ON FINITENESS AND RIGIDITY OF J-HOLOMORPHIC CURVES IN SYMPLECTIC THREE-FOLDS

EAMAN EFTEKHARY

Abstract. Given a symplectic three-fold (M,ω) we show that for a generic almost complex structure J compatible with ω there are finitely many J-holomorphic curves in M of genus g representing the homology class β for every $g \geq 0$ and every $\beta \in \mathrm{H}_2(M,\mathbb{Z})$ such that $c_1(M)\beta = 0$ and the divisibility of β is at most 4 (i.e. if $\beta = n\alpha$ with $\alpha \in \mathrm{H}_2(M,\mathbb{Z})$ and $n \in \mathbb{Z}$, then $n \leq 4$). Moreover, every such curve is embedded and 4-rigid.

1. Introduction

Let (M,ω) be a symplectic three-fold, and J be an element in the space $\mathcal{J}^\infty(M,\omega)$ of smooth almost complex structure on M which are compatible with ω. For every homology class $\beta \in \mathrm{H}_2(M,\mathbb{Z})$ and every genus $g \geq 0$, the virtual dimension of the moduli space of J-holomorphic curves representing β which have genus g is equal to $c_1(M)\beta$. In particular, if $c_1(M)\beta = 0$ (e.g. if $c_1(M) = 0$) this moduli space is expected to be zero-dimensional. In fact, a conjecture of Ionel and Parker predicts that for a generic almost complex structure J all such moduli spaces are compact zero dimensional manifolds (see Section 7.4 of [3]). In this paper, we present a proof when the homology class β is not divisible by an integer greater than 4.

Definition 1.1. *For a homology class $\beta \in \mathrm{H}_2(M,\mathbb{Z})$, define the* divisibility *$|\beta|$ to be the largest integer n such that $\beta = n\alpha$ for some non-zero $\alpha \in \mathrm{H}_2(M,\mathbb{Z})$.*

The following is the main result of this paper:

Theorem 1.2. *For every J in a subset $\mathcal{J}^\infty_{rigid}(M,\omega) \subset \mathcal{J}^\infty(M,\omega)$ of the second category, every genus $g \geq 0$, and every homology class $\beta \in \mathrm{H}_2(M,\mathbb{Z})$ which satisfies $c_1(M)\beta = 0$ and $|\beta| \leq 4$, the moduli space $\mathcal{M}_g(M,\beta,J)$ of somewhere injective J-holomorphic curves of genus g representing the homology class β consists of finitely many elements. Moreover, every curve in this moduli space is embedded and 4-rigid with respect to the almost complex structure J.*

An embedded J-holomorphic curve $C \subset M$ with holomorphic normal bundle N_C is called *n-rigid* if for every holomorphic branched covering map $\pi : \Sigma \to C$ from a smooth Riemann surface Σ to C of degree less than or equal to n there is no non-zero section $X \in \Gamma(\Sigma, \pi^* N_C)$ satisfying the linearized Cauchy-Riemann equation

$$\mathcal{K}_\pi(X) := \nabla X + J\nabla_{j_\Sigma} X + (\nabla_X J) d\pi j_\Sigma = 0. \tag{1}$$

Here ∇ is the covariant derivative associated with the Levi-Civita connection for the metric $\omega(.,J.)$, j_Σ is the complex structure on Σ, and the equation takes place in the vector space $\Gamma(\Sigma, \Omega^{0,1}_\Sigma \otimes_J \pi^* N_C)$. Such a curve is called *rigid* if it is 1-rigid, and *super-rigid* if it is n-rigid for all $n \in \mathbb{Z}^+$. For integrable J, this notion of super-rigidity agrees with the notions of super-rigidity from [1].

The n-rigidity of a J-holomorphic curve $C \subset M$ implies that for every genus $g \geq 0$ and every homology class $\beta \in \mathrm{H}_2(M, \mathbb{Z})$ which is of the form $d[C]$ for some integer $0 < d \leq n$, the compactified moduli space $\overline{\mathcal{M}}_g(M, \beta, J)$ (of stable J-holomorphic curves $f : \Sigma \to M$ of genus g representing the homology class β) has a topological component identified with the moduli space $\overline{\mathcal{M}}_g(C, d[C])$ of genus g, degree d branched covers of C. Thus, the contribution of C to the Gromov-Witten invariants $N_g(M, \beta)$ is well-defined, and is independent of the normal bundle if J is integrable (see page 290 from [1]). When J-holomorphic curves are super-rigid, global Gromov-Witten theory of M is thus reduced to the local Gromov-Witten theory of such J-holomorphic curves. The significance of super-rigid curves was observed by Pandharipande [11], and also by Bryan and Pandharipande [1] in the study of a local version of Gopakumar-Vafa conjecture [5]. Nevertheless, Bryan and Pandharipande showed later that if the genus of a J-holomorphic curve C is at least 2, for any choice of stable (holomorphic) normal bundle, C cannot be 3-rigid [2]. So if a curve C is n-rigid for some (non-integrable) almost complex structure, its contribution to the Gromov-Witten invariants may a priori depend on the normal bundle.

Below we outline the proof of Theorem 1.2. For a generic almost complex structure J on M which is compatible with ω the moduli space $\mathcal{M}_g(M, \beta, J)$ is known to be a zero dimensional manifold if $c_1(M)\beta = 0$. The study of the possible limits of the elements of such moduli spaces reduces the proof of Theorem 1.2 to the study of embedded J-holomorphic curves $C \subset M$ with normal bundle N_C which host a branched covering map $\pi : \Sigma \to C$ such that the kernel of $\mathcal{K}_\pi$ is non-trivial. In Section 2 we show that it is enough to consider the branched covering maps π which come from the action of a subgroup $\mathfrak{G}$ of the permutation group S_n (with $n = |\beta|$) on Σ, so that $C = \Sigma/\mathfrak{G}$. The group ring $\mathbb{R}_\mathfrak{G}$ acts from the left on $\mathrm{Ker}(\mathcal{K}_\pi)$. There is a finite set $I(\mathfrak{G})$ (independent of the map π) of the maximal left ideals of $\mathbb{R}_\mathfrak{G}$ such that for some $\mathfrak{m} \in I(\mathfrak{G})$ and some non-zero $X \in \mathrm{Ker}(\mathcal{K}_\pi)$, $\mathfrak{m}$ annihilates X. When $|\beta| \leq 4$ the codimension of every $\mathfrak{m} \in I(\mathfrak{G})$ in $\mathbb{R}_\mathfrak{G}$ is at most 3. These reductions are discussed in Section 2.

In Section 4 we prove that the space of all tuples consisting of a C^ℓ almost complex structure J, an embedded J-holomorphic curve C of genus h and representing a class $\alpha = \beta/|\mathfrak{G}| \in \mathrm{H}_2(M,\mathbb{Z})$, a surface Σ admitting the action of $\mathfrak{G}$ with $\Sigma/\mathfrak{G} = C$ and $\pi : \Sigma \to C$ the corresponding quotient map, and $X \in \mathrm{Ker}(\mathcal{K}_\pi)$ a non-zero element annihilated by $\mathfrak{m}$ form a Banach manifold $\mathcal{N}^{\mathfrak{m}}_{[\mathfrak{G}]}(M,\beta)$ for fixed $g, h, \alpha, \mathfrak{G} \subset S_4, \mathfrak{m} \in I(\mathfrak{G})$ (see Definition 4.5 for the precise definition). This Banach manifold parametrizes the unwanted situations. In order to show that $\mathcal{N}^{\mathfrak{m}}_{[\mathfrak{G}]}(M,\beta)$ is a manifold at a point corresponding to a tuple $(J, \pi : \Sigma \to C = \Sigma/\mathfrak{G}, X)$ as above we need to prove that if $X \in \mathrm{Ker}(\mathcal{K}_\pi)$ and $\delta \in \mathrm{Ker}(\mathcal{K}^*_\pi)$ are annihilated by $\mathfrak{m} \in I(\mathfrak{G})$ then $\langle \nabla_X u, \delta\rangle \neq 0$ for some section u of the vector bundle of $(0,1)$-homomorphisms from the tangent space T_C of C to N_C. Here $\mathcal{K}^*_\pi$ denotes the adjoint of the operator $\mathcal{K}_\pi$. It is important in this argument that the codimension of $\mathfrak{m}$ in $\mathbb{R}_{\mathfrak{G}}$ is small.

The projection map $\Pi^{\mathfrak{m}}_{[\mathfrak{G}]}(\beta)$ from $\mathcal{N}^{\mathfrak{m}}_{[\mathfrak{G}]}(M,\beta)$ to the space of almost complex structures (of class C^ℓ) is Fredholm. In Section 3 we show that the index of $\Pi^{\mathfrak{m}}_{[\mathfrak{G}]}(\beta)$ is at most zero. For a generic almost complex structure J we conclude that $\Pi^{\mathfrak{m}}_{[\mathfrak{G}]}(\beta)^{-1}(J)$ is empty. The detour to the space of C^ℓ complex structures is completed by extending the results to smooth almost complex structures in Section 5.

The main technical issue in extending the results of the current paper for arbitrary homology classes is that for an arbitrary group $\mathfrak{G}$ the large size of the irreducible representations of $\mathfrak{G}$ damages the transversality argument of Section 4, which is the heart of our proof. Although this obstacle seems hard to overcome, the author hopes that the techniques and the setup used in this paper may be applied in other moduli problems.

Acknowledgements. The author would like to thank Gang Tian for his continuous advice and support and for sharing his insight.

2. Moduli space of embedded Riemann surfaces

Fix the integers $k, p > 1$ and $\ell > k$. We will sometimes drop these integers from our notation for simplicity.

Let (M,ω) be a compact symplectic three-fold and fix a second homology class $\beta \in \mathrm{H}_2(M,\mathbb{Z})$. Let $\mathcal{J} = \mathcal{J}^\ell(M,\omega)$ be the C^ℓ completion of the space of smooth almost complex structures on M compatible with ω. Let $\mathcal{T}_J$ be the tangent space to $\mathcal{J}$ at J. It consists of the linear homomorphisms $u : TM \to TM$ (of class $C^{\ell-1}$) which satisfy

$$u \circ J + J \circ u = 0 \quad \text{and} \quad \omega_x(u(X), Y) + \omega_x(X, u(Y)) = 0 \ \ \forall X, Y \in T_xM.$$

Let $\mathcal{M}_g$ denote the moduli space of Riemann surfaces of genus g. For a smooth surface Σ let $W^{k,p}(\Sigma, M)$ denote the space of maps of class $W^{k,p}$ from Σ to M. Define

$$\mathcal{M}_g(M,\beta,J) := \left\{ f : (\Sigma, j_\Sigma) \to (M,J) \;\middle|\; \begin{array}{c} (\Sigma, j_\Sigma) \in \mathcal{M}_g, f \in W^{k,p}(\Sigma, M) \\ \overline{\partial}_{J,j_\Sigma}(f) = df + J \circ df \circ j_\Sigma = 0, \\ f \text{ is somewhere injective} \end{array} \right\}$$

$$\mathcal{M}_g(M,\beta) := \left\{ (f, j_\Sigma, J) \;\middle|\; \begin{array}{c} J \in \mathcal{J}^\ell(M,\omega), \\ (f, j_\Sigma) \in \mathcal{M}_g(M,\beta,J) \end{array} \right\} = \bigcup_{J \in \mathcal{J}} \mathcal{M}_g(M,\beta,J).$$

Let $\mathcal{M}_g^\circ(M,\beta,J)$ and $\mathcal{M}_g^\circ(M,\beta)$ denote the open subsets consisting of the embeddings. The proof of the following (well-known) proposition is basically a slight modification of the proof of Theorem 3.4.1 and Proposition 3.2.1 from [9] (by letting the complex structure on the domain curve vary in a finite cover of $\mathcal{M}_g$).

