.

Degenerations of types 1 and 2 correspond to the coefficient of $\mathbf{y}$ in $(I^{\mathfrak{s}}\circ g_1^{\mathfrak{s}})(\mathbf{x})$ and $(X^{\mathfrak{s}}\circ H_{h_1}^{\mathfrak{s}})(\mathbf{x})$, respectively, where

$$I^{\mathfrak{s}}(\mathbf{z}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'},\mathfrak{s}(\mathbf{y})=\mathfrak{s}} \quad \sum_{\square\in\pi_2^{-1}(\mathbf{z},\Theta_{f_1},\Theta_\infty,\mathbf{y};u,v,z)} \#\left(\mathcal{M}(\square)\right).\mathbf{y} \quad \text{and}$$
$$X^{\mathfrak{s}}(\mathbf{z}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_{\beta'},\mathfrak{s}(\mathbf{y})=\mathfrak{s}} \quad \sum_{\Delta\in\pi_2^{0}(\mathbf{z},\Theta_\infty,\mathbf{y};u,v,z)} \#\left(\mathcal{M}(\Delta)\right).\mathbf{y}.$$

Considering the local multiplicities around $\lambda_\infty\cap\lambda'_\infty$ one concludes that there are no triangle classes $\Delta\in\pi_2^0(\mathbf{z},\Theta_\infty,\mathbf{y};u,v,z)$ with positive domain. In particular, $X^{\mathfrak{s}}$ is trivial. There are no triangle classes which contribute in the degenerations of type 3. The contributing triangles in degenerations of type 4 come in canceling pairs. Thus the total number of boundary ends corresponding to degenerations of types 3 and 4 is zero. There is a unique square class in $\pi_2^{-1}(\Theta_{g_1},\Theta_{f_1},\Theta_\infty,\Theta;u,v,z)$ with non-trivial contribution to degenerations of type 5. For this square class $\Theta\in\mathbb{T}_{\beta_1}\cap\mathbb{T}_{\beta'}$ is the top generator, and $\square$ has a unique holomorphic representative. Using the generator Θ we define $\mathfrak{f}_0^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_1,\mathfrak{s})\to\widehat{\mathrm{CFK}}(K,\mathfrak{s})$. The contribution of the degenerations of type 5 thus corresponds to the coefficient of $\mathbf{y}$ in $\mathfrak{f}_0^{\mathfrak{s}}(\mathbf{x})$. The map on homology induced by $\mathfrak{f}_0^{\mathfrak{s}}$ coincides with the map used in the splicing formula of [Ef4].

Define the maps $F_0^{\mathfrak{s}}: M(f_1^{\mathfrak{s}})\to\widehat{\mathrm{CFK}}(K,\mathfrak{s})$ and $F_\infty^{\mathfrak{s}}: M(f_0^{\mathfrak{s}})\to M(f_1^{\mathfrak{s}})$ by

$$F_0^{\mathfrak{s}}(\mathbf{x}_1,\mathbf{x}_2) := I^{\mathfrak{s}}(\mathbf{x}_1), \qquad \forall\ \mathbf{x}_1\in C_{1,m-1}(\mathfrak{s}),\ \forall\ \mathbf{x}_2\in\widehat{\mathrm{CF}}(Y) \quad \text{and}$$
$$F_\infty^{\mathfrak{s}}(\mathbf{x}_1,\mathbf{x}_2) := (G_\infty^{\mathfrak{s}}(\mathbf{x}_1),-\mathbf{x}_2), \quad \forall\ \mathbf{x}_1\in C_{0,m}(\mathfrak{s}), \quad \forall\ \mathbf{x}_2\in\widehat{\mathrm{CF}}(Y).$$

With this notation fixed, the outcome of the above observations, together with Lemma 3.1 and Lemma 3.2 is the following theorem.

Theorem 3.3. *With the above notation fixed and up to chain homotopy, the following diagram is commutative:*

$$\begin{array}{ccccc}
\widehat{\mathrm{CFK}}(K_0,\mathfrak{s}) & \xrightarrow{\mathfrak{f}_\infty^{\mathfrak{s}}} & \widehat{\mathrm{CFK}}(K_1,\mathfrak{s}) & \xrightarrow{\mathfrak{f}_0^{\mathfrak{s}}} & \widehat{\mathrm{CFK}}(K,\mathfrak{s}) \\
\downarrow{\imath_0^{\mathfrak{s}}} & & \downarrow{\imath_1^{\mathfrak{s}}} & & \downarrow{Id} \\
M(f_0^{\mathfrak{s}}) & \xrightarrow{F_\infty^{\mathfrak{s}}} & M(f_1^{\mathfrak{s}}) & \xrightarrow{F_0^{\mathfrak{s}}} & \widehat{\mathrm{CFK}}(K,\mathfrak{s})
\end{array} \qquad (7)$$

Proof. By the preceding discussion, $\mathfrak{f}_0^{\mathfrak{s}}+F_0^{\mathfrak{s}}\circ\imath_1^{\mathfrak{s}}=d\circ Q^{\mathfrak{s}}+Q^{\mathfrak{s}}\circ d$. This proves the commutativity of the right-hand-side square upto chain homotopy. To prove the commutativity of the left-hand-side square, define $R^{\mathfrak{s}}:\widehat{\mathrm{CFK}}(K_0,\mathfrak{s})\to M(f_1^{\mathfrak{s}})$ by $R^{\mathfrak{s}}(\mathbf{x}):=(H^{\mathfrak{s}}(\mathbf{x}),P^{\mathfrak{s}}(\mathbf{x}))$. We thus find

$$\begin{aligned}
(d\circ R^{\mathfrak{s}}+R^{\mathfrak{s}}\circ d)(\mathbf{x}) &= d(H^{\mathfrak{s}}(\mathbf{x}),P^{\mathfrak{s}}(\mathbf{x}))+(R^{\mathfrak{s}}\circ d)(\mathbf{x})\\
&= \big((G_\infty^{\mathfrak{s}}\circ g_0^{\mathfrak{s}}+g_1^{\mathfrak{s}}\circ\mathfrak{f}_\infty^{\mathfrak{s}})(\mathbf{x}),(d\circ P^{\mathfrak{s}}+P^{\mathfrak{s}}\circ d+f_1^{\mathfrak{s}}\circ H^{\mathfrak{s}})(\mathbf{x})\big)\\
&= \big((g_1^{\mathfrak{s}}\circ\mathfrak{f}_\infty^{\mathfrak{s}}+G_\infty^{\mathfrak{s}}\circ g_0^{\mathfrak{s}})(\mathbf{x}),(H_{h_1}^{\mathfrak{s}}\circ\mathfrak{f}_\infty^{\mathfrak{s}}+H_{h_0}^{\mathfrak{s}})(\mathbf{x})\big)\\
&= (F_\infty^{\mathfrak{s}}\circ\imath_0^{\mathfrak{s}}+\imath_1^{\mathfrak{s}}\circ\mathfrak{f}_\infty^{\mathfrak{s}})(\mathbf{x}).
\end{aligned}$$

The second equality follows from Lemma 3.2, while the third equality follows from Lemma 3.1. This observation completes the proof of Theorem 3.3. $\square$

4. Surgery and splicing formulas for knots

4.1. **Surgery formulas.** Theorem 2.1 implies that $\widehat{\mathrm{CFK}}(K_n,\mathfrak{s})$ is quasi-isomorphic, for m sufficiently large, to the mapping cone of $f_n^{\mathfrak{s}}: C_{n,m}(\mathfrak{s})\to\widehat{\mathrm{CF}}(Y)$. When the curve λ_{n+m} is very close to the juxtaposition of λ and $(n+m)\beta_g$, and it cuts β_g almost in the middle of the winding region, this mapping cone has a particularly easy description, which follows. With this choice, we may assume that associated with every generator $\mathbf{x}$ for the Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u)$, which in turn is a generator of $\widehat{\mathrm{CF}}(Y)$, we obtain $n+m$ generators for $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m};u,v)$. These $n+m$ generators will be denoted by $\mathbf{x}_{1-l},\mathbf{x}_{2-l},...,\mathbf{x}_{m+n-l}$, where $l=\lfloor m/2\rfloor$ and $\mathbf{x}_i$ is on the left of β_g if $i<0$ and is on the right of β_g otherwise. The rest of generators for the Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_{n+m};u,v)$ are in correspondence with the generators $\mathbf{y}$ of $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}_0;u,v)$. Every such generator will be denote by $\widehat{\mathbf{y}}$. With this notation fixed we have

$$\mathfrak{s}(\mathbf{x}_i)=\begin{cases}\mathfrak{s}(\mathbf{x})+i & \text{if } i\geq 0\\ \mathfrak{s}(\mathbf{x})+n+m+i & \text{if } i<0\end{cases}\quad\text{and}\quad \mathfrak{s}(\widehat{\mathbf{y}})=\mathfrak{s}(\mathbf{y})+n+\left\lceil\frac{m}{2}\right\rceil.$$