Proposition 2.1. *There is a subset $\mathcal{J}^\ell_{reg}(M,\omega) \subset \mathcal{J}^\ell(M,\omega)$ of the second category such that for every $J \in \mathcal{J}^\ell_{reg}(M,\omega)$ the following are true.*

- *If $c_1(M)\beta < 0$ for some $\beta \in \mathrm{H}_2(M,\mathbb{Z})$ the moduli space $\mathcal{M}_g(J,\beta)$ is empty for every genus g.*
- *If $c_1(M)\beta = 0$ for some $\beta \in \mathrm{H}_2(M,\mathbb{Z})$ the moduli space $\mathcal{M}_g(M,\beta)$ is a separable $C^{\ell-k}$-Banach manifold and J is a regular value for the Fredholm projection map $\pi_{g,\beta} : \mathcal{M}_g(M,\beta) \to \mathcal{J}$ which has index zero. Then $\mathcal{M}_g(M,\beta,J)$ is a zero dimensional manifold and consists of embeddings.*
- *If $c_1(M)\beta = c_1(M)\beta' = 0$ for $\beta, \beta' \in \mathrm{H}_2(M,\mathbb{Z})$ and $f \in \mathcal{M}_g(M,\beta,J)$, $f' \in \mathcal{M}_{g'}(M,\beta',J)$ have distinct images, their images are disjoint.*

Suppose $J \in \mathcal{J}^\ell_{reg}(M,\omega) \cap \mathcal{J}^\infty(M,\omega)$. Let $\beta \in \mathrm{H}_2(M,\mathbb{Z})$ be a homology class with $c_1(M)\beta = 0$. Let $(f_i : \Sigma_i \to M)_i$ be an infinite sequence of embedded J-holomorphic curves in $\mathcal{M}_g(M,\beta,J)$. After passing to a subsequence, f_i converge to a stable J-holomorphic curve $f : \Sigma \to M$ with $f_*[\Sigma] = \beta$. The domain Σ may be nodal, while f can collapse some of the components and have non-trivial degree over other components. Since $J \in \mathcal{J}^\ell_{reg}(M,\omega)$, the image of f lies over a smooth embedded J-holomorphic curve C, and $\beta = d[C]$ for some positive integer d. The domain $\Sigma = \cup_{i=1}^N \Sigma^i$ is a union of the components $\Sigma^{N_0+1}, ..., \Sigma^N$ which are collapsed under f, with the components $\Sigma^i,\ i = 1, ..., N_0$ where f is the composition of a branched covering map $\pi^i : \Sigma^i \to C$ with the J-holomorphic embedding $\imath_C$ of C in the manifold M.

Lemma 6.3 and Proposition 6.6 from [7] give a section $X \in \Gamma(\Sigma, \pi^* N_C)$ with supremum norm equal to 1, consisting of a union of sections

$$X^i \in \Gamma^\infty\left(\Sigma^i, (\pi^i)^* N_C\right) \subset \Gamma^{k,p}\left(\Sigma^i, (\pi^i)^* N_C\right), \quad i = 1, ..., N_0,$$

and satisfying the following equation

$$\mathcal{K}_{\pi^i}(X^i) := \nabla X^i + J\nabla_{j_{\Sigma^i}} X^i + (\nabla_{X^i} J)\circ d\pi^i \circ j_{\Sigma^i} = 0, \quad i = 1, ..., N_0,$$

in $\Gamma^{k-1,p}(\Sigma^i, \Omega^{0,1}_{\Sigma^i} \otimes_J (\pi^i)^* N_C)$. Although the proofs in [7] are presented in dimension 4 where the normal bundle is a complex line bundle, this assumption is not used in the proof. At least one of the sections X^i for $i = 1, ..., N_0$ is non-zero, since the supremum norm of X is equal to 1 and the Riemann surface Σ is connected.

Thus, the existence of a sequence as above which converges to an embedded J-holomorphic curve $C \subset M$ of genus $h \le g$ implies that there is a branched covering map $\pi_S : S \to C$ (with $S = \Sigma^i$ and $\pi_S = \pi^i$ for some i), and a non-trivial section X in the kernel of the elliptic operator

$$\mathcal{K}_{\pi_S} : \Gamma^{k,p}(S, \pi_S^* N_C) \to \Gamma^{k-1,p}(S, \Omega^{0,1}_S \otimes_J (\pi_S)^* N_C).$$

Based on this observation, we make the following definition.

Definition 2.2. *An embedded J-holomorphic curve C of genus $h \ge 0$ with normal bundle N_C is called (n, m)-*rigid *for the integers $m, n \in \mathbb{Z}^+$ if for every branched covering map $\pi_S : S \to C$ from a smooth curve S of genus $g \le m + h$ to C with $\deg(\pi_S) \le n$, $\mathrm{Ker}(\mathcal{K}_{\pi_S}) = 0$. The curve C is called n-*rigid *if it is (n, m)-rigid for all $m \in \mathbb{Z}^+$, and* super-rigid *if it is n-rigid for every integer $n \in \mathbb{Z}^+$.*

This definition does not depend on the particular choice of the integers $k, p \ge 2$ by elliptic regularity.

We now study a few possible reductions of the concept of rigidity, which are useful in our later considerations.

Reduction 1. Suppose that $\pi : \Sigma \to C$ is an arbitrary branched covering map between smooth Riemann surfaces of degree n. Suppose that $B \in \mathrm{Div}(\Sigma)$ is the branching divisor of π. For a regular value $q \in C - \pi(B)$ the group $\pi_1(C - \pi(B), q)$ acts on $\pi^{-1}(q)$. This gives a homomorphism

$$\rho : \pi_1(C - \pi(B), q) \longrightarrow S_n$$

to the group S_n of permutations in n letters. The kernel $\mathrm{Ker}(\rho)$ determines a finite regular covering map $\pi^\circ : \Sigma^\circ \to C - \pi(B)$, which may be extended to a branched covering map $\widetilde{\pi} : \widetilde{\Sigma} \to C$ from a compact Riemann surface $\widetilde{\Sigma}$ to C. Moreover, this map decomposes as $\widetilde{\pi} = \pi \circ \tau$ for some branched covering map $\tau : \widetilde{\Sigma} \to \Sigma$. Since $\mathrm{Ker}(\rho)$ is normal in the fundamental group, the group of deck transformations for π° may be computed as

$$\mathrm{Dec}(\pi^\circ) = \frac{\pi_1(C - \pi(B), q)}{\pi^\circ_*(\pi_1(\Sigma^\circ, p^\circ))} = \frac{\pi_1(C - \pi(B), q)}{\mathrm{Ker}(\rho)} \simeq \mathrm{Im}(\rho) \subset S_n.$$

Denote this later subgroup of S_n by $\mathfrak{G}$. The degree of $\widetilde{\pi}$ is equal to the order of the finite group $\mathfrak{G}$. Every deck transformation of Σ° may be extended to $\widetilde{\Sigma}$ as an automorphism with possible fixed points, and C may be realized as the quotient $\widetilde{\Sigma}/\mathfrak{G}$ The reader is referred to [10] (in particular, Theorem 4.9) for a more detailed discussion.

Suppose that the curve C is an embedded J-holomorphic curve in M for some almost complex structure $J \in \mathcal{J}^\ell(M,\omega)$. Fix the branched covering map $\pi : \Sigma \to C$. If $X \in \mathrm{Ker}(\mathcal{K}_\pi)$ is non-trivial, then $\tau^* X \in \mathrm{Ker}(\mathcal{K}_{\widetilde{\pi}})$ is non-trivial as well. Thus, if an embedded J-holomorphic curve C is not n-rigid, there is a Riemann surface $S = \widetilde{\Sigma}$ admitting an action of a subgroup $\mathfrak{G}$ of S_n, with $\pi_S : S \to C = S/\mathfrak{G}$ the corresponding branched covering map, so that the kernel of $\mathcal{K}_{\pi_S}$ is non-trivial.

Reduction 2. Let $\pi = \pi_S : S \to S/\mathfrak{G} = C$ be a branched covering map coming from the action of a finite group and $X \in \mathrm{Ker}(\mathcal{K}_\pi)$ be a non-trivial section. For any $\sigma \in \mathfrak{G}$, $\sigma^* X$ is also in $\mathrm{Ker}(\mathcal{K}_\pi)$. Thus, the left action of the group ring $\mathbb{R}_{\mathfrak{G}}$ on $\Gamma^{k,p}(S, \pi^* N_C)$ defined by

$$\mathfrak{a}Y := \sum_{\sigma\in\mathfrak{G}} a_\sigma \sigma^* Y \quad \forall\ \mathfrak{a} = \sum_{\sigma\in\mathfrak{G}} a_\sigma \sigma^{-1} \in \mathbb{R}_{\mathfrak{G}}, \quad Y \in \Gamma^{k,p}(S, \pi^* N_C)$$

induces an action of $\mathbb{R}_{\mathfrak{G}}$ on $\mathrm{Ker}(\mathcal{K}_\pi)$.

Let us denote by $\mathfrak{m}_X$ the left ideal of $\mathbb{R}_{\mathfrak{G}}$ consisting of the elements $\mathfrak{a} \in \mathbb{R}_{\mathfrak{G}}$ such that $\mathfrak{a}X = 0$. Let $\widehat{I}(\mathfrak{G})$ be the set of irreducible representations ρ of $\mathfrak{G}$ over $\mathbb{R}$. The group ring $\mathbb{R}_{\mathfrak{G}}$ decomposes (using Artin-Wedderburn and Maschke theorems) as

$$\begin{aligned} \mathbb{R}_{\mathfrak{G}} &\simeq \bigoplus_{\rho\in\widehat{I}(\mathfrak{G})} M_{\ell(\rho)}(R_\rho), \quad R_\rho \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}, \\ \sigma &\mapsto \left(\rho(\sigma)\right)_{\rho\in\widehat{I}(\mathfrak{G})} \end{aligned} \tag{2}$$

where $M_\ell(R)$ denotes the space of $\ell \times \ell$ matrices with entries in the ring R, $\ell(\rho)$ is the dimension of the representation ρ as an algebra over R_ρ, and $\mathbb{H}$ denotes the ring of quaternions. The reader is referred to Proposition 3.29, Theorem 3.37 and Exercise 3.41 from [4] for a proof of the above decomposition theorem. Denote by $I_\rho \in \mathbb{R}_{\mathfrak{G}}$ the identity matrix in the matrix algebra associated with the representation ρ.

We thus have

$$1_{\mathbb{R}_{\mathfrak{G}}} = \sum_{\rho\in\widehat{I}(\mathfrak{G})} I_\rho \quad\Rightarrow\quad X = \sum_{\rho\in\widehat{I}(\mathfrak{G})} I_\rho X =: \sum_{\rho\in\widehat{I}(\mathfrak{G})} X_\rho.$$

Since X is in $\mathrm{Ker}(\mathcal{K}_\pi)$, so is X_ρ for every $\rho \in \widehat{I}(\mathfrak{G})$. Moreover, at least one of the sections X_ρ is non-zero. Fix such a representation $\rho \in \widehat{I}(\mathfrak{G})$ and let

$R = R_\rho$ and $\ell = \ell(\rho)$. Let $\epsilon_i,\ i = 1, ..., \ell$, denote the matrix in $M_\ell(R)$ with 1 as its (i,i) entry and zeros elsewhere. We may think of ϵ_i as an element of $\mathbb{R}_{\mathfrak{G}}$. Since

$$0 \neq X_\rho = I_\rho X = (\epsilon_1 + ... + \epsilon_\ell)X,$$

at least one of the sections $\epsilon_i X$ is a non-zero element in $\mathrm{Ker}(\mathcal{K}_\pi)$. Denote one such section by Y. The left ideal $\mathfrak{m} = \mathfrak{m}_Y$ is then a maximal left ideal $\mathfrak{m}_{\rho,i}$ which consists of all elements of $\mathbb{R}_{\mathfrak{G}}$ which are characterized using (2) as those matrices which have zeros in the i-th column in the matrix presentation corresponding to the representation ρ. We denote the finite set of the maximal left ideals of $\mathbb{R}_{\mathfrak{G}}$ of the form $\mathfrak{m} = \mathfrak{m}_{\rho,i}$ by $I(\mathfrak{G})$. There is a natural map $\imath : I(\mathfrak{G}) \to \widehat{I}(\mathfrak{G})$. For $\mathfrak{m}$ as above define $\ell(\mathfrak{m}) = \ell$ and $R_{\mathfrak{m}} = R$.

The maximal left ideal $\mathfrak{m}$ determines a restriction of the operator $\mathcal{K}_\pi$:

$$\mathcal{K}_\pi^{\mathfrak{m}} : \Gamma_{\mathfrak{m}}^{k,p}(S, \pi^* N_C) \longrightarrow \Gamma_{\mathfrak{m}}^{k-1,p}\left(S, \Omega_S^{0,1} \otimes_J \pi^* N_C\right).$$

Here $\Gamma_{\mathfrak{m}}^{\bullet}(S, \star)$ denotes the vector space of sections Z of the bundle $\star$ such that either $Z = 0$ or the left ideal $\mathfrak{m}_Z$ is $\mathfrak{m}$. The section Y is then in the kernel of the operator $\mathcal{K}_\pi^{\mathfrak{m}}$.

The kernel $\mathrm{Ker}(\rho)$ of the representation $\rho : \mathfrak{G} \to M_\ell(R)$ is a normal subgroup of $\mathfrak{G}$ and decomposes the covering map $\pi : S \to C = S/\mathfrak{G}$ as a composition

$$S \xrightarrow{\pi_1} S' = S/\mathrm{Ker}(\rho) \xrightarrow{\pi_2} C = S/\mathfrak{G}.$$

Since $\mathfrak{H} = \mathrm{Ker}(\rho)$ is a normal subgroup of $\mathfrak{G}$, the quotient $\mathfrak{G}/\mathfrak{H}$ is a group, is isomorphic to $\mathrm{Im}(\rho)$ and there is a well-defined action of this quotient on $S' = S/\mathfrak{H}$. Moreover, the quotient of S' by the action of $\mathrm{Im}(\rho)$ is precisely the surface $C = S/\mathfrak{G}$. Moreover, since the action of $\mathrm{Ker}(\rho)$ preserve the section Y of the bundle $\pi^* N_C$, $Y = \pi_1^* Z$ for some non-zero $Z \in \Gamma^{k,p}(S', \pi_2^* N_C)$. We may thus replace S by S', $\mathfrak{G}$ by $\mathfrak{G}' = \mathrm{Im}(\rho)$, and the section Y by Z. The action of $\mathfrak{G}'$ extends the action of the group of deck transformations of a regular which is obtained from $S' \to C$ by removing the branched locus from C and its pre-image from S'. The above considerations imply the following lemma.

Lemma 2.3. *Suppose that an embedded J-holomorphic curve C in the symplectic Clabi-Yau threefold (M, ω) is not n-rigid. Then there is a Riemann surface S admitting an action of a finite group $\mathfrak{G} \subset S_n$ (corresponding to a branched covering map $\pi : S \to C = S/\mathfrak{G}$) and a maximal left ideal $\mathfrak{m} \in I(\mathfrak{G})$ of the group ring $\mathbb{R}_{\mathfrak{G}}$ so that the kernel of the operator*

$$\mathcal{K}_\pi^{\mathfrak{m}} : \Gamma_{\mathfrak{m}}^{k,p}(S, \pi^* N_C) \longrightarrow \Gamma_{\mathfrak{m}}^{k-1,p}\left(S, \Omega_S^{0,1} \otimes_J \pi^* N_C\right)$$

is non-trivial and the representation $\rho = \imath(\mathfrak{m}) : \mathfrak{G} \to M_{\ell(\mathfrak{m})}(R_{\mathfrak{m}})$ is faithful.

Before moving to the next step, note that the operator $\mathcal{K}_\pi^{\mathfrak{m}}$ considered above is Fredholm:

Lemma 2.4. *With the above notation the operator $\mathcal{K}_\pi^{\mathfrak{m}}$ is Fredholm, and its adjoint is the operator*

$$(\mathcal{K}_\pi^{\mathfrak{m}})^* : \Gamma_{\mathfrak{m}}^{k,q}\left(S, \Omega_S^{0,1} \otimes_J \pi^* N_C\right) \longrightarrow \Gamma_{\mathfrak{m}}^{k-1,q}\left(S, \pi^* N_C\right),$$
$$(\mathcal{K}_\pi^{\mathfrak{m}})^* := \mathcal{K}_\pi^*|_{\Gamma_{\mathfrak{m}}^{k,q}(\Omega_S^{0,1}\otimes_J \pi^* N_C)},$$

with $\frac{1}{p}+\frac{1}{q}=2$. In particular $\mathrm{Coker}\,(\mathcal{K}_\pi^{\mathfrak{m}}) \simeq \mathrm{Ker}\,((\mathcal{K}_\pi^{\mathfrak{m}})^*)$.

Proof. Theorem C.1.10 from [9] implies that $\mathcal{K}_\pi$ is Fredholm with an adjoint operator $\mathcal{K}_\pi^*$, and that $\mathrm{Ker}(\mathcal{K}_\pi^*) \simeq \mathrm{Coker}(\mathcal{K}_\pi)$. Let

$$E_{\mathfrak{m}} := \Gamma_{\mathfrak{m}}^{k,p}(S, \pi^* N_C), \qquad E := \Gamma^{k,p}(S, \pi^* N_C) = \bigoplus_{\mathfrak{m}\in I(\mathfrak{G})} E_{\mathfrak{m}},$$

$$F_{\mathfrak{m}} := \Gamma_{\mathfrak{m}}^{k-1,p}(S, \Omega_S^{0,1} \otimes_J \pi^* N_C) \text{ and } F := \Gamma^{k-1,p}(S, \Omega_S^{0,1} \otimes_J \pi^* N_C) = \bigoplus_{\mathfrak{m}\in I(\mathfrak{G})} F_{\mathfrak{m}}.$$

The subspaces $E_{\mathfrak{m}} \subset E$ and $F_{\mathfrak{m}} \subset F$ are closed. For every element $\sigma \in \mathfrak{G}$ and every section $X \in E$, $\mathcal{K}_\pi(\sigma^* X) = \sigma^*(\mathcal{K}_\pi(X))$. Thus, $\mathcal{K}(E_{\mathfrak{m}})$ is a subset of $F_{\mathfrak{m}}$ and the operator $\mathcal{K}_\pi$ has a diagonal presentation in the above decomposition of E and F as a direct sum of the operators $\mathcal{K}_\pi^{\mathfrak{m}}$ for $\mathfrak{m} \in I(\mathfrak{G})$. In particular, we have

$$\mathrm{Ker}\,(\mathcal{K}_\pi^{\mathfrak{m}}) = \mathrm{Ker}(\mathcal{K}_\pi) \cap E_{\mathfrak{m}} \quad \text{and} \quad \mathrm{Coker}\,(\mathcal{K}_\pi^{\mathfrak{m}}) = \frac{F_{\mathfrak{m}}}{F_m \cap \mathrm{Im}(\mathcal{K}_\pi)}.$$

In particular, the kernel and the cokernel of the bounded operator $\mathcal{K}_\pi^{\mathfrak{m}}$ are both finite dimensional (i.e. $\mathcal{K}_\pi^{\mathfrak{m}}$ is Fredholm). The adjoint $\mathcal{K}_\pi^*$ of $\mathcal{K}_\pi$ satisfies $\mathcal{K}_\pi^*(\sigma^*(\delta)) = \sigma^*(\mathcal{K}_\pi^*(\delta))$ for every $\sigma \in \mathfrak{G}$ and $\delta \in F$. We may thus write $\mathcal{K}_\pi^*$ as a direct sum

$$\mathcal{K}_\pi^* = \bigoplus_{\mathfrak{m}\in I(\mathfrak{G})} (\mathcal{K}_\pi^*)^{\mathfrak{m}}.$$

For every $X \in E_{\mathfrak{m}}$ and $\delta \in F_{\mathfrak{m}}$ we have

$$\begin{aligned}\int_S \langle \delta, \mathcal{K}_\pi^{\mathfrak{m}}(X)\rangle d\mathrm{vol}_S &= \int_S \langle \delta, \mathcal{K}_\pi(X)\rangle d\mathrm{vol}_S \\ &= \int_S \langle \mathcal{K}_\pi^*(\delta), X\rangle d\mathrm{vol}_S = \int_S \langle (\mathcal{K}_\pi^*)^{\mathfrak{m}}(\delta), X\rangle d\mathrm{vol}_S\end{aligned}$$

Thus, $(\mathcal{K}_\pi^*)^{\mathfrak{m}}$ is the formal adjoint of $\mathcal{K}_\pi^{\mathfrak{m}}$. The second claim of the lemma follows from this observation. □

Reduction 3. In order to study 4-rigidity we only need to consider the case where the group $\mathfrak{G}$ is isomorphic to a subgroup of S_4.