Restricting our attention to the relative Spin^c classes $\mathfrak{s}$ and $\mathfrak{s}+m$ we find

$$\begin{aligned}\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s})&=\left\langle \mathbf{x}_{\mathfrak{s}-\mathfrak{s}(\mathbf{x})}\ \middle|\ \mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\ \text{ and }\ \mathfrak{s}(\mathbf{x})\leq\mathfrak{s}\right\rangle,\\ \widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s}+m)&=\left\langle \mathbf{x}_{\mathfrak{s}-\mathfrak{s}(\mathbf{x})-n}\ \middle|\ \mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\ \text{ and }\ \mathfrak{s}(\mathbf{x})>\mathfrak{s}-n\right\rangle,\end{aligned}$$

If the curve λ_{n+m} is sufficiently close to the juxtaposition $\lambda\star(m+n)\beta_g$ the first complex is identified with the sub-complex

$$\left\langle \mathbf{x}\ \middle|\ \mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\ \text{ and }\ \mathfrak{s}(\mathbf{x})\leq\mathfrak{s}\right\rangle$$

of $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u)$, while the restriction of the map $f^{\mathfrak{s}}$ to $\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s})$ is identified with the inclusion of the above sub-complex in $\widehat{\mathrm{CF}}(Y)$ (c.f. proof of Theorem 4.4 in [OS3]). Similarly, the second complex is identified with the sub-complex

$$\left\langle \mathbf{x}\ \middle|\ \mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\ \text{ and }\ \mathfrak{s}(\mathbf{x})>\mathfrak{s}-n\right\rangle$$

of $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};s)$ while the restriction of the map $\overline{f}^{\mathfrak{s}}$ to $\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s}+m)$ is identified with the inclusion of the aforementioned sub-complex in $\widehat{\mathrm{CF}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};s)$.

Let $C=C_K$ denote the $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain complex generated by triples $[\mathbf{x},i,j]$ with $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta,\ i,j\in\mathbb{Z}$ and $\mathfrak{s}(\mathbf{x})-i+j=0$. The differential of C is defined by

$$d[\mathbf{x},i,j]=\sum_{\substack{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\\ \phi\in\pi_2^1(\mathbf{x},\mathbf{y})}}\#\left(\widehat{\mathcal{M}}(\phi)\right)[\mathbf{y},i-n_u(\phi),j-n_s(\phi)]=:\sum_{a,b=0}^{\infty}[d^{a,b}(\mathbf{x}),i-a,j-b].$$

Since $d\circ d=0$ we conclude that $d^{0,0}\circ d^{0,0}=0$, while

$$\begin{aligned}&d^{0,1}\circ d^{0,0}+d^{0,0}\circ d^{0,1}=0,\qquad d^{1,0}\circ d^{0,0}+d^{0,0}\circ d^{1,0}=0\\ &\text{and}\quad d^{1,1}\circ d^{0,0}+d^{0,0}\circ d^{1,1}+d^{0,1}\circ d^{1,0}+d^{1,0}\circ d^{0,1}=0.\end{aligned}\tag{8}$$

Following [OS4] (or the notation of introduction) $\widehat{\mathrm{CF}}(Y)$ is identified as $C\{j=0\}$, while $\widehat{\mathrm{CFK}}(K_{n+m},\mathfrak{s})$ and $\widehat{\mathrm{CF}}(K_{n+m},\mathfrak{s}+m)$ are identified with $C\{i\leq\mathfrak{s},j=0\}$ and $C\{i=0,j\leq n-\mathfrak{s}-1\}$, respectively. There is a chain homotopy equivalence Ξ from $C\{i=0\}$ to $C\{j=0\}$. The following is thus a re-statement of Theorem 2.1.

Theorem 4.1. *For every $\mathfrak{s} \in \mathbb{Z} = \underline{\mathrm{Spin}}^c(Y,K)$ and $n \in \mathbb{Z}$ the chain complex $\widehat{\mathrm{CFK}}(K_n,\mathfrak{s})$ is quasi-isomorphic to the mapping cone $M(i_n^{\mathfrak{s}})$ of*

$$i_n^{\mathfrak{s}} : C\{i \le \mathfrak{s}, j=0\} \oplus C\{i=0, j \le n-\mathfrak{s}-1\} \longrightarrow C\{j=0\},$$
$$i_n^{\mathfrak{s}}([\mathbf{x},i,0],[\mathbf{y},0,j]) := [\mathbf{x},i,0] + \Xi[\mathbf{y},0,j].$$

4.2. **The bypass homomorphisms.** We now turn to understanding the maps $F_0^{\mathfrak{s}}$ and $F_\infty^{\mathfrak{s}}$ (which will be called the *bypass homomorphisms*) under the above identifications. To understand $F_0^{\mathfrak{s}}$, one should identify $I^{\mathfrak{s}}$ on

$$\widehat{\mathrm{CFK}}(K_m,\mathfrak{s}) \oplus \widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m-1) = C\{i \le \mathfrak{s}, j=0\} \oplus C\{i=0, j \le -\mathfrak{s}\}.$$

Let $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{x}_i$ be the corresponding generator in $\widehat{\mathrm{CFK}}(K_m)$ and suppose that $\square \in \pi_2^{-1}(\mathbf{x}_i, \Theta_{f_1}, \Theta_\infty, \mathbf{y}; u, v, z)$ contributes to $I^{\mathfrak{s}}$. Looking at local coefficients in the regions pictured in Figure 1 implies that $i=-1$. In particular, $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}(\mathbf{y}) = \mathfrak{s}$ and $\mathbf{x}_{-1}$ corresponds to the generator $[\mathbf{x},0,-\mathfrak{s}] \in C\{i=0, j \le -\mathfrak{s}\}$. There is a particular class $\square \in \pi_2(\mathbf{x}_{-1}, \Theta_{f_1}, \Theta_\infty, \mathbf{x})$ with small domain and non-trivial contribution to $I^{\mathfrak{s}}$. Modifying $\widehat{\mathrm{CFK}}(K,\mathfrak{s}) = C\{i=0, j=-\mathfrak{s}\}$ by the chain map $I^{\mathfrak{s}}|_{C\{i=0,j=-\mathfrak{s}\}}$, which is a change of basis using the energy filtration, we may thus assume that $F_0^{\mathfrak{s}}$ is induced by projecting the factor $C\{i=0, j \le -\mathfrak{s}\}$ in the mapping cone of $i_1^{\mathfrak{s}}$ over the quotient complex $C\{i=0, j=-\mathfrak{s}\} = \widehat{\mathrm{CFK}}(K,\mathfrak{s})$.

We next study $G^{\mathfrak{s}} : \widehat{\mathrm{CFK}}(K_0,\mathfrak{s}) \to \widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m-1)$ in order to understand $F_\infty^{\mathfrak{s}}$. Local considerations imply that for a triangle class $\Delta \in \pi_2^0(\mathbf{x}, \Theta'_{g_0}, \mathbf{y})$ with non-trivial contribution to $G^{\mathfrak{s}}$ we have $\mathbf{y} = \mathbf{z}_i$ with $\mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $i \le -2$. Every such Δ corresponds to a triangle class $\Delta' \in \pi_2^0(\mathbf{x}, \Theta_{g_0}, \mathbf{z}_{i+1}; u, v, w)$, and if λ_m is sufficiently close to $\lambda \star m\beta_g$ and m is sufficiently large, $\mathcal{M}(\Delta)$ and $\mathcal{M}(\Delta')$ may in fact be identified. Such Δ' are the classes which contribute to the holomorphic triangle map $g_{0,2}^{\mathfrak{s}}$, and $\mathfrak{s}(\mathbf{z}_{i+1}) = \mathfrak{s}(\mathbf{z}_i) + 1 = \mathfrak{s} + m$. The image of $G^{\mathfrak{s}}$ is thus in

$$C\{i=0, j \le -\mathfrak{s}-1\} \subset \widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m-1) = C\{i=0, j \le -\mathfrak{s}\}.$$

If $J^{\mathfrak{s}}$ from $\widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m) = C\{j < -\mathfrak{s}\}$ to $\widehat{\mathrm{CFK}}(K_m,\mathfrak{s}+m-1) = C\{j \le -\mathfrak{s}\}$ denotes the inclusion, we find $G^{\mathfrak{s}}(\mathbf{x}) = J^{\mathfrak{s}}(g_{0,2}^{\mathfrak{s}}(\mathbf{x}))$, implying the following theorem.

Theorem 4.2. *Under the identification of $\widehat{\mathrm{CFK}}(K_\bullet,\mathfrak{s})$ with $M(i_\bullet^{\mathfrak{s}})$ for $\bullet = 0,1$, $F_\infty^{\mathfrak{s}}$ is given by the inclusion of $M(i_0^{\mathfrak{s}})$ in $M(i_1^{\mathfrak{s}})$ as a sub-complex, while $F_0^{\mathfrak{s}}$ is given by the quotient map. In particular, we have a short exact sequence*

$$0 \longrightarrow M(i_0^{\mathfrak{s}}) \xrightarrow{F_\infty^{\mathfrak{s}} = \hookrightarrow} M(i_1^{\mathfrak{s}}) \xrightarrow{F_0^{\mathfrak{s}}} \widehat{\mathrm{CFK}}(K,\mathfrak{s}) = \frac{M(i_1^{\mathfrak{s}})}{M(i_0^{\mathfrak{s}})} \longrightarrow 0.$$

Theorem 4.2 implies that the second row in (7) is part of a short exact sequence. The discussion preceding Theorem 4.6 in [Ef4] implies that the initial Heegaard diagram may be chosen so that the first row is also completed to a short exact sequence. We thus have the following commutative diagram (upto chain homotopy):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathrm{CFK}}(K_0,\mathfrak{s}) & \xrightarrow{\mathfrak{f}_\infty^{\mathfrak{s}}} & \widehat{\mathrm{CFK}}(K_1,\mathfrak{s}) & \xrightarrow{\mathfrak{f}_0^{\mathfrak{s}}} & \widehat{\mathrm{CFK}}(K,\mathfrak{s}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i_0^{\mathfrak{s}}} & & \downarrow{\scriptstyle i_1^{\mathfrak{s}}} & & \downarrow{\scriptstyle Id} & & \\ 0 & \longrightarrow & M(i_0^{\mathfrak{s}}) & \xrightarrow{F_\infty^{\mathfrak{s}}} & M(i_1^{\mathfrak{s}}) & \xrightarrow{F_0^{\mathfrak{s}}} & \widehat{\mathrm{CFK}}(K,\mathfrak{s}) & \longrightarrow & 0 \end{array} \quad . \tag{9}$$

In particular, in the level of homology, the connecting homomorphism of the short exact sequence in the second row of (9) is identified with the connecting homomorphism $\mathfrak{f}_1^{\mathfrak{s}}$ of the first row, which is used in the splicing formula of [Ef4]. A completely similar argument identifies $\bar{\mathfrak{f}}_\infty^{\mathfrak{s}}$ with the inclusion map $\overline{F}_\infty^{\mathfrak{s}}$ from $M(i_0^{\mathfrak{s}-1})$ to $M(i_1^{\mathfrak{s}})$ and $\bar{\mathfrak{f}}_0^{\mathfrak{s}}$ with the quotient map $\overline{F}_0^{\mathfrak{s}}$ to $\widehat{\mathrm{CFK}}(K,\mathfrak{s})$, while $\bar{\mathfrak{f}}_1^{\mathfrak{s}}$ is identified with the connecting homomorphism of the short exact sequence

$$0 \longrightarrow M(i_0^{\mathfrak{s}-1}) \xrightarrow{\overline{F}_\infty^{\mathfrak{s}}} M(i_1^{\mathfrak{s}}) \xrightarrow{\overline{F}_0^{\mathfrak{s}}} \widehat{\mathrm{CFK}}(K,\mathfrak{s}) \longrightarrow 0. \tag{10}$$

Proof. (of Theorem 1.6) Let $C_\bullet(K) = \bigoplus_{\mathfrak{s}\in\mathbb{Z}} C_\bullet(K,\mathfrak{s})$, where $C_\bullet(K,\mathfrak{s}) = M(i_\bullet^{\mathfrak{s}})$ for $\bullet = 0,1$ and $C_\infty(K,\mathfrak{s}) = C\{i=\mathfrak{s}, j=0\}$. The maps $F_\infty, \overline{F}_\infty\colon C_0(K) \to C_1(K)$ and $F_0, \overline{F}_0\colon C_1(K) \to C_\infty(K)$ sit in the short exact sequences

$$0 \longrightarrow C_0(K) \xrightarrow{F_\infty} C_1(K) \xrightarrow{F_0} C_\infty(K) \longrightarrow 0 \qquad \text{and}$$

$$0 \longrightarrow C_0(K) \xrightarrow{\overline{F}_\infty} C_1(K) \xrightarrow{\overline{F}_0} C_\infty(K) \longrightarrow 0$$

The maps induced by $F_\bullet$ and $\overline{F}_\bullet$ are $\mathfrak{f}_\bullet$ and $\bar{\mathfrak{f}}_\bullet$, respectively. Thus, Proposition 7.2 from [Ef4] may be applied here to complete the proof of Theorem 1.6. □

5. The linear algebra of bypass homomorphisms

5.1. **Nilpotent compositions.** Let K be a knot inside the homology sphere Y.

Lemma 5.1. *Let $\mathbf{x} \in \widehat{\mathrm{CFK}}(K,\mathfrak{s})$ be a closed element and $[\mathbf{x}]$ denote the class represented by $\mathbf{x}$ in $\widehat{\mathrm{HFK}}(K,\mathfrak{s})$. Then*

$$(\overline{F}_0^{\mathfrak{s}-1} \circ F_\infty^{\mathfrak{s}-1} \circ \overline{F}_1^{\mathfrak{s}})[\mathbf{x}] = [d^{1,0}(\mathbf{x})] \quad \textit{and} \quad (F_0^{\mathfrak{s}+1} \circ \overline{F}_\infty^{\mathfrak{s}+1} \circ F_1^{\mathfrak{s}})[\mathbf{x}] = [d^{0,1}(\mathbf{x})].$$

Proof. Since $\overline{F}_1^{\mathfrak{s}}$ is the connecting homomorphism associated with the short exact sequence (10), to compute $\overline{F}_1^{\mathfrak{s}}[\mathbf{x}]$ note that $\mathbf{x}$ is the image of $([\mathbf{x},\mathfrak{s},0],0,0) \in M(i_1^{\mathfrak{s}})$ under the quotient map. The differential of $M(i_1^{\mathfrak{s}})$ takes this element to

$$\left(\sum_{i=0}^{\infty}[d^{i,0}(\mathbf{x}),\mathfrak{s}-i,0],0,[\mathbf{x},\mathfrak{s},0]\right) \in M(i_1^{\mathfrak{s}}).$$

Since $d^{0,0}(\mathbf{x}) = 0$ this latter element is in $M(i_0^{\mathfrak{s}-1})$. $F_\infty^{\mathfrak{s}-1}$ is the inclusion, thus

$$\left(F_\infty^{\mathfrak{s}-1} \circ \overline{F}_1^{\mathfrak{s}}\right)[\mathbf{x}] = \left(\sum_{i=1}^{\infty}[d^{i,0}(\mathbf{x}),\mathfrak{s}-i,0],0,[\mathbf{x},\mathfrak{s},0]\right) \in M(i_1^{\mathfrak{s}-1})$$

The projection map $\overline{F}_0^{\mathfrak{s}-1}$ takes this latter element to the closed element $d^{1,0}(\mathbf{x})$ in $\widehat{\mathrm{CFK}}(K,\mathfrak{s}-1)$. The second claim is proved similarly. □

Corollary 5.2. *For every relative* Spin^c *class $\mathfrak{s}$ the map*

$$\bar{\mathfrak{f}}_0 \circ \mathfrak{f}_\infty \circ \bar{\mathfrak{f}}_1 \circ \mathfrak{f}_0 \circ \bar{\mathfrak{f}}_\infty \circ \mathfrak{f}_1\big|_{\widehat{\mathrm{HFK}}(K,\mathfrak{s})} : \widehat{\mathrm{HFK}}(K,\mathfrak{s}) \longrightarrow \widehat{\mathrm{HFK}}(K,\mathfrak{s})$$

is nilpotent.

Proof. It suffices to show that $F=\overline{F}_0\circ F_\infty\circ\overline{F}_1\circ F_0\circ\overline{F}_\infty\circ F_1$ is nilpotent. However, by Lemma 5.1, for $\mathbf{x}\in\widehat{\mathrm{HFK}}(K,\mathfrak{s})$ we have

$$\begin{aligned}&F[\mathbf{x}]=[d^{1,0}(d^{0,1}(\mathbf{x}))]\\ \Rightarrow\quad &F^n[\mathbf{x}]=\left[\left(d^{1,0}\circ d^{0,1}\right)^n(\mathbf{x})\right]=\left[\left(\left(d^{1,0}\right)^n\circ\left(d^{0,1}\right)^n\right)(\mathbf{x})\right],\end{aligned}$$

where the last equality follows by an inductive use of (8). Since $[(d^{0,1})^n(\mathbf{x})]$ is in $\widehat{\mathrm{HFK}}(K,\mathfrak{s}+n)$, which is trivial for large values of n, it follows that F^n is trivial if n is sufficiently large (e.g. if $n>2g$ where g is the genus of K). □

5.2. **Block decomposition for bypass homomorphisms.** Let us assume that the chain complex C is defined from the Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta};u,v)$. Changing the role of punctures gives the duality maps $\tau_\bullet=\tau_\bullet(K):\mathbb{H}_\bullet(K)\to\mathbb{H}_\bullet(K)$ for $\bullet\in\{0,1,\infty\}$, where $\tau_\bullet$ takes $\mathbb{H}_\bullet(K,\mathfrak{s})$ to $\mathbb{H}_\bullet(K,-\mathfrak{s})$ if $\bullet=1,\infty$, and to $\mathbb{H}_0(K,1-\mathfrak{s})$ when $\bullet=0$. Furthermore, we have $\tau_\bullet\circ\tau_\bullet=Id$. Following the notation of [Ef4], in a basis for $\mathbb{H}_\bullet(K)$ where $\mathfrak{f}_\bullet$ takes the block form $\begin{pmatrix}0&0\\ I&0\end{pmatrix}$, we assume that

$$\tau_\bullet=\begin{pmatrix}A_\bullet&B_\bullet\\ C_\bullet&D_\bullet\end{pmatrix}\quad \bullet\in\{0,1,\infty\}. \tag{11}$$

As observed in [Ef4], one can then compute $\bar{\mathfrak{f}}_0=\tau_\infty\circ\mathfrak{f}_0\circ\tau_1$, $\bar{\mathfrak{f}}_1=\tau_0\circ\mathfrak{f}_1\circ\tau_\infty$ and $\bar{\mathfrak{f}}_\infty=\tau_1\circ\mathfrak{f}_\infty\circ\tau_0$. Define $X_\bullet=X_\bullet(K)$ by $X_0=B_1B_0B_\infty, X_1=B_\infty B_1B_0$ and $X_\infty=B_0B_\infty B_1$. Denote the rank of $F_\bullet$ by $a_\bullet=a_\bullet(K)$. Thus a_1,a_∞ and a_0+1 have the same parity. Note that B_0,B_1 and B_∞ are matrices of sizes $a_\infty\times a_1$, $a_0\times a_\infty$ and $a_1\times a_0$, respectively.