Lemma 2.5. *If $\mathfrak{G}$ is a subgroup of S_4 for any $\mathfrak{m} \in I(\mathfrak{G})$ the associated matrix algebra is one of $M_i(\mathbb{R}),\ i = 1, 2, 3$ or $M_1(\mathbb{C}) = \mathbb{C}$.*

Proof. The subgroups of S_4 are isomorphic to one of the following groups (c.f. page 160 of [12]):

$$\mathbb{Z}/k\mathbb{Z},\quad k=1,2,3,4,\quad \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z},\quad S_3,\quad D_8,\quad A_4,\quad S_4.$$

Except for the last 4 in this list, these groups are abelian and the matrix algebras corresponding to their irreducible real representations are either $M_1(\mathbb{R})$ or $M_1(\mathbb{C})$. The group A_4 has 3 irreducible real representations with the corresponding matrix algebras equal to $M_1(\mathbb{R}), M_1(\mathbb{C})$, and $M_3(\mathbb{R})$. For D_8, the dihedral group, we may distinguish 4 one dimensional irreducible representations with the associated matrix algebra equal to $M_1(\mathbb{R})$, and one two dimensional irreducible representation with the corresponding matrix algebra equal to $M_2(\mathbb{R})$. Finally for the groups S_3 and S_4 the associated matrix algebra of every irreducible real representation is one of $M_1(\mathbb{R}), M_2(\mathbb{R})$ and $M_3(\mathbb{R})$. Most of these claims follow from the discussion in sections 2.3, 3.1 and 3.5 from [4], and are easy exercises in representation theory. This completes the proof. □

3. Index computation

Let J be an almost complex structure on M and $C\subset M$ be an embedded J-holomorphic curve. Fix a holomorphic map $\pi:\Sigma\to C$ of degree $n>1$ which is determined by the action of a group $\mathfrak{G}$ on a Riemann surface Σ (so that $C=\Sigma/\mathfrak{G}$). Let $E\to C$ be a holomorphic vector bundle over C and $\pi^*E\to\Sigma$ be its pull-back over Σ. The group ring $\mathbb{R}_{\mathfrak{G}}$ acts on the cohomology groups $\mathrm{H}^i(\Sigma,\pi^*E)$. For $\eta\in\mathrm{H}^i(\Sigma,\pi^*E)$, we set $\mathfrak{m}_\eta$ to be the left ideal of $\mathbb{R}_{\mathfrak{G}}$ consisting of the elements $\mathfrak{a}\in\mathbb{R}_{\mathfrak{G}}$ such that $\mathfrak{a}.\eta=0$. For every $\mathfrak{m}\in I(\mathfrak{G})$, let

$$\mathrm{H}^i_{\mathfrak{m}}(\pi^*E):=\big\{\eta\in\mathrm{H}^i(\Sigma,\pi^*E)\ \big|\ \mathfrak{m}\subset\mathfrak{m}_\eta\big\},\qquad i=0,1.$$

Define $h^i_{\mathfrak{m}}(\pi^*E)=\dim_{\mathbb{C}}(\mathrm{H}^i_{\mathfrak{m}}(\pi^*E)),\ i=0,1$, and let

$$\chi_{\mathfrak{m}}(\pi^*E)=h^0_{\mathfrak{m}}(\pi^*E)-h^1_{\mathfrak{m}}(\pi^*E).$$

For every $\mathfrak{m}\in I(\mathfrak{G})$, the Fredholm map $\mathcal{K}^{\mathfrak{m}}_\pi$ is a zero-order perturbation of the operator

$$\overline{\partial}_{\mathfrak{m}}:\Gamma^{k,p}_{\mathfrak{m}}(\Sigma,\pi^*N_C)\to\Gamma^{k-1,p}_{\mathfrak{m}}(\Sigma,\Omega^{0,1}_\Sigma\otimes_J\pi^*N_C).$$

The index of $\mathcal{K}^{\mathfrak{m}}_\pi$ may thus be computed by computing the index of $\overline{\partial}_{\mathfrak{m}}$. In this case $\mathrm{Ker}(\overline{\partial})=\mathrm{H}^0(\Sigma,\pi^*N_C)$, $\mathrm{Ker}(\overline{\partial}^*)=\mathrm{H}^1(\Sigma,\pi^*N_C)$, and thus

$$\begin{aligned}\mathrm{Ker}(\overline{\partial}_{\mathfrak{m}})&=\mathrm{Ker}(\overline{\partial})\cap\Gamma_{\mathfrak{m}}(\Sigma,\pi^*N_C)=\mathrm{H}^0_{\mathfrak{m}}(\Sigma,\pi^*N_C),\\ \mathrm{Coker}(\overline{\partial}_{\mathfrak{m}})&=\mathrm{Ker}(\overline{\partial}^*)\cap\Gamma_{\mathfrak{m}}(\Sigma,\Omega^{0,1}_\Sigma\otimes\pi^*N_C)=\mathrm{H}^1_{\mathfrak{m}}(\Sigma,\pi^*N_C).\end{aligned}$$

Since N_C may be deformed to $\mathcal{O}_C\oplus K_C$, where K_C is the canonical bundle of C,

$$\frac{1}{2}\mathrm{Index}(\overline{\partial}_{\mathfrak{m}})=\chi_{\mathfrak{m}}(\pi^*K_C)+\chi_{\mathfrak{m}}(\pi^*\mathcal{O}_C).$$

Let B denote the branching divisor of the map π and $\mathcal{O}_B$ denote the corresponding sheaf over Σ with support in B. The group ring $\mathbb{R}_{\mathfrak{G}}$ also acts on $\mathrm{H}^0(\Sigma,\mathcal{O}_B)$ and we can thus define $\mathrm{H}^0_{\mathfrak{m}}(\mathcal{O}_B)$ and $h^0_{\mathfrak{m}}(\mathcal{O}_B)$ as above.

Lemma 3.1. *For every* $\mathfrak{m}\in I(\mathfrak{G})$, $\mathrm{Index}(\overline{\partial}_{\mathfrak{m}})=-2h^0_{\mathfrak{m}}(\mathcal{O}_B)$.

Proof. Kodaira-Serre duality implies that for a complex line bundle $L\to\Sigma$ and for $q=0,1$, the Hodge star operator $\star_L$

$$\begin{aligned}\star_L:\Gamma(\Sigma,\Omega^{0,q}\otimes L)&\longrightarrow\Gamma(\Sigma,\Omega^{1,1-q}\otimes(-L))\\ \langle\phi,\psi\rangle=\int_\Sigma\phi\wedge\star_L(\psi)&\quad\forall\phi,\psi\in\Gamma(\Sigma,\Omega^{0,q}\otimes L)\end{aligned}$$

gives an isomorphism from $\mathrm{H}^q(\Sigma,L)$ to $\big(\mathrm{H}^{1-q}(\Sigma,K_\Sigma-L)\big)^*$ (c.f. page 153 of [6]). If L is the pull-back of a line bundle over C, for every $\sigma\in\mathfrak{G}$ and every $\phi,\psi\in\Gamma(\Sigma,\Omega^{0,q}\otimes L)$, $\langle\sigma^*\phi,\sigma^*\psi\rangle=\langle\phi,\psi\rangle$. Since the above equation uniquely determines the Hodge operator,

$$\sigma^*\circ\star_L=\star_L\circ\sigma^*\qquad\forall\ \sigma\in\mathfrak{G}.$$

Since the isomorphism in the Kodaira-Serre duality theorem is induced by the Hodge star operator and the group action is preserved by the latter operator, for every line bundle $L=\pi^*L'$ we obtain the refined isomorphisms

$$\mathrm{H}^q_{\mathfrak{m}}(\Sigma,L)\simeq\big(\mathrm{H}^{1-q}_{\mathfrak{m}}(\Sigma,K_\Sigma-L)\big)^*\quad\forall\ \ \mathfrak{m}\in I(\mathfrak{G}),\quad q=0,1.$$

In particular, $\chi_{\mathfrak{m}}(\pi^*K_C)=-\chi_{\mathfrak{m}}(K_\Sigma-\pi^*K_C)=-\chi_{\mathfrak{m}}(B)$. The line bundle associated with B sits in a short exact sequence

$$0\longrightarrow\mathcal{O}_\Sigma\longrightarrow B\longrightarrow\mathcal{O}_B\longrightarrow 0.$$

The homomorphisms in the corresponding cohomology long exact sequence also preserve the action of the group $\mathfrak{G}$. The cohomology long exact sequence thus refines to exact sequences corresponding to every $\mathfrak{m}\in I(\mathfrak{G})$. The exact sequence corresponding to $\mathfrak{m}\in I(\mathfrak{G})$ gives $\chi_{\mathfrak{m}}(B)=\chi_{\mathfrak{m}}(\mathcal{O}_\Sigma)+h^0_{\mathfrak{m}}(\mathcal{O}_B)$. Since $\pi^*\mathcal{O}_C=\mathcal{O}_\Sigma$, the proof is complete. □

Proposition 3.2. *With the above notation, if* $\mathfrak{m}\in I(\mathfrak{G})$ *corresponds to a faithful representation* $\rho=\imath(\mathfrak{m})\in\widehat{I}(\mathfrak{G})$, *the index* $\mathrm{Index}(\mathcal{K}^{\mathfrak{m}}_\pi)$ *is at most* $-2l$, *where* l *is the number of points in the image* $\pi(B)$ *of the branched locus of* π.

Proof. By Lemma 3.1 the computation of the index of the operator $\mathcal{K}^{\mathfrak{m}}_\pi$ is reduced to the computation of $h^0_{\mathfrak{m}}(\mathcal{O}_B)$. Let $x\in C$ be a point in the image of the branched locus of π. The branching over x gives a partition of $n=|\mathfrak{G}|$ as $n=d_1+d_2+...+d_r$ with $d_1\ge d_2\ge...\ge d_r$. Associated with each d_i is a point y_i in $\pi^{-1}(x)$ which has multiplicity d_i and is a branched point if $d_i>1$. Let B_x be the divisor $(d_1-1)y_1+...+(d_r-1)y_r$. Since π comes from the action of a group $d_1=...=d_r=d>1$, and the degree of B_x is $r(d-1)$.

Note that $B = \sum_x B_x$ and $\mathcal{O}_B = \sum_x \mathcal{O}_{B_x}$. Since $\mathrm{H}^0_{\mathfrak{m}}(\mathcal{O}_B) = \oplus_x \mathrm{H}^0_{\mathfrak{m}}(\mathcal{O}_{B_x})$, it suffices to show $h^0_{\mathfrak{m}}(\mathcal{O}_{B_x}) \geq 1$.