Lemma 5.3. *If K is a knot of genus $g>0$ then $B_\bullet\neq 0$ for $\bullet\in\{0,1,\infty\}$. In particular, $a_\bullet>0$.*

Proof. Since $H_*(M(i_0^g))=0$ by Theorem 4.1, the map $F_0^g:\mathbb{H}_1(K,g)\to\mathbb{H}_\infty(K,g)$ is an isomorphism. From here and by duality $\overline{F}_0^{-g}$ is also an isomorphism. Similarly,

$$H_*\left(M\left(i_1^{-g}\right)\right)\simeq\widehat{\mathrm{HFK}}(K,-g)\quad\text{and}\quad H_*\left(M\left(i_0^{-g}\right)\right)\simeq\widehat{\mathrm{HFK}}(K,-g)\oplus\widehat{\mathrm{HFK}}(K,-g).$$

Thus F_∞^{-g} is surjective, i.e. F_0^{-g} is trivial, implying that $\mathrm{Ker}(F_0)\setminus\mathrm{Ker}(\overline{F}_0)$ and $\mathrm{Im}(\overline{F}_0)\setminus\mathrm{Im}(F_0)$ are both non-empty. The first claim implies that

$$\exists\ \begin{pmatrix}a\\ b\end{pmatrix}\in\mathbb{H}_1(K)\ \text{ s.t. }\qquad \begin{pmatrix}0&0\\ I&0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=0\ \text{ and }\ \begin{pmatrix}A_\infty&B_\infty\\ C_\infty&D_\infty\end{pmatrix}\begin{pmatrix}0&0\\ I&0\end{pmatrix}\begin{pmatrix}A_1&B_1\\ C_1&D_1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\neq 0.$$

Thus $a=0$ and $\begin{pmatrix}B_\infty B_1b\\ D_\infty B_1b\end{pmatrix}\neq 0$. In particular, $B_1\neq 0$. Similarly, the second claim above implies that $\mathrm{Ker}(\overline{F}_1)\setminus\mathrm{Ker}(F_1)$ is non-empty and thus $B_\infty\neq 0$. For non-triviality of B_0, choose $x\in\mathbb{H}_\infty(K,g)$, which may be represented by $y=[x,g,0]$ in $C\{j=0\}$. Thus, $d^{*,0}y\in C\{i<g,j=0\}$ and $\overline{F}_1^g(x)=(d^{*,0}y,y,0)\in M(i_0^{g-1})$ is in the kernel of $\overline{F}_\infty$. If $F_\infty^{g-1}(\overline{F}_1^g(x))=0$ then $\overline{F}_1^g(x)=F_1^{g-1}(x')$ for some $x'\in\mathbb{H}_\infty(K,g-1)$. In other words, if we denote the dual of $[x',0,1-g]$ by $z\in C\{i\le 1-g,j=0\}$, the above equality implies

$$\exists\ \begin{cases}\overline{y}\in C\{i<g,j=0\},\\ \overline{z}\in C\{i<1-g,j=0\}\end{cases}\qquad\text{s.t}\ \begin{cases}d^{*,0}(y-\overline{y})=0\\ d^{*,0}(z-\overline{z})=0\\ (y-\overline{y})+(z-\overline{z})\text{ is exact.}\end{cases}$$

Note that $-\overline{y}+z-\overline{z}\in C\{i<g,j=0\}$ while y represents a non-trivial element in the homology of the quotient $C\{i=g,j=0\}=C\{i\le g,j=0\}/C\{i<g,j=0\}$. Thus $(y-\overline{y})+(z-\overline{z})$ can not be exact, and $\mathrm{Ker}(F_\infty)\setminus\mathrm{Ker}(\overline{F}_\infty)$ can not be trivial. From here, an argument similar to the preceding two cases implies $B_0\neq 0$. □

Lemma 5.4. *For every knot K, $X_\bullet=X_\bullet(K)$ is nilpotent for $\bullet\in\{0,1,\infty\}$. In particular, if K is non-trivial the kernel and the cokernel of $X_\bullet$ are non-trivial.*

Proof. The first claim is a direct consequence of Corollary 5.2 once we represent $F_\bullet=F_\bullet(K)$ as $\begin{pmatrix}0&0\\I&0\end{pmatrix}$ and note that $\overline{F}_\bullet=\overline{F}_\bullet(K)$ are given by $\overline{F}_0=\tau_\infty F_0\tau_1$, $\overline{F}_1=\tau_0F_1\tau_\infty$ and $\overline{F}_\infty=\tau_1F_\infty\tau_0$, respectively. This implies

$$\overline{F}_0F_\infty\overline{F}_1F_0\overline{F}_\infty F_1=\begin{pmatrix}(X_1)^2&0\\ \star&0\end{pmatrix}.$$

Thus $X_1^N=0$ for N sufficiently large. As a consequence $X_\bullet^{N+1}=0$ for $\bullet=0,1,\infty$. The second claim is a consequence of the first, since $a_\bullet>0$ by Lemma 5.3. □

Definition 5.5. *The knot K inside the homology sphere Y is called* full-rank *if all three matrices $B_0(K),B_1(K)$ and $B_\infty(K)$ are full rank.*

If $P_\bullet$ is an invertible $a_\bullet\times a_\bullet$ matrix and the matrices $Y_\bullet$ are arbitrary matrices of correct size, we may choose a change of basis for either of $\mathbb{H}_0(K),\mathbb{H}_1(K)$ and $\mathbb{H}_\infty(K)$ which is given by the invertible matrices

$$\mathbb{P}_0=\begin{pmatrix}P_\infty&0\\Y_0&P_1\end{pmatrix},\quad \mathbb{P}_1=\begin{pmatrix}P_0&0\\Y_1&P_\infty\end{pmatrix}\ \text{and}\ \mathbb{P}_\infty=\begin{pmatrix}P_1&0\\Y_\infty&P_0\end{pmatrix},\tag{12}$$

respectively. The block forms $F_\bullet=\begin{pmatrix}0&0\\I&0\end{pmatrix}$ remain unchanged under such a change of basis. A simultaneous change of basis of the form illustrated in (12) is called an *admissible* change of basis. The following lemma will be useful through our forthcoming discussions.

Lemma 5.6. *Suppose that K is a knot in a homology sphere and for $\bullet\in\{0,1,\infty\}$ let $\tau_\bullet$ denote $\tau_\bullet(K)$ and $X_\bullet$ denote the matrix $X_\bullet(K)$. Choose*

$$(\circ,\bullet,*)\in\{(0,1,\infty),(1,\infty,0),(\infty,0,1)\}.$$

1. *If $B_\circ(K),B_\bullet(K)$ are injective and $B_*(K)$ is surjective, after an admissible change of basis we may assume that*

$$\tau_\circ=\left(\begin{array}{cc|c}0&0&I\\0&\star&0\\\hline I&0&0\end{array}\right),\quad \tau_\bullet=\left(\begin{array}{ccc|cc}0&0&0&I&0\\0&0&0&0&I\\0&0&\star&0&0\\\hline I&0&0&0&0\\0&I&0&0&0\end{array}\right)\ \text{and}\ \tau_*=\left(\begin{array}{c|ccc}0&X_\bullet&\star&\star\\\hline\star&\star&\star&\star\\\star&\star&\star&\star\\\star&\star&\star&\star\end{array}\right)\tag{13}$$

2. *If $B_\circ(K),B_\bullet(K)$ are surjective and $B_*(K)$ is injective, after an admissible change of basis we may assume that*

$$\tau_\bullet=\left(\begin{array}{c|cc}0&0&I\\\hline 0&\star&0\\I&0&0\end{array}\right),\quad \tau_\circ=\left(\begin{array}{cc|ccc}0&0&0&I&0\\0&0&0&0&I\\\hline 0&0&\star&0&0\\I&0&0&0&0\\0&I&0&0&0\end{array}\right)\ \text{and}\ \tau_*=\left(\begin{array}{ccc|c}\star&\star&\star&\star\\\star&\star&\star&\star\\\star&\star&\star&X_\circ\\\hline\star&\star&\star&0\end{array}\right)\tag{14}$$

Proof. The proof consists of straight-forward linear algebra. □

6. Splicing and homology sphere L-spaces

6.1. **Special pairs.** Given an arbitrary matrix M denote the rank of $\mathrm{Ker}(M)$ by $k(M)$, denote the rank of $\mathrm{Coker}(M)$ by $c(M)$ and set $i(M) = k(M) + c(M)$. The matrices M_1 and M_2 are called *equivalent* if $k(M_1) = k(M_2)$ and $c(M_1) = c(M_2)$. If $M^\star \in M_{n_\star \times m_\star}(\mathbb{F})$ for $\star = 1, 2$ are a pair of matrices, $M^1 \otimes M^2 \in M_{n_1n_2 \times m_1m_2}(\mathbb{F})$ is the associated map from $\mathbb{F}^{m_1m_2} = \mathbb{F}^{m_1} \otimes \mathbb{F}^{m_2}$ to $\mathbb{F}^{n_1n_2} = \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2}$.

Let $Y = Y(K_1, K_2)$ denote the three-manifold obtained by splicing the complements of $K_1 \subset Y_1$ and $K_2 \subset Y_2$, where Y_1 and Y_2 are homology spheres. For $\square \in \{A, B, C, D, X, \tau\}$, $\bullet \in \{0, 1, \infty\}$ and $\star \in \{1, 2\}$ let $\square^\star_\bullet = \square_\bullet(K_\star)$. Proposition 5.4 from [Ef4] and the discussion following it give the following.

Proposition 6.1. *If K_i is a knot inside the homology sphere Y_i for $i = 1, 2$,*

$$\mathrm{rnk}\ \widehat{\mathrm{HF}}(Y(K_1, K_2); \mathbb{F}) = i(\mathfrak{D}(K_1, K_2)),$$

where the matrix $\mathfrak{D}(K_1, K_2)$ is given by

$$\begin{pmatrix}
D^1_\infty B^1_1 \otimes B^2_1 A^2_0 & B^1_1 A^1_0 \otimes I & B^1_1 B^1_0 \otimes I & D^1_\infty A^1_1 \otimes B^2_1 A^2_0 & I \otimes B^2_1 B^2_0 & 0 \\
I \otimes B^2_\infty B^2_1 & D^1_1 A^1_0 \otimes B^2_\infty A^2_1 & D^1_1 B^1_0 \otimes B^2_\infty A^2_1 & 0 & B^1_0 B^1_\infty \otimes I & B^1_0 A^1_\infty \otimes I \\
I \otimes D^2_\infty B^2_1 & \begin{matrix} I \otimes I + \\ D^1_1 A^1_0 \otimes D^2_\infty A^2_1 \end{matrix} & D^1_1 B^1_0 \otimes D^2_\infty A^2_1 & 0 & 0 & 0 \\
B^1_\infty B^1_1 \otimes I & 0 & I \otimes B^2_0 B^2_\infty & B^1_\infty A^1_1 \otimes I & \begin{matrix} D^1_0 B^1_\infty \otimes B^2_0 A^2_\infty \\ +X^1_1 B^1_\infty \otimes B^2_0 X^2_1 \end{matrix} & \begin{matrix} D^1_0 A^1_\infty \otimes B^2_0 A^2_\infty \\ +X^1_1 A^1_\infty \otimes B^2_0 X^2_1 \end{matrix} \\
D^1_\infty B^1_1 \otimes D^2_1 A^2_0 & 0 & 0 & \begin{matrix} I \otimes I + \\ D^1_\infty A^1_1 \otimes D^2_1 A^2_0 \end{matrix} & I \otimes D^2_1 B^2_0 & 0 \\
0 & 0 & I \otimes D^2_0 B^2_\infty & 0 & \begin{matrix} D^1_0 B^1_\infty \otimes D^2_0 A^2_\infty \\ +X^1_1 B^1_\infty \otimes D^2_0 X^2_1 \end{matrix} & \begin{matrix} I \otimes I + \\ D^1_0 A^1_\infty \otimes D^2_0 A^2_\infty \\ +X^1_1 A^1_\infty \otimes D^2_0 X^2_1 \end{matrix}
\end{pmatrix},$$

Definition 6.2. *The pair (K_1, K_2) is called a* special pair *if* $\widehat{\mathrm{HF}}(Y(K_1, K_2); \mathbb{F}) = \mathbb{F}$.

Suppose, throughout this section, that (K_1, K_2) is a special pair. Let $k^\bullet_\star = k(B^\bullet_\star)$ and $c^\bullet_\star = c(B^\bullet_\star)$, for $\star \in \{0, 1, \infty\}$ and $\bullet = 1, 2$. Define $\imath : \{0, 1, \infty\} \to \{0, 1, \infty\}$ by $\imath(0) = \infty, \imath(1) = 1$ and $\imath(\infty) = 0$. For $\mathfrak{D} = \mathfrak{D}(K_1, K_2)$ the cokernel and kernel of $\mathfrak{D}$ include subspaces $\mathrm{C}(\mathfrak{D})$ and $\mathrm{K}(\mathfrak{D})$ (respectively) which are isomorphic to

$$\bigoplus_{\bullet \in \{0,1,\infty\}} \mathrm{Coker}(B^1_\bullet) \otimes \mathrm{Coker}(B^2_{\imath(\bullet)}) \quad \text{and} \quad \bigoplus_{\bullet \in \{0,1,\infty\}} \mathrm{Ker}(B^1_\bullet) \otimes \mathrm{Ker}(B^2_{\imath(\bullet)})$$

respectively, and correspond to the first, second and fourth rows, and to the first, third and fifth columns, respectively. Moreover, if $A^1_\infty \otimes D^2_0 + D^1_0 \otimes A^2_\infty = 0$ (which may be assumed after an admissible change of basis if $c^1_\infty k^2_0 = k^1_0 c^2_\infty = 0$) the cokernel also includes a subspace isomorphic to $\mathrm{Coker}(B^1_\infty) \otimes \mathrm{Coker}(B^2_\infty)$ and the kernel includes a subspace isomorphic to $\mathrm{Ker}(B^1_0) \otimes \mathrm{Ker}(B^2_0)$. Denote the ranks of $\mathrm{K}(\mathfrak{D})$ and $\mathrm{C}(\mathfrak{D})$ by $\widehat{k}(\mathfrak{D})$ and $\widehat{c}(\mathfrak{D})$, respectively. Thus $k(\mathfrak{D}) + c(\mathfrak{D}) \leq 1$ and

$$\widehat{k}(\mathfrak{D}) = \sum_{\bullet \in \{0,1,\infty\}} k^1_\bullet k^2_{\imath(\bullet)} \leq k(\mathfrak{D}) \quad \text{and} \quad \widehat{c}(\mathfrak{D}) = \sum_{\bullet \in \{0,1,\infty\}} c^1_\bullet c^2_{\imath(\bullet)} \leq c(\mathfrak{D}).$$

Proposition 6.3. *If (K_1, K_2) is a special pair, then possibly after interchanging K_1 and K_2, one of the following is the case:*

- **(G)** *K_1 is full-rank.*
- **(S-1)** *The matrix B^2_0 is invertible, B^1_0 is surjective and B^1_1 and B^2_∞ are injective.*
- **(S-2)** *The matrix B^2_0 is invertible, B^1_0 is injective and B^1_1 and B^2_∞ are surjective.*

Proof. We assume that (K_1, K_2) is a special pair, while none of K_1 and K_2 is full-rank. Let us first assume that both $\widehat{k}(\mathfrak{D})$ and $\widehat{c}(\mathfrak{D})$ are zero. From the above assumption we find $k^1_\bullet k^2_{\imath(\bullet)} = c^1_\bullet c^2_{\imath(\bullet)} = 0$ for $\bullet = 0, 1, \infty$. If $B^1_\bullet$ is not a full rank matrix then both $c^1_\bullet$ and $k^1_\bullet$ are non-zero. From here $k^2_{\imath(\bullet)} = c^2_{\imath(\bullet)} = 0$, i.e. $B^2_{\imath(\bullet)}$ is invertible. Since the parity of a^2_0 is different from the parity of a^2_1 and a^2_∞, the matrices B^2_1 and B^2_∞ can not be square matrices. Thus $\imath(\bullet) = 0$ and $\bullet = \infty$. In other words, we conclude that B^1_0 and B^1_1 are full-rank and B^2_0 is invertible, while B^2_∞ is not full-rank. Similarly, we may conclude that B^2_1 is full-rank and B^1_0 is invertible, while B^1_∞ is not full-rank. Moreover, since $c^1_1 c^2_1 = k^1_1 k^2_1 = 0$, precisely one of B^1_1 and B^2_1 is injective, and the other one is surjective. Without loosing on generality we may thus assume that:

- B^1_0 and B^2_0 are invertible, B^1_1 is injective and B^2_1 is surjective.
- None of B^1_∞ and B^2_∞ is full-rank.