The elements of $\mathrm{H}^0(\mathcal{O}_{B_x})$ are the germs $\phi = \sum_{i=1}^r \sum_{j=1}^{d-1} A_{i,j}/z_i^j$, where z_i is the pull-back of the local coordinate z around x to a neighbourhood of y_i. If $\zeta = \exp(2\pi\sqrt{-1}/d)$, there is a unique $\tau \in \mathfrak{G}$ such that $\tau(z_1) = \zeta z_1$ is satisfied near y_1. We may also assume that there are elements $\sigma_1, ..., \sigma_r \in \mathfrak{G}$ such that $\sigma_j(z_1) = z_j$ is satisfied for $j = 1, ..., r$. Note that σ_1 is the identity element of $\mathfrak{G}$. The group $\mathfrak{G}$ is then identified (as a set) as

$$\mathfrak{G} = \left\{ \sigma(i,j) := \sigma_i \tau^j \;\middle|\; i = 1, ..., r,\ j = 1, ..., d \right\}.$$

For $(i,j) \in \{1, ..., r\} \times \{1, ..., d\}$ define $\mathfrak{x}_{i,j} := \sum_{p=1}^{d-1} \zeta^{pj} z_i^{-p} \in \mathrm{H}^0(\mathcal{O}_{B_x})$. Since the matrix sending the column $\left(z_i^{-j}\right)_{j=1,...,d-1}$ to $(\mathfrak{x}_{i,j})_{j=1,...,d}$ is a rank-$(d-1)$ Vandermonde matrix, the rd sections $\mathfrak{x}_{i,j}$ generate $\mathrm{H}^0(\mathcal{O}_{B_x})$. For $\sigma \in \mathfrak{G}$, $\sigma(z_i) = \zeta^a z_{\sigma(i)}$, where $a \in \{1, ..., d\}$ and $\sigma(i) \in \{1, ..., m\}$ are integers which depend on σ and i. With this assumption,

$$\sigma(\mathfrak{x}_{i,j}) = \sum_{p=1}^{d-1} \zeta^{pj} \left(\zeta^a z_{\sigma(i)}\right)^{-p} = \sum_{p=1}^{d-1} \zeta^{p(j-a)} z_{\sigma(i)}^{-p} = \mathfrak{x}_{\sigma(i), j-a}.$$

Thus, $\mathfrak{G}$ permutes the elements $\mathfrak{x}_{i,j}$ among themselves. If $\sigma(\mathfrak{x}_{i,j}) = \mathfrak{x}_{i,j}$ then $\sigma(z_i) = z_i$, and σ is thus the identity. If we denote $\sigma(\mathfrak{x}_{1,d})$ by $\mathfrak{x}_\sigma$,

$$\sigma_1(\mathfrak{x}_{\sigma_2}) = \mathfrak{x}_{\sigma_1\sigma_2} \quad \forall\ \sigma_1, \sigma_2 \in \mathfrak{G}.$$

The above observation implies that the rd elements $\{\mathfrak{x}_\sigma\}_{\sigma\in\mathfrak{G}}$ are all distinct and the sets $\{\mathfrak{x}_\sigma\}_{\sigma\in\mathfrak{G}}$ and $\{\mathfrak{x}_{i,j}\}_{\substack{i=1,...,r \\ j=1,...,d}}$ are identical. In particular, $\{\mathfrak{x}_\sigma\}_{\sigma\in\mathfrak{G}}$ generate $\mathrm{H}^0(\mathcal{O}_{B_x})$. Thus, there are precisely r relations among the sections $\mathfrak{x}_\sigma$, indexed by $i = 1, ..., r$:

$$\sum_{j=1}^{d} \mathfrak{x}_{\sigma(i,j)} = \sigma_i \left(\sum_{j=1}^{d} \sum_{p=1}^{d-1} \tau^j (z_1)^{-p} \right) = \sigma_i \left(\sum_{p=1}^{d-1} z_1^{-p} \left(\sum_{j=1}^{d} \zeta^{-jp} \right) \right) = 0.$$

An element $\phi \in \mathrm{H}^0(\mathcal{O}_{B_x})$ of the form $\phi = \sum_{\sigma\in\mathfrak{G}} a_\sigma \mathfrak{x}_\sigma$ is in $\mathrm{H}^0_{\mathfrak{m}}(\mathcal{O}_{B_x})$ if and only if

$$\sum_{\sigma\in\mathfrak{G}} b_\sigma a_\sigma \mathfrak{x}_\sigma = 0 \quad \forall\ \mathfrak{b} = \sum_{\sigma\in\mathfrak{G}} b_\sigma \sigma^{-1} \in \mathfrak{m}.$$

This means that $\mathrm{H}^0_{\mathfrak{m}}(\mathcal{B}_x)$ is non-trivial if and only if the sum of the subspace $\mathfrak{m}$ of the vector space $\mathbb{R}_{\mathfrak{G}}$ with the subspace

$$\left\langle \alpha(i) := \sum_{j=1}^{d} \sigma(i,j)^{-1} \;\middle|\; i = 1, ..., r \right\rangle_{\mathbb{R}}$$

is not all of $\mathbb{R}_{\mathfrak{G}}$.

Let us assume that the vector space $\mathfrak{m} \subset \mathbb{R}_{\mathfrak{G}}$ is of codimension ν. The subspace of $\mathbb{R}_{\mathfrak{G}}/\mathfrak{m}$ generated by $\sigma_1^{-1}, ..., \sigma_r^{-1}$ is at most of dimension ν, and we may assume that it is generated by $\sigma_1^{-1}, ..., \sigma_\nu^{-1}$. Thus, there are real numbers $c_{i,k}$, $i = 1, ..., r$, $k = 1, ..., \nu$ such that $\sigma_i^{-1} = \sum_{k=1}^{\nu} c_{i,k}\sigma_k^{-1}$ for $i = 1, ..., r$, where the equality takes place in $\mathbb{R}_{\mathfrak{G}}/\mathfrak{m}$. Thus,

$$\alpha(i) = \sum_{k=1}^{\nu} c_{i,k}\alpha(k) \mod \mathfrak{m} \qquad \forall\ i = 1, ..., m,$$

which implies that $\mathrm{H}^0_{\mathfrak{m}}(\mathcal{B}_x) \neq 0$ if and only if

$$\mathfrak{m} + \langle \alpha(i) \mid i = 1, ..., \nu\rangle_{\mathbb{R}} \neq \mathbb{R}_{\mathfrak{G}}.$$

It thus suffices to show that there is some non-zero $a = (a_1, ..., a_\nu) \in \mathbb{R}^\nu$ such that

$$\sum_{i=1}^{\nu} a_i\alpha(i) = \left(\sum_{j=0}^{d-1} \tau^j\right)\left(\sum_{i=1}^{\nu} a_i\sigma_i^{-1}\right) \in \mathfrak{m}.$$

Let $\rho : \mathbb{R}_{\mathfrak{G}} \to M_k(R)$ denote the irreducible representation $\imath(\mathfrak{m})$ for an integer $k = \nu, \nu/2$ or $\nu/4$ depending on whether $R = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, respectively. Thus, the above condition is satisfied if $\rho\left(\sum_{i=1}^{\nu} a_i\alpha(i)\right) \in \rho(\mathfrak{m})$. The ideal $\mathfrak{m}$ determines a left ideal of $M_k(R)$, which may be specified by a vector $v_{\mathfrak{m}} \in R^k$ in the sense that

$$\rho(\mathfrak{m}) \cap M_k(R) = \{A \in M_k(R) \mid Av_{\mathfrak{m}} = 0\}.$$

Let A be the matrix in $M_k(R)$ which corresponds to $(1 + \tau + ... + \tau^{d-1})/d$. Let $S_1, ..., S_\nu$ denote the matrices in $M_k(R)$ which correspond to $\sigma_1, ..., \sigma_\nu$. If $\mathfrak{m} + \langle \alpha(i)\rangle_i = \mathbb{R}_{\mathfrak{G}}$, for all non-zero $(a_1, ..., a_\nu) \in \mathbb{R}^\nu$, $A\left(\sum_{i=1}^{\nu} a_iS_i^{-1}\right)$ is non-zero in $\mathbb{R}_{\mathfrak{G}}/\mathfrak{m}$. Thus $e_i = S_iv_{\mathfrak{m}}$, for $i = 1, ..., \nu$, generate R^k over $\mathbb{R}$. It thus suffices to show that A has non-trivial kernel. Suppose otherwise that A is non-singular. Since $A^2 = A$, $A = I_{M_k(R)}$ is the identity matrix in $M_k(R)$. Let $T = \rho(\tau)$. Then,

$$\tau\frac{1 + \tau + ... + \tau^{d-1}}{d} = \frac{1 + \tau + ... + \tau^{d-1}}{d} \ \Rightarrow\ TA = A \ \Rightarrow\ T = I_{M_k(R)}.$$

In other words, τ is in the kernel of ρ, which contradicts the assumption that ρ is faithful. The contradiction completes the proof. □

4. Moduli space of multiply covered curves

Suppose that $\pi : \Sigma \to C$ is a branched covering map from a Riemann surface Σ of genus g to a Riemann surface C of genus h and $B \in \mathrm{Div}(\Sigma)$ is the branching divisor. Let $Q(\pi) = \{q_1, ..., q_l\}$ denote the points in the image of B (i.e. the critical values for π), and the injection $o : Q(\pi) \hookrightarrow \mathbb{Z}$ defined by $o(q_i) = i$ denote the ordering of the critical values of π.

Definition 4.1. *With the above notation, the pair*

$$(\pi : \Sigma \to C, o : Q(\pi) \hookrightarrow \mathbb{Z})$$

is called an ordered *branched covering map. Let*

$$(\pi_i : \Sigma_i \to C_i,\ o_i : Q(\pi_i) \hookrightarrow \mathbb{Z}) \quad i = 1, 2$$

be a pair of ordered branched covering maps between Riemann surfaces. We say that (π_1, o_1) *and* (π_2, o_2) *are of the same* topological type *if there are diffeomorphisms* $f : \Sigma_1 \to \Sigma_2$ *and* $g : C_1 \to C_2$ *such that* $g \circ \pi_1 = \pi_2 \circ f$ *and* $o_2 = o_1 \circ g$.

The ordering $o : Q(\pi) \hookrightarrow \mathbb{Z}$ is usually implicit in our discussions, and is dropped from the notation. We will sometimes abuse the notation and denote the topological type of the pair (π, o) by $[\pi]$. When the ordered branched covering map $\pi : \Sigma \to C = \Sigma/\mathfrak{G}$ is determined by the action of a group $\mathfrak{G}$ over the surface Σ, we will further abuse the notation and denote the topological type by $[\mathfrak{G}]$. In this case, we say that $\mathfrak{G}$ is the underlying group of $[\mathfrak{G}]$.

Fix the topological type $[\pi]$ of an ordered branched covering map, and let h denote the genus of the target, while l denotes the number of critical values. Associated with every ordered branched covering map of topological type $[\pi]$, a forgetful map assigns a complex curve C of genus h and an ordered set $q_1, \dots, q_l$ of marked points over it. The points $q_1, ..., q_l$ (as an ordered set of points on C), the conjugacy class of the monodromy map corresponding to π around these points, and the complex structure on the complement of the points $q_1, \dots, q_l$ determines the branched covering map π. Moreover, corresponding to every l-pointed genus h complex curve $(C, q_1, \dots, q_l)$ there are only finitely many ordered branched coverings of topological type $[\pi]$ which correspond to $(C, q_1, \dots, q_l)$ under the aforementioned forgetful map. Moreover, for every finite group $\mathfrak{G}$ and every genus h there are only finitely many topological types $[\mathfrak{G}]$ with target genus h and the underlying group $\mathfrak{G}$.