In particular, $k^1_\infty > c^1_\infty > 0$ and $c^2_\infty > k^2_\infty > 0$. Since B^1_0 and B^2_0 are both invertible we may assume that $D^1_0 = 0$ and $D^2_0 = 0$. From here the cokernel of $\mathfrak{D}$ includes a subspace isomorphic to $\mathrm{Coker}(B^1_\infty) \otimes \mathrm{Coker}(B^2_\infty)$, which is of size $c^1_\infty c^2_\infty \geq 2$. This implies that (K_1, K_2) is not special.

From this contradiction, we conclude that one of $\widehat{k}(\mathfrak{D})$ and $\widehat{c}(\mathfrak{D})$ is non-zero. Suppose that $\widehat{c}(\mathfrak{D}) = 1$ and $\widehat{k}(\mathfrak{D}) = 0$. For some $\bullet \in \{0, 1, \infty\}$ we thus have $c^1_\bullet = c^2_{\imath(\bullet)} = 1$ while $k^1_\bullet k^2_{\imath(\bullet)} = 0$ and for $\star \neq \bullet$ we have $c^1_\star c^2_{\imath(\star)} = k^1_\star k^2_{\imath(\star)} = 0$. Without loosing on generality we may assume that $k^1_\bullet = 0$. Thus $B^1_\bullet$ is injective with a 1-dimensional cokernel. In particular, the parity of the number of rows and the number of columns for $B^1_\bullet$ are different, i.e. $\bullet \neq 0$. Thus $c^1_0 c^2_\infty = k^1_0 k^2_\infty = 0$. Since B^2_∞ is not a square matrix, at least one of c^2_∞ and k^2_∞ is non-zero, implying that at least one of c^1_0 and k^1_0 is zero, i.e. B^1_0 is full-rank. The assumption that K_1 is not full-rank implies that $B^1_\star$ is not full-rank, where $\{\star\} = \{1, \infty\} \setminus \{\bullet\}$. From here $c^1_\star, k^1_\star > 0$. Together with $c^1_\star c^2_{\imath(\star)} = k^1_\star k^2_{\imath(\star)} = 0$ this implies that $c^2_{\imath(\star)} = k^2_{\imath(\star)} = 0$, i.e. $B^2_{\imath(\star)}$ is invertible. Thus, $\imath(\star) = 0, \star = \infty$ and $\bullet = 1$. We thus conclude

- B^2_0 is invertible, B^1_0 is full-rank, B^1_1 is injective and B^1_∞ is not full-rank.
- $c^1_1 = c^2_1 = 1$.

Since B^2_0 is invertible, we may assume that $A^2_0 = D^2_0 = 0$. If B^1_0 is injective, we may also assume that $D^1_0 = 0$ and that $\mathrm{Coker}(\mathfrak{D})$ includes a subspace isomorphic to $\mathrm{Coker}(B^1_\infty) \otimes \mathrm{Coker}(B^2_\infty)$ and of size $c^1_\infty c^2_\infty$. Since $c^1_\infty \neq 0$ we conclude that B^2_∞ is surjective. From here $a^2_\infty = a^2_1 \leq a^2_0 - 1$ and $1 - k^2_1 = c^2_1 - k^2_1 = a^2_0 - a^2_\infty \geq 1$. We thus find $k^2_1 = 0$ and K_2 is full-rank, a contradiction. Thus $k^1_0 > 0$ and $c^1_0 = 0$. From $k^1_0 k^2_\infty = 0$ we find $k^2_\infty = 0$, i.e. B^2_∞ is injective and the conditions of (S-1) are satisfied. A similar argument reduces the case $\widehat{k}(\mathfrak{D}) = 1$ and $\widehat{c}(\mathfrak{D}) = 0$ to (S-2). □

Proposition 6.4. *Given the pair of knots (K_1, K_2) where K_1 is full-rank and $(\circ, \bullet, *) \in \{(0, 1, \infty), (1, \infty, 0), (\infty, 0, 1)\}$,*

(K) *If $B^1_\circ, B^1_\bullet$ are injective and B^1_* is surjective then*

$$c(\mathfrak{D}) \geq c^1_\bullet c^2_{\imath(\bullet)} + c^1_\circ c^2_{\imath(\circ)} \quad \text{and} \quad k(\mathfrak{D}) \geq k(X^1_\bullet) k(B^2_{\imath(*)} X^2_{\imath(\bullet)}).$$

(C) *If $B^1_\circ, B^1_\bullet$ are surjective and B^1_* is injective then*

$$k(\mathfrak{D}) \geq k^1_\bullet k^2_{\imath(\bullet)} + k^1_\circ k^2_{\imath(\circ)} \text{ and } c(\mathfrak{D}) \geq c(X^1_\bullet) c(X^2_{\imath(\bullet)} B^2_{\imath(*)}).$$

Proof. The first claim in either of cases (K) and (C) is already observed in our earlier discussions. We thus need to prove the second claim in each case. The proofs are very similar. In fact, the proof of claim (C) for $(\circ,\bullet,*)$ is almost identical to the proof of claim (K) for $(\imath(\bullet),\imath(\circ),\imath(*))$ because of the symmetry in the block presentation of $\mathfrak{D}$. We will only go through the proof for $(\circ,\bullet,*)=(0,1,\infty)$.

In case (K), after an admissible change of basis, we may assume that $\tau_0(K_1),\tau_1(K_1)$ and $\tau_\infty(K_1)$ take the standard form of (13). Since $D_0^1=D_1^1=A_\infty^1=0$, the $(3,2)$ entry and the $(6,6)$ entry of the matrix $\mathfrak{D}$ are both the identity matrix. The matrix $\mathfrak{D}$ is thus equivalent to the matrix

$$\begin{pmatrix} \begin{matrix} D_\infty^1B_1^1\otimes B_1^2A_0^2+\\ B_1^1A_0^1\otimes D_\infty^2B_1^2\end{matrix} & B_1^1B_0^1\otimes I & D_\infty^1A_1^1\otimes B_1^2A_0^2 & I\otimes B_1^2B_0^2\\ I\otimes B_\infty^2B_1^2 & 0 & 0 & B_0^1B_\infty^1\otimes I\\ B_\infty^1B_1^1\otimes I & I\otimes B_0^2B_\infty^2 & B_\infty^1A_1^1\otimes I & X_1^1B_\infty^1\otimes B_0^2X_1^2\\ D_\infty^1B_1^1\otimes D_1^2A_0^2 & 0 & \begin{matrix} I\otimes I+\\ D_\infty^1A_1^1\otimes D_1^2A_0^2\end{matrix} & I\otimes D_1^2B_0^2 \end{pmatrix}.$$

Replacing the block forms for $\tau_\star(K_1)$ gives the following presentation of the above matrix

$$\begin{pmatrix} \star & \star & I\otimes I & 0 & 0 & \star & I\otimes B_1^2B_0^2 & \star & \star\\ \star & \star & 0 & 0 & 0 & \star & 0 & \star & \star\\ \star & \star & 0 & 0 & 0 & \star & 0 & \star & \star\\ \star & \star & 0 & 0 & 0 & \star & X_1^1\otimes I & \star & \star\\ \star & \star & 0 & 0 & 0 & \star & 0 & \star & \star\\ \star & \star & I\otimes B_0^2B_\infty^2 & 0 & 0 & \star & X_1^1X_1^1\otimes B_0^2X_1^2 & \star & \star\\ \star & \star & 0 & I\otimes I & 0 & \star & \star & \star & \star\\ \star & \star & 0 & 0 & I\otimes I & \star & 0 & \star & \star\\ \star & \star & 0 & 0 & 0 & \star & 0 & \star & \star \end{pmatrix}.$$

After subtracting $I\otimes B_0^2B_\infty^2$ times the first row from the sixth row, the identity matrices which appear in the entries $(1,3),(7,4)$ and $(8,5)$ of the above matrix become the only non-zero entries of their respective columns. They may thus be used for the cancellation of the third, the fourth and the fifth columns against the first, the seventh and the eighth rows. We thus arrive at a 6×6 matrix equivalent to $\mathfrak{D}$, which is of the form

$$\begin{pmatrix} \star & \star & \star & 0 & \star & \star\\ \star & \star & \star & 0 & \star & \star\\ \star & \star & \star & X_1^1\otimes I & \star & \star\\ \star & \star & \star & 0 & \star & \star\\ \star & \star & \star & (I+X_1^1X_1^1)\otimes B_0^2X_1^2 & \star & \star\\ \star & \star & \star & 0 & \star & \star \end{pmatrix}.$$

Since the kernel of $\mathfrak{D}$ includes a subspace which is isomorphic to the kernel corresponding to the fourth column we find $k(\mathfrak{D})\geq k(X_1^1)k(B_0^2X_1^2)$.