Fix a topological type $[\mathfrak{G}]$ represented by the ordered branched covering map

$$(\pi_0 : \Sigma_0 \to C_0 = \Sigma_0/\mathfrak{G}, o_0 : Q(\pi_0) \to \mathbb{Z}).$$

Let h denote the target genus and l denote the number of critical values corresponding to $[\mathfrak{G}]$. Let $\Delta^l(M) \subset M^l$ denotes the closed subspace of M^l which consists of the l-tuples $(q_1, ..., q_l)$ with $q_i = q_j$ for some $i \neq j$ and set $M^l_\Delta = M^l - \Delta^l(M)$. Choose $\alpha \in \mathrm{H}_1(M, \mathbb{Z})$ such that $c_1(M)\alpha = 0$ and let $\beta = |\mathfrak{G}|\alpha$. Let $\mathcal{M}_{[\mathfrak{G}]}(M, \beta)$ denote the set of all tuples (J, π) where J is an almost complex structure over M and $\pi : \Sigma \to C$ (together with an ordering) is an ordered branched covering map of topological type $[\mathfrak{G}]$ such that $(C, J) \in \mathcal{M}^\circ_h(M, \alpha)$ is an embedded J-holomorphic curve. The forgetful map discussed earlier gives a finite covering map

$$\mathcal{M}_{[\mathfrak{G}]}(M,\beta) \longrightarrow \left\{(C,J,q_1,...,q_l) \in \mathcal{M}_h^\circ(M,\alpha) \times M_\Delta^l \;\middle|\; q_1,...,q_l \in C\right\}.$$

The assumption that $\imath_C : C \to M$ is an embedding guarantees that

$$\left\{(C,J,q_1,...,q_l) \in \mathcal{M}_h^\circ(M,\alpha) \times M_\Delta^l \;\middle|\; q_1,...,q_l \in C\right\},$$

and consequently $\mathcal{M}_{[\mathfrak{G}]}(M,\beta)$, are $C^{\ell-k}$ Banach manifolds. Moreover, we obtain a Fredholm projection map

$$q_{[\mathfrak{G}]} : \mathcal{M}_{[\mathfrak{G}]}(M,\beta) \to \mathcal{M}_h^\circ(M,\alpha)$$

of Fredholm index $2l$. The proof is similar to the proof of Theorem 3.4.1 and Proposition 3.4.2 from [9]. We denote the composition of $q_{[\mathfrak{G}]}$ with the projection map from $\mathcal{M}_h(M,\alpha)$ to $\mathcal{J}^\ell(M,\omega)$ by $\Pi_{[\mathfrak{G}]}$, which is again a Fredholm map of index $2l$.

For every maximal left ideal $\mathfrak{m} \in I(\mathfrak{G})$ we define the bundle $\overline{\mathcal{F}}^{\mathfrak{m}}$ over $\mathcal{M}^\circ_{[\mathfrak{G}]}(M,\beta)$ by defining the fiber at a point $\phi \in \mathcal{M}^\circ_{[\mathfrak{G}]}(M,\beta)$ (which corresponds to an almost complex structure J on M, a J-holomorphic embedding $\imath_C : C \to M$, and an ordered branched covering map $\pi : \Sigma \to C$ of topological type $[\mathfrak{G}]$) by

$$\overline{\mathcal{F}}^{\mathfrak{m}}_\phi = \Gamma^{k,p}_{\mathfrak{m}}(\Sigma, \pi^* N_C).$$

Let $\mathcal{F}^{\mathfrak{m}}$ denote the complement of the zero section in $\overline{\mathcal{F}}^{\mathfrak{m}}$. We define the bundle $\mathcal{E}^{\mathfrak{m}}$ over $\mathcal{F}^{\mathfrak{m}}$ by defining the fiber over $X \in \mathcal{F}^{\mathfrak{m}}_\phi \subset \mathcal{F}^{\mathfrak{m}}$ by

$$\mathcal{E}^{\mathfrak{m}}_X = \Gamma^{k-1,p}_{\mathfrak{m}}(\Sigma, \Omega^{0,1}_\Sigma \otimes_J \pi^* N_C).$$

The operators $\mathcal{K}^{\mathfrak{m}}_\pi$ defines a section $\mathcal{K}^{\mathfrak{m}} : \mathcal{F}^{\mathfrak{m}} \to \mathcal{E}^{\mathfrak{m}}$ of the bundle $\mathcal{E}^{\mathfrak{m}} \to \mathcal{F}^{\mathfrak{m}}$.

Proposition 4.2. *With notation as above, for every maximal left ideal $\mathfrak{m}$ of $\mathbb{R}_{\mathfrak{G}}$ such that the associated irreducible representation is faithful and the corresponding matrix algebra is either $M_i(\mathbb{R})$ for $i = 1,2,3$ or $M_1(\mathbb{C}) = \mathbb{C}$, the intersection of $\mathcal{K}^{\mathfrak{m}} : \mathcal{F}^{\mathfrak{m}} \to \mathcal{E}^{\mathfrak{m}}$ with the zero section of the vector bundle $\mathcal{E}^{\mathfrak{m}} \to \mathcal{F}^{\mathfrak{m}}$ is transverse.*

Proof. Suppose that ϕ is a point of $\mathcal{M}_{[\mathfrak{G}]}(M,\beta)$ which corresponds to an almost complex structure J, a J-holomorphic embedding $\imath_C : C \to M$ and an ordered branched covering map $\pi : \Sigma \to C$ of type $[\mathfrak{G}]$ and that X is a point of $\mathcal{F}^{\mathfrak{m}}_\phi \subset \mathcal{F}^{\mathfrak{m}}$ such that $\mathcal{K}^{\mathfrak{m}}_\pi(X) = 0$. We need to show that the differential of $\mathcal{K}^{\mathfrak{m}}$, projected over the fiber, defines a surjective operator

$$d\mathcal{K}^{\mathfrak{m}} : T_X\mathcal{F}^{\mathfrak{m}} \simeq T_\phi \mathcal{M}_{[\mathfrak{G}]}(M,\beta) \oplus \Gamma^{k,p}_{\mathfrak{m}}(\pi^* N_C) \to \Gamma^{k-1,p}_{\mathfrak{m}}(\Omega^{0,1}_\Sigma \otimes_J \pi^* N_C).$$

Definition 4.3. *Let J be an almost complex structure over the symplectic manifold (M,ω) and suppose that $\imath_C : C \to M$ is a J-holomorphic embedding of the Riemann surface C in M as above. Define $\mathcal{H}_J(\imath_C)$ to be the set of all sections u of $\mathcal{T}_J = T_J\mathcal{J}^\ell(M,\omega)$ which vanish over C, i.e. such that $u|_{\mathrm{Im}(\imath_C)} = 0$.*

The tangent space $T_\phi\mathcal{M}_{[\mathfrak{G}]}(M,\beta)$ has a subspace which is isomorphic to $\mathcal{H}_J(\imath_C)$ (the tangent vectors in this subspace are trivial in the direction of $\mathcal{X}$). The restriction of $d\mathcal{K}^{\mathfrak{m}}$ to $\mathcal{H}_J(\imath_C) \oplus \Gamma^{k,p}_{\mathfrak{m}}(\Sigma,\pi^*N_C)$ may be easily computed as

$$d\mathcal{K}^{\mathfrak{m}}(u,Y) = \mathcal{K}^{\mathfrak{m}}_\pi(Y) + (\nabla_X u)\circ d\pi \circ j_\Sigma \quad \forall \;\; u \in \mathcal{H}_J(\imath_C),\; Y \in \Gamma^{k,p}_{\mathfrak{m}}(\pi^*N_C).$$

The following lemma then completes the proof of Proposition 4.2. □

Lemma 4.4. *With notation as above, suppose that $0 \neq X \in \mathrm{Ker}(\mathcal{K}_\pi)$ and $\mathfrak{m} = \mathfrak{m}_X$ is a maximal left ideal of $\mathbb{R}_{\mathfrak{G}}$ such that the associated irreducible representation of $\mathfrak{G}$ is faithful and the corresponding matrix algebra is either $M_i(\mathbb{R})$ for $i = 1,2,3$ or $M_1(\mathbb{C}) = \mathbb{C}$. Then the operator*

$$\nabla_X : \mathcal{H}_J(\imath_C) \to \mathrm{Coker}(\mathcal{K}^{\mathfrak{m}}_\pi)$$

defined by $\nabla_X(u) = (\nabla_X u)\circ d\pi \circ j_\Sigma$ is surjective, where $\mathcal{H}_J(\imath_C)$ is defined in Definition 4.3.

Proof. We present the proof in the cases where the associated matrix algebra of $\mathfrak{m}$ is $M_3(\mathbb{R})$ or $M_1(\mathbb{C})$. The other two cases are in fact easier (and similar). Let us denote the linear operator $\mathcal{K}^{\mathfrak{m}}_\pi$ by L, and the normal bundle N_C by N for simplicity. Suppose that ∇_X is not surjective. Identify the cokernel of L with the kernel of its adjoint L^* (this is possible according to Lemma 2.4). Choose the non-zero section δ in $\mathrm{Ker}(L^*)$ so that it is orthogonal to the image of ∇_X. Since $L^*(\delta) = 0$, δ is not identically zero on any open subset of Σ. Choose an open ball $U \subset C^\circ = C - \pi(B)$. Let $p \in U$ and $\{p_\sigma\}_{\sigma\in\mathfrak{G}}$ be the set of pre-images, which are indexed so that $\tau(p_\sigma) = p_{\tau\sigma}$ for all $\sigma,\tau \in \mathfrak{G}$. We may assume that $\pi^{-1}(U) = \cup_{\sigma\in\mathfrak{G}} U_\sigma$, with U_σ a ball around p_σ such that the balls U_σ are disjoint and $\pi|_{U_\sigma} : U_\sigma \to U$ is a diffeomorphism. Denote the inverse of $\pi|_{U_\sigma}$ by $\jmath_\sigma$. Let z denote the local coordinate over U and z_σ denote the corresponding local coordinate over U_σ. After shrinking U, we may assume that X is nowhere zero in U_e (where e denotes the identity element of $\mathfrak{G}$). Let $\mathrm{Hom}^{0,1}_U(N_C,N_C)$ denote the space of $C^{\ell-1}$ sections $A \in \Gamma(U_e,(\pi^*N_C)^*\otimes\pi^*N_C)$ with $JA + AJ = 0$. If U is small enough, $\mathrm{Hom}^{0,1}_U(N_C,N_C)$ is trivialized as $U \times \mathbb{C}^4$. Under this identification, the restriction of the elements $u \in \mathcal{H}_J(\imath_C)$ to U gives a surjection from $\mathcal{H}_J(\imath_C)$ to $U \times \mathbb{C}^4$. Thus, the induced map

$$\nabla_X : \mathcal{H}_J(\imath_C) \longrightarrow \mathrm{Hom}^{0,1}_U(N_C,N_C),$$

which sends u to $\nabla_{X\circ \jmath_e} u$, is surjective for every nowhere zero section X.