For case (C), using Lemma 5.6 choose the standard block form of (14) for K_1. In particular, A_0^1, A_1^1 and D_∞^1 are all zero. The entries $(3,2)$ and $(5,4)$ of $\mathfrak{D}$ are thus identity matrices which may be used for cancellation. Add $B_\infty^1B_1^1\otimes B_0^2X_1^2$ times the second row of the resulting matrix to its third row, add $B_\infty^1B_1^1\otimes D_0^2X_1^2$ times the second row to the last row, and note that $B_1^1D_1^1=0$ to arrive at the following matrix, which is equivalent to $\mathfrak{D}$:

$$\begin{pmatrix} 0 & B_1^1B_0^1\otimes I & I\otimes B_1^2B_0^2 & 0\\ I\otimes B_\infty^2B_1^2 & D_1^1B_0^1\otimes B_\infty^2A_1^2 & B_0^1B_\infty^1\otimes I & B_0^1A_\infty^1\otimes I\\ B_\infty^1B_1^1\otimes(I+X_0^2X_0^2) & I\otimes B_0^2B_\infty^2 & D_0^1B_\infty^1\otimes B_0^2A_\infty^2 & D_0^1A_\infty^1\otimes B_0^2A_\infty^2\\ B_\infty^1B_1^1\otimes D_0^2X_1^2B_\infty^2B_1^2 & I\otimes D_0^2B_\infty^2 & D_0^1B_\infty^1\otimes D_0^2A_\infty^2 & \begin{matrix} I\otimes I+\\ D_0^1A_\infty^1\otimes D_0^2A_\infty^2\end{matrix} \end{pmatrix}.$$

Replacing the block forms of (14) for $\tau_0(K_1),\tau_1(K_1)$ and $\tau_\infty(K_1)$ we arrive at a matrix of the form

$$\begin{pmatrix}
0 & 0 & 0 & 0 & I\otimes I & I\otimes B_1^2B_0^2 & 0 & 0 & 0\\
\star & \star & \star & \star & 0 & \star & \star & \star & \star\\
\star & \star & \star & \star & 0 & \star & \star & \star & \star\\
\star & \star & \star & \star & 0 & \star & \star & \star & \star\\
\star & \star & \star & \star & 0 & \star & \star & \star & \star\\
0 & X_0^1\otimes(I+X_0^2X_0^2) & 0 & 0 & I\otimes B_0^2B_\infty^2 & 0 & 0 & 0 & 0\\
\star & \star & \star & \star & 0 & \star & \star & \star & \star\\
\star & \star & \star & \star & 0 & \star & \star & \star & \star\\
\star & \star & \star & \star & 0 & \star & \star & \star & \star
\end{pmatrix},$$

which is in turn equivalent to a matrix of the form

$$\begin{pmatrix}
\star & \star & \star & \star & \star & \star & \star & \star\\
\star & \star & \star & \star & \star & \star & \star & \star\\
\star & \star & \star & \star & \star & \star & \star & \star\\
\star & \star & \star & \star & \star & \star & \star & \star\\
0 & X_0^1\otimes(I+X_0^2X_0^2) & 0 & 0 & I\otimes X_\infty^2B_0^2 & 0 & 0 & 0\\
\star & \star & \star & \star & \star & \star & \star & \star\\
\star & \star & \star & \star & \star & \star & \star & \star\\
\star & \star & \star & \star & \star & \star & \star & \star
\end{pmatrix},$$

In particular, we conclude $c(\mathfrak{D})\geq c(X_0^1)c(X_\infty^2B_0^2)$. This completes the proof of case (C) when $(\circ,\bullet,*)=(0,1,\infty)$. $\square$

6.2. **The special cases (S-1) and (S-2).**

Lemma 6.5. *If (K_1,K_2) is a special pair of type (S-1) or (S-2) then one of the knots K_1 or K_2 is trivial.*

Proof. Suppose otherwise that (K_1,K_2) is a special pair of type (S-1) and that both K_1 and K_2 are non-trivial. After an admissible change of basis, assume that

$$(15)\qquad \tau_0^2=\left(\begin{array}{cc|cc}0&0&I&0\\0&0&0&I\\\hline I&0&0&0\\0&I&0&0\end{array}\right),\quad \tau_\infty^2=\left(\begin{array}{cc|c}0&0&I\\0&\star&0\\\hline I&0&0\end{array}\right)\quad\text{and}\quad \tau_1^2=\left(\begin{array}{c|cc}\star&X_\infty^2&\star\\\hline\star&\star&\star\\\star&\star&\star\end{array}\right)$$

In particular, A_0^2,D_0^2 and D_∞^2 are zero. We may also assume that

$$(16)\qquad \tau_0^1=\left(\begin{array}{c|cc}0&I&0\\\hline I&0&0\\0&0&\star\end{array}\right),\quad \tau_1^1=\left(\begin{array}{cc|c}0&0&I\\0&\star&0\\\hline I&0&0\end{array}\right)\quad\text{and}\quad \tau_\infty^1=\left(\begin{array}{cc|cc}\star&\star&X_\infty^1&\star\\\star&\star&\star&\star\\\hline\star&\star&\star&\star\\\star&\star&\star&\star\end{array}\right)$$

In particular, A_0^1 and D_1^1 are zero. The identity matrices which appear as entries $(3,2)$, $(5,4)$ and $(6,6)$ in $\mathfrak{D}(K_1,K_2)$ may be used for cancellation to obtain the equivalent matrix

$$\begin{pmatrix}
0 & B_1^1B_0^1\otimes I & I\otimes B_1^2B_0^2\\
I\otimes B_\infty^2B_1^2 & 0 & B_0^1B_\infty^1\otimes I\\
B_\infty^1B_1^1\otimes I & I\otimes B_0^2B_\infty^2 & \begin{array}{c}D_0^1B_\infty^1\otimes B_0^2A_\infty^2\\+X_1^1B_\infty^1\otimes B_0^2X_1^2\\+B_\infty^1A_1^1\otimes D_1^2B_0^2\end{array}
\end{pmatrix}.$$

Subtracting $X_1^1B_\infty^1\otimes B_0^2B_\infty^2$ times the first row from the third row we arrive at the equivalent matrix

$$\begin{pmatrix}
0 & B_1^1B_0^1\otimes I & I\otimes B_1^2B_0^2\\
I\otimes B_\infty^2B_1^2 & 0 & B_0^1B_\infty^1\otimes I\\
B_\infty^1B_1^1\otimes I & (I+X_1^1X_1^1)\otimes B_0^2B_\infty^2 & \begin{array}{c}D_0^1B_\infty^1\otimes B_0^2A_\infty^2\\+B_\infty^1A_1^1\otimes D_1^2B_0^2\end{array}
\end{pmatrix}.$$

Replacing the block forms of (15) and (16), the above matrix takes the form

$$\begin{pmatrix}
0 & \star & I\otimes I & 0 & I\otimes X^2_\infty & \star & \star & \star\\
0 & \star & 0 & 0 & 0 & \star & \star & \star\\
I\otimes X^2_\infty & \star & 0 & 0 & X^1_\infty\otimes I & \star & \star & \star\\
0 & \star & 0 & 0 & 0 & \star & \star & \star\\
X^1_\infty\otimes I & \star & (I+X^1_\infty X^1_\infty)\otimes I & 0 & 0 & \star & \star & \star\\
0 & \star & 0 & 0 & 0 & \star & \star & \star\\
\star & \star & \star & I\otimes I & 0 & \star & \star & \star\\
0 & \star & 0 & 0 & 0 & \star & \star & \star
\end{pmatrix}.$$

Subtract $(I+X^1_\infty X^1_\infty)\otimes I$ times the first row from the fifth row and use the identity matrices which appear as $(1,3)$ and $(7,4)$ entries of the above matrix for cancellation to arrive at the following equivalent matrix

$$\begin{pmatrix}
0 & \star & 0 & \star & \star & \star\\
I\otimes X^2_\infty & \star & X^1_\infty\otimes I & \star & \star & \star\\
0 & \star & 0 & \star & \star & \star\\
X^1_\infty\otimes I & \star & (I+X^1_\infty X^1_\infty)\otimes X^2_\infty & \star & \star & \star\\
0 & \star & 0 & \star & \star & \star\\
0 & \star & 0 & \star & \star & \star
\end{pmatrix}.$$

From the above presentation we conclude

$$k(\mathfrak{D}) \geq 2k(X^1_\infty)k(X^2_\infty) \geq 2.$$

This contradiction rules out the case (S-1). Ruling out the case (S-2) is similar. □

7. Incompressible tori in homology spheres

7.1. **The main theorem.**

Theorem 7.1. *Suppose that K_i is a non-trivial knot in the homology sphere Y_i for $i=1,2$. Let $Y=Y(K_1,K_2)$ denote the three-manifold obtained by splicing the complements of K_1 and K_2. Then the rank of $\widehat{\mathrm{HF}}(Y)$ is bigger than one.*

Proof. Suppose otherwise that Y is a L-space. Thus (K_1,K_2) is a special pair. By Proposition 6.3 and Lemma 6.5 we may assume that K_1 is full-rank. In particular, one of the cases (K) or (C) from Proposition 6.4 will happen. Note that in case (K) the kernel of $\mathfrak{D}$ is necessarily non-trivial by Lemma 5.4, while in case (C) the cokernel of $\mathfrak{D}$ is non-trivial.
Let us assume that (K) is the case. Thus $c(\mathfrak{D})=0$ and $k(X^1_\bullet)=k(B^2_{\imath(*)}X^2_{\imath(\bullet)})=1$. Note that $\mathrm{Ker}(B^2_{\imath(*)})\subset \mathrm{Ker}(B^2_{\imath(*)}X^2_{\imath(\bullet)})$, which implies that either $B^2_{\imath(*)}$ is injective or $\mathrm{Ker}(B^2_{\imath(*)})=\mathrm{Ker}(B^2_{\imath(*)}X^2_{\imath(\bullet)})$. If the latter happens, we find

$$\mathrm{Ker}(B^2_{\imath(*)})=\mathrm{Ker}(B^2_{\imath(*)}X^2_{\imath(\bullet)})=\mathrm{Ker}(B^2_{\imath(*)}X^2_{\imath(\bullet)}X^2_{\imath(\bullet)})=\cdots=\mathrm{Ker}(0),$$

since $X^2_{\imath(\bullet)}$ is nilpotent by Lemma 5.4. Since $B^2_{\imath(*)}\neq 0$ this can not happen and we conclude that $B^2_{\imath(*)}$ is injective.
Let us first assume that B^1_0 is not invertible. Then $c^1_\bullet, c^1_\circ \neq 0$. Since $c^1_\bullet c^2_{\imath(\bullet)} = c^1_\circ c^2_{\imath(\circ)}=0$ we conclude that $B^2_{\imath(\circ)}$ and $B^2_{\imath(\bullet)}$ are both surjective. Thus K_2 is full-rank and by part (C) of Proposition 6.4 $c(\mathfrak{D})>0$. This contradiction implies that B^1_0 is invertible. Moreover, the argument implies that $0\in\{\circ,\bullet\}$ and at least one of $c^2_{\imath(\circ)}$ and $c^2_{\imath(\bullet)}$ is trivial. It is easy to conclude from here that we are then either in case (S-1) or case (S-2) of Proposition 6.3, which are both excluded by Lemma 6.5. The contradiction rules out case (K) of Proposition 6.4. Excluding the case (C) is completely similar. □

Corollary 7.2. *If the homology sphere Y contains an incompressible torus then* $\text{rnk}(\widehat{\text{HF}}(Y,\mathbb{F}) > 1$*.*

Proof. If Y contains an incompressible torus T, T will be separating and there will be a pair of curves λ and μ on T such that λ is homologically trivial on one side of T and μ is homologically trivial on the other side of T. Since Y is a homology sphere, the intersection number of μ and λ is one. Let U_1 and U_2 be the two components of $Y-T$ and let U_1 be the component containing a surface which bounds λ. Capping off $\mu \subset T = \partial U_1$ by a disk and then gluing a three-ball gives a three-manifold Y_1. The simple closed curve λ represents a knot $K_1 \subset Y_1$. Similarly capping off $\lambda \subset T = \partial U_2$ by a disk and then gluing a three-ball gives a three-manifold Y_2 and μ represents a knot $K_2 \subset Y_2$. Both Y_1 and Y_2 are homology spheres and Y is obtained by splicing K_1 and K_2. Since T is incompressible, both K_1 and K_2 are non-trivial and Theorem 7.1 completes the proof of this corollary. □

7.2. **Applications.** We may use the relation between Khovanov homology of a knot inside the standard sphere and the Heegaard Floer homology of its branched double-cover, discovered by Ozsváth and Szabó [OS5], to show the non-triviality of Khovanov homology for certain classes of knots. We emphasize again that the results presented here are all special cases of the the theorem of Kronheimer and Mrowka [KM] that Khovanov homology is an unknot detector.

Definition 7.3. *A prime knot $K \subset S^3$ is an n-*string composite *if there is an embedded 2-sphere intersecting the knot transversely which separates (S^3, K) into prime n-string tangles. A 2-string composite knot is called a* doubly composite knot*.*

We refer the reader to [Blei] for more on doubly composite and doubly prime knots, and only quote the following lemma from that paper:

Lemma 7.4. *A prime knot $K \subset S^3$ is a doubly composite knot if and only if the double cover $\Sigma(K)$ of S^3 branched over the knot K contains an incompressible torus T which is invariant under the non-trivial covering translation and meets the fixed point set of this map precisely in 4 points, and separates $\Sigma(K)$ into irreducible boundary irreducible pieces.*

Corollary 7.5. *If the prime knot $K \subset S^3$ is doubly composite, the rank of its reduced Khovanov homology group $\widetilde{\text{Kh}}(K)$ is bigger than 1.*

Proof. If K is doubly composite, by Lemma 7.4 there exists an incompressible torus T inside the three-manifold $\Sigma(K)$. Thus the rank of $\widehat{\text{HF}}(\Sigma(K),\mathbb{F})$ is bigger than 1. By the main theorem of [OS5] there is a spectral sequence whose E^2-term consists of Khovanov's reduced homology $\widetilde{\text{Kh}}(K)$ of the mirror of K with coefficients in $\mathbb{F}$ which converges to $\widehat{\text{HF}}(\Sigma(K),\mathbb{F})$, and is of rank greater than 1 by Theorem 7.1. Thus the rank of $\widetilde{\text{Kh}}(K)$ is bigger than 1 as well. □

Furthermore, if K is a prime satellite knot, we will have an incompressible torus in the complement of K. This torus gives an incompressible torus in the double cover $\Sigma(K)$ of S^3 branched over the knot K. Thus, Heegaard Floer homology of $\Sigma(K)$ will be non-trivial. We thus have the following corollary:

Corollary 7.6. *If $K \subset S^3$ is a prime satellite knot the rank of its reduced Khovanov homology group $\widetilde{\mathrm{Kh}}(K)$ is greater than* 1.

In fact, we may prove a slightly more general statement:

Proposition 7.7. *If the rank of the reduced Khovanov homology $\widetilde{\mathrm{Kh}}(K)$ of a non-trivial knot $K \subset S^3$ is one, the double cover $\Sigma(K)$ of S^3, branched over the knot K, is hyperbolic.*

Proof. Note that if a knot K is doubly composite Corollary 7.5 implied that the rank of $\widetilde{\mathrm{Kh}}(K)$ is bigger than 1. Thus, K has to be doubly prime. By Thurston's orbifold geometrization theorem (see [BP] and [CHK]) the branched double cover $\Sigma(K)$ is a geometric manifold and there are three possible cases.
1- $\Sigma(K)$ is a Lens space and thus admits a spherical structure. If $\widehat{\mathrm{HF}}(\Sigma(K))$ is one dimensional, $\Sigma(K)$ is forced to be the standard sphere and K is trivial. Thus in this case, the rank of $\widetilde{\mathrm{Kh}}(K)$ is bigger than 1 only if K is trivial.
2- $\Sigma(K)$ admits a Seifert fibration and K is a Montesinos knot with at most three rational tangles. If $\Sigma(K)$ is not a homology sphere, $\widetilde{\mathrm{Kh}}(K)$ is clearly different from $\mathbb{F}$, and if it is a homology sphere which admits a Seifert fibration and $\widehat{\mathrm{HF}}(\Sigma(K)) = \mathbb{F}$, we know (see [Rus] or [Ef5]) that $\Sigma(K)$ is either the standard sphere, or the Poincaré sphere. Moreover, for $\Sigma(K)$ to be the Poincaré sphere we should have $K = T(3,5)$, i.e. K is the $(3,5)$-torus knot, or equivalently $(-2,3,5)$-pretzel knot, which is 10_{124} in Rolfsen's table (see [HW] and [Rolf]). $\widetilde{\mathrm{Kh}}(T(3,5))$ has rank 7 by direct computation [Shu].
3- $\Sigma(K)$ admits a hyperbolic structure which is invariant under the deck transformation.
Having ruled out the first two possibilities, the proof is complete. □

The knots K with the property that $\Sigma(K)$ admits a hyperbolic structure which is invariant under the involution of $\Sigma(K)$ are called π-*hyperbolic*. The hyperbolic structure comes from a hyperbolic structure on $S^3 - K$ which becomes a singular folding with angle π around K. Thus in particular, π-hyperbolic knots are hyperbolic.

Suppose K is not the unknot. By Proposition 7.7, if $\widetilde{\mathrm{Kh}}(K) = \mathbb{F}$, the branched double cover $\Sigma(K)$ is hyperbolic. Conjecture 1.2 then implies that $\widehat{\mathrm{HF}}(\Sigma(K))$ is non-trivial, and by the correspondence of [OS5],

$$1 = \mathrm{rnk}\big(\widetilde{\mathrm{Kh}}(K)\big) \geq \mathrm{rnk}(\widehat{\mathrm{HF}}(\Sigma(K))) > 1.$$

In particular, if Conjecture 1.2 is true then for every non-trivial knot K the reduced Khovanov homology $\widetilde{\mathrm{Kh}}(K)$ is non-trivial (i.e. different from $\mathbb{F}$).

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School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P. O. Box 19395-5746, Tehran, Iran

E-mail address: eaman@ipm.ir