Fix a bump function λ with support inside U and lift it to Σ, keeping the same name λ for it. For every $u \in \mathcal{H}_J(\imath_C)$ the section $\lambda.u$ is then supported on $\pi^{-1}(U)$. Since $u|_{\mathrm{Im}(\imath_C)} = 0$, $\nabla_X(\lambda u) = \lambda \nabla_X u$ and

$$0 = \langle \nabla_X(\lambda u), \delta \rangle = \frac{1}{2\sqrt{-1}} \sum_{\sigma \in \mathfrak{G}} \int_{U_\sigma} \lambda(z_\sigma) \langle \nabla_X(u), \delta \rangle_{z_\sigma} dz_\sigma \wedge d\overline{z}_\sigma$$

The complex anti-linear 1-form δ can locally be written as $\delta = Zds - JZdt$ where $Z \in \Gamma^{k,q}(\pi^{-1}(U), \pi^* N)$ and $z = s + t\sqrt{-1}$. Thus, for every $p \in C^\circ$ and every linear transformation $B : N_p \to N_p$, by letting λ converge to the delta function supported above p and choosing u so that $((\nabla_X u) \circ d\pi \circ j)\,(p_e)$ is equal to $B(X(p_e))ds - JB(X(p_e))dt$

$$F_B(p) := \sum_{\sigma \in \mathfrak{G}} \langle BX, Z \rangle_{p_\sigma} = 0. \tag{3}$$

If U is small, N may be trivialized over U as $U \times \mathbb{C}^2$. In this trivialization $X = (X_1, X_2)$ and $Z = (Z_1, Z_2)$. Varying the matrix B in (3) we conclude that for all $i, j \in \{1, 2\}$ and all $p \in U$

$$F_{ij}(p) := \sum_{\sigma \in \mathfrak{G}} (X_i Z_j)(p_\sigma) = 0, \quad \overline{F}_{ij}(p) := \sum_{\sigma \in \mathfrak{G}} (X_i \overline{Z}_j)(p_\sigma) = 0. \tag{4}$$

Suppose that the matrix algebra associated with $\mathfrak{m}$ is $M_1(\mathbb{C}) = \mathbb{C}$. Since the corresponding representation is faithful

$$\mathfrak{G} = \mathbb{Z}/n\mathbb{Z} = \langle \sigma \rangle, \quad \text{with } \rho(\sigma) = \zeta \in S^1 - \{\pm 1\} \subset \mathbb{C}^*,$$

where ζ is a primitive n-th root of unity. There is thus a second section $Y \in \mathrm{Ker}(L)$ such that

$$\begin{pmatrix} (\sigma^m)^* X \\ (\sigma^m)^* Y \end{pmatrix} = \begin{pmatrix} \mathrm{Re}(\zeta^m) & \mathrm{Im}(\zeta^m) \\ -\mathrm{Im}(\zeta^m) & \mathrm{Re}(\zeta^m) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \quad \forall \;\; m \in \mathbb{Z}. \tag{5}$$

Similarly, there is a section $\epsilon \in \mathrm{Ker}(L^*)$ such that

$$\begin{gathered} \epsilon|_{\pi^{-1}(U)} = Wds - JWdt, \quad W \in \Gamma^{k,q}(\pi^{-1}(U), \pi^* N), \\ \begin{pmatrix} (\sigma^m)^* Z \\ (\sigma^m)^* W \end{pmatrix} = \begin{pmatrix} \mathrm{Re}(\zeta^m) & \mathrm{Im}(\zeta^m) \\ -\mathrm{Im}(\zeta^m) & \mathrm{Re}(\zeta^m) \end{pmatrix} \begin{pmatrix} Z \\ W \end{pmatrix} \quad \forall \;\; m \in \mathbb{Z}. \end{gathered} \tag{6}$$

Rewriting (4) using (5) and (6) we obtain

$$\begin{aligned} 0 &= \left(\sum_{m=0}^{n-1} \mathrm{Re}(\zeta^m)^2 \right) (X_i Z_j)_{p_e} + \left(\sum_{m=0}^{n-1} \mathrm{Im}(\zeta^m)^2 \right) (Y_i W_j)_{p_e} \\ &\quad + \left(\sum_{m=0}^{n-1} \mathrm{Im}(\zeta^m) \mathrm{Re}(\zeta^m) \right) (X_i W_j + Y_i Z_j)_{p_e} \\ &= \left(\sum_{m=0}^{n-1} \mathrm{Re}(\zeta^m)^2 \right) (X_i Z_j + Y_i W_j)_{p_e}, \qquad\qquad \forall \;\; p \in U. \end{aligned}$$

The last equality follows from $\sum_{m=0}^{n-1}\zeta^{2m}=0$ since $n\neq 2$ and

$$\zeta^{2m}=\left(\mathrm{Re}(\zeta^m)^2+\mathrm{Im}(\zeta^m)^2\right)-2\sqrt{-1}\left(\mathrm{Re}(\zeta^m)\mathrm{Im}(\zeta^m)\right).$$

Since $\sum_{m=0}^{n-1}\mathrm{Re}(\zeta^m)^2$ is a positive real number, we conclude that $X_iZ_j+Y_iW_j$ is identically zero on U_e. Similarly, from the second equality in (4) we conclude that

$$\left(X_i\overline{Z_j}+Y_i\overline{W_j}\right)\big|_{U_e}\cong 0.$$

These two relations imply that

$$(7)\qquad \left(\mathrm{Re}(Z_j)X+\mathrm{Re}(W_j)Y\right)(z_e)=0,\quad \forall\ z_e\in U_e.$$

Since X and Y are both in the kernel of L and the zeros of both of them are isolated (7) implies that $\overline{\partial}_w\left(\frac{\mathrm{Re}(W_j)}{\mathrm{Re}(Z_j)}\right)=0$ on U_e and hence $\mathrm{Re}(W_j)/\mathrm{Re}(Z_j)$ is constant on U_e (as a real valued holomorphic function). This means that Y is a constant multiple of X over U_e, and hence on all of Σ, unless $\mathrm{Re}(Z_j)=\mathrm{Re}(W_j)=0$. Since the former can not happen (and the same thing may be proved for $\mathrm{Im}(Z_j)$ and $\mathrm{Im}(W_j)$) $\delta=0$ on U_e, which is also a contradiction. This completes the proof when the matrix algebra is $\mathbb{C}$.

Suppose the matrix algebra associated with $\mathfrak{m}$ is $M_3(\mathbb{R})$. Then associated with every $\sigma\in\mathfrak{G}$ is an orthogonal matrix $A_\sigma=\rho(\sigma)\in O_3(\mathbb{R})\subset M_3(\mathbb{R})$. With $X\in\mathrm{Ker}(L)$ and $Z\in\Gamma_{\mathfrak{m}}^{k,p}(\pi^{-1}(U),\pi^*N)$ as above, define

$$\begin{aligned}\alpha(X)&:=\sum_{\sigma\in\mathfrak{G}}\sigma^*X\otimes A_\sigma\in\mathrm{Ker}(\mathcal{K}_\pi)\otimes M_3(\mathbb{R}),\\ \alpha(Z)&:=\sum_{\sigma\in\mathfrak{G}}\sigma^*Z\otimes A_\sigma\in\Gamma_{\mathfrak{m}}^{k,p}(\pi^{-1}(U),\pi^*N)\otimes M_3(\mathbb{R}).\end{aligned}$$

For every $\tau\in\mathfrak{G}$ and every $\mathfrak{a}=\sum_{\sigma\in\mathfrak{G}}a_\sigma\sigma^{-1}\in\mathbb{R}_{\mathfrak{G}}$

$$(8)\qquad\begin{aligned}\tau^*\alpha(X)&=\sum_{\sigma\in\mathfrak{G}}(\tau^*\sigma^*X)\otimes A_\sigma=\alpha(X)A_\tau^t,\\ \alpha(\mathfrak{a}.X)&=\left(\sum_{\sigma\in\mathfrak{G}}a_\sigma {A_\sigma}^t\right)\left(\sum_{\tau\in\mathfrak{G}}\tau^*X\otimes A_\tau\right)=\rho(\mathfrak{a})\alpha(X),\end{aligned}$$

where the multiplication of A_τ^t on the right-hand-side of the first equation and the multiplication of $\rho(\mathfrak{a})$ on the right-hand-side of the second equation in (8) are done in $M_3(\mathbb{R})$. Assuming $\mathfrak{m}$ corresponds to the matrices with zero in the first column, the second equation in (8) implies that $\alpha(X)$ may be written as

$$\alpha(X)=X^1\otimes\alpha_1+X^2\otimes\alpha_2+X^3\otimes\alpha_3,\quad\text{where}$$

$$\alpha_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\ \alpha_2=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\ \alpha_3=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},$$

$$X^i\in\langle\sigma^*X\mid\sigma\in\mathfrak{G}\rangle_{\mathbb{R}}\subset\mathrm{Ker}(\mathcal{K}_\pi)\quad\text{for}\quad i=1,2,3.$$

Similarly, we may write

$$\begin{aligned}&\alpha(Z) = Z^1 \otimes \alpha_1 + Z^2 \otimes \alpha_2 + Z^3 \otimes \alpha_3, \quad \text{where}\\ &Z^i \in \langle \sigma^* Z \mid \sigma \in \mathfrak{G}\rangle_{\mathbb{R}} \subset \Gamma^{k,q}(\pi^{-1}(U), \pi^* N), \quad \text{for} \quad i = 1,2,3.\end{aligned}$$

In fact, if e_i, $i = 1,2,3$ are the standard basis vectors of $\mathbb{R}^3$ and $v^i_\sigma = e_i A_\sigma e_i^t \in \mathbb{R}$ for $\sigma \in \mathfrak{G}$,

$$X^i = \sum_{\sigma\in\mathfrak{G}} v^i_\sigma(\sigma^* X), \quad Z^i = \sum_{\sigma\in\mathfrak{G}} v^i_\sigma(\sigma^* Z) \quad i = 1,2,3.$$

The first equation in (8) implies that

$$\alpha(X)\rho(\mathfrak{a}) = (\mathfrak{a}X^1) \otimes \alpha_1 + (\mathfrak{a}X^2) \otimes \alpha_2 + (\mathfrak{a}X^3) \otimes \alpha_3.$$

If $\mathfrak{a}$ is an element such that the i-th column of $\rho(\mathfrak{a})$ is zero, the i-th column of $\alpha(X)\rho(\mathfrak{a})$ becomes zero, implying that $\mathfrak{a}X^i = 0$. Thus, $\mathfrak{m}_{X^i}$ contains the elements $\mathfrak{a} \in \mathbb{R}_{\mathfrak{G}}$ such that the i-th column of $\rho(\mathfrak{a})$ is zero.

Setting $\beta(X) = (X^1, X^2, X^3)^t$ and $\beta(Z) = (Z^1, Z^2, Z^3)^t$

$$\sigma^*\beta(X) = A_\sigma\beta(X), \quad \sigma^*\beta(Z) = A_\sigma\beta(Z) \quad \forall\, \sigma \in \mathfrak{G}.$$

If $w\beta(X) = 0$ for some $w \in \mathbb{R}^3$,

$$0 = \sigma^*\left(w\beta(X)\right) = (wA_\sigma)\beta(X) \quad \forall\, \sigma \in \mathfrak{G} \quad \Rightarrow \quad \beta(X) = 0,$$

which implies $X = 0$. Thus X^1, X^2, X^3 are linearly independent over $\mathbb{R}$ and generate the three-dimensional subspace $\langle \sigma^* X \mid \sigma \in \mathfrak{G}\rangle_{\mathbb{R}}$ of $\mathrm{Ker}(\mathcal{K}_\pi)$. Since $\mathfrak{m}_{X^1} = \mathfrak{m}_X$, we may assume that $X^1 = X$, i.e. $X = e_1\beta(X)$. Similarly, Z^1, Z^2 and Z^3 are independent and may assume that $Z = e_1\beta(Z)$.

We let Z^j_i be the i-th component of Z^j in the trivialization of N over $\pi^{-1}(U)$, and

$$\begin{aligned}\beta(\mathrm{Re}(Z^j)) &:= (\mathrm{Re}(Z^j_1), \mathrm{Re}(Z^j_2), \mathrm{Re}(Z^j_3)),\\ \beta(\mathrm{Im}(Z^j)) &:= (\mathrm{Im}(Z^j_1), \mathrm{Im}(Z^j_2), \mathrm{Im}(Z^j_3)).\end{aligned}$$

Equation (4) reads as

$$\begin{aligned}0 &= \sum_{\sigma\in\mathfrak{G}} \mathrm{Re}(Z_i)(p_\sigma) X(p_\sigma)\\ &= \beta(X)^t(p_e)\left(\sum_{\sigma\in\mathfrak{G}} A_\sigma^t e_1^t e_1 A_\sigma\right)\beta(\mathrm{Re}(Z_i))(p_e)\\ &=: \beta(X)^t(p_e) B \beta(\mathrm{Re}(Z_i))(p_e) \quad \text{for} \quad i = 1,2,\\ 0 &= \beta(X)^t(p_e) B \beta(\mathrm{Im}(Z_i))(p_e) \quad \text{for} \quad i = 1,2.\end{aligned} \tag{9}$$

For the 3×3 matrix B and every $\tau \in \mathfrak{G}$,

$$A_\tau^{-1} B A_\tau = A_\tau^t B A_\tau = \sum_{\sigma\in\mathfrak{G}} A^t_{\sigma\tau} e_1^t e_1 A_{\sigma\tau} = B \;\Rightarrow\; BA_\tau = A_\tau B, \quad \forall\, \tau \in \mathfrak{G}.$$

Since $\{A_\tau\}_{\tau\in\mathfrak{G}}$ generate the algebra $M_3(\mathbb{R})$ the matrix B is a multiple of the identity. On the other hand,

$$\begin{aligned} \operatorname{tr}(B) &= \sum_{\sigma\in\mathfrak{G}} \operatorname{tr}(A_\sigma^t e_1^t e_1 A_\sigma) = \sum_{\sigma\in\mathfrak{G}} \operatorname{tr}(e_1 A_\sigma A_\sigma^t e_1^t) = |\mathfrak{G}| \neq 0 \\ \Rightarrow \quad B &= \frac{|\mathfrak{G}|}{3} I_{3\times 3} \neq 0. \end{aligned} \tag{10}$$

By (9) and (10) the matrices $\beta(X)\beta(\mathrm{Re}(Z_i))^t$ and $\beta(X)\beta(\mathrm{Im}(Z_i))^t$ are identically zero over $\pi^{-1}(U)$ for $i = 1, 2$.

The three vectors $X^1(p_e), X^2(p_e)$ and $X^3(p_e)$ thus linearly depend on each other (over the point p_e). If $r(p_e)$ is the rank of the vector space spanned by these three vectors, $r(p_e) \in \{1, 2\}$ for a generic choice of $p_e \in U_e$. If $r(p_e) = 2$, then for points in an open neighbourhood of p_e the same is true. For every point z_e in this open neighbourhood, the four vectors

$$\beta(\mathrm{Re}(Z_i))(z_e), \beta(\mathrm{Im}(Z_i))(z_e) \in \mathbb{R}^3, \quad i = 1, 2$$

are thus multiples of one-another, and Z^1 is thus a multiple of Z^2 over this open neighbourhood, i.e. $Z^1(z_e) = \lambda(z_e)Z^2(z_e)$ for some real valued function λ over a neighbourhood of p_e. Since Z^1 and Z^2 satisfy perturbed Cauchy-Riemann equations, this means that $\partial_{z_e}(\lambda) = 0$, implying that λ is constant (since it is real valued). Repeating this argument for the other pairs we find that there is $w \in \mathbb{R}^3 - \{0\}$ such that $w\beta(X) = 0$ over a small open neighbourhood on Σ, and hence on all of Σ. This contradicts the linear independence of X^1, X^2 and X^3 over $\mathbb{R}$ which was proved above.

From this contradiction $r(q) = 1$ for a generic point q on Σ. If $r \leq 1$ in a neighbourhood of q, it is implied that $X^2(z) = \lambda(z)X^1(z)$ for z near q and for a real valued function λ. Since X^1 and X^2 satisfy perturbed Cauchy-Riemann equations, this means that $\overline{\partial}_z(\lambda) = 0$, implying that λ is constant near q. The equation $X^2(z) = \lambda X^1(z)$ thus extends to all of Σ, and the same argument as above may be repeated for $w = (-\lambda, 1, 0)$. The resulting contradiction then completes the proof. □

Let $\alpha \in \mathrm{H}_2(M, \mathbb{Z})$ be a class which satisfies $c_1(M)\alpha = 0$ and let $g \geq h \geq 0$ be integers. Let the ordered branched covering map $\pi : \Sigma \to \Sigma/\mathfrak{G} = C$ denote a representative of a topological type $[\mathfrak{G}]$ where Σ is a Riemann surface of genus g and with quotient a surface C of genus h. Let $\beta = |\mathfrak{G}|\alpha \in \mathrm{H}_2(M, \mathbb{Z})$ and fix $\mathfrak{m} \in I(\mathfrak{G})$.

Definition 4.5. *With the above notation fixed, the* moduli space of unwanted sections *corresponding to* $[\mathfrak{G}], \mathfrak{m}$ *and* β*, denoted by* $\mathcal{N}^{\mathfrak{m}}_{[\mathfrak{G}]}(\beta)$*, is the zero locus of the section* $\mathcal{K}^{\mathfrak{m}} : \mathcal{F}^{\mathfrak{m}} \to \mathcal{E}^{\mathfrak{m}}$*.*

Theorem 4.6. *For every* J *in a subset* $\mathcal{J}_4^\ell(M, \omega) \subset \mathcal{J}_{reg}^\ell(M, \omega)$ *which is of the second category as a subset of* $\mathcal{J}^\ell(M, \omega)$*, every* $h \geq 0$*, and every* *homology class $\alpha \in \mathrm{H}_2(M,\mathbb{Z})$ which satisfies $c_1(M)\alpha = 0$, all the curves in the moduli space $\mathcal{M}_h(\alpha, J)$ are 4-rigid embeddings.*

Proof. Let $\alpha \in \mathrm{H}_2(M,\mathbb{Z})$ be a class which satisfies $c_1(M)\alpha = 0$ and $g \geq h \geq 0$ be integers. Let $\pi : \Sigma \to \Sigma/\mathfrak{G} = C$ denote a representative of a topological type $[\mathfrak{G}]$ where Σ is a Riemann surface of genus g and with quotient a surface C is of genus h. Furthermore, assume that the underlying group $\mathfrak{G}$ of $[\mathfrak{G}]$ is a subgroup of S_4. Let $\beta = |\mathfrak{G}|\alpha \in \mathrm{H}_2(M,\mathbb{Z})$. Fix $\mathfrak{m} \in I(\mathfrak{G})$ and assume that the corresponding representation $\imath(\mathfrak{m}) \in \widehat{I}(\mathfrak{G})$ is faithful. The moduli spaces $\mathcal{N}^{\mathfrak{m}}_{[\mathfrak{G}]}(\beta)$ of unwanted sections are fibered over $\mathcal{J}^{\ell}(M,\omega)$. By Proposition 4.2, each $\mathcal{N}^{\mathfrak{m}}_{[\mathfrak{G}]}(\beta)$ is a Banach manifold. The set of regular values $\mathcal{J}^{\ell}_{reg}([\mathfrak{G}],\mathfrak{m},\alpha)$ of the Fredholm map

$$\Pi^{\mathfrak{m}}_{[\mathfrak{G}]}(\beta) : \mathcal{N}^{\mathfrak{m}}_{[\mathfrak{G}]}(\beta) \to \mathcal{J}^{\ell}(M,\omega)$$

is of the second category by Sard-Smale theorem [13]. Set

$$\mathcal{J}^{\ell}_4(M,\omega) = \mathcal{J}^{\ell}_{reg}(M,\omega) \bigcap \left(\bigcap_{[\mathfrak{G}],\mathfrak{m}} \quad \bigcap_{\substack{\alpha\in \mathrm{H}_2(M,\mathbb{Z}) \\ c_1(M).\alpha=0,\ |\alpha|\leq 4}} \mathcal{J}^{\ell}_{reg}([\mathfrak{G}],\mathfrak{m},\alpha) \right).$$

Here, the first intersection inside the parentheses is over all the topological types $[\mathfrak{G}]$ with the underlying group $\mathfrak{G}$ a subgroup of S_4 and all $\mathfrak{m} \in I(\mathfrak{G})$ which correspond to faithful representations.

For every almost complex structure $J \in \mathcal{J}^{\ell}_4(M,\omega)$ and every homology class $\alpha \in \mathrm{H}_2(M,\mathbb{Z})$ satisfying $c_1(M)\alpha = 0$, $\mathcal{M}_h(\alpha, J)$ is a zero-dimensional manifold since $\mathcal{J}_4(M,\omega) \subset \mathcal{J}^{\ell}_{reg}(M,\omega)$. If an embedded curve $C \in \mathcal{M}_h(\alpha, J)$ is not 4-rigid, we may construct a branched covering map $\pi : \Sigma \to \Sigma/\mathfrak{G} = C$ for a subgroup $\mathfrak{G} \subset S_4$ so that the kernel of $\mathcal{K}^{\mathfrak{m}}_{\pi}$ is non-trivial for some $\mathfrak{m} \in I(\mathfrak{G})$ with $\imath(\mathfrak{m})$ faithful. Together with an ordering of the critical values of the branched covering map π, the action of $\mathfrak{G}$ on Σ determined a topological type $[\mathfrak{G}]$. The aforementioned non-trivial kernel gives a non-empty subset of $\mathcal{N}^{\mathfrak{m}}_{[\mathfrak{G}]}(\beta, J)$. The moduli space $\mathcal{N}^{\mathfrak{m}}_{[\mathfrak{G}]}(\beta, J)$ is a smooth manifold of dimension $\mathrm{Index}\left(\Pi^{\mathfrak{m}}_{[\mathfrak{G}]}(\beta)\right)$, by regularity of the almost complex structure J. This later index is equal to the sum of the indices of the projection map from $\mathcal{M}_{[\mathfrak{G}]}(M,\beta)$ to $\mathcal{J}^{\ell}(M,\omega)$ and the projection map from $\mathcal{N}^{\mathfrak{m}}_{[\mathfrak{G}]}(\beta)$ to $\mathcal{M}_{[\mathfrak{G}]}(M,\beta)$. The latter projection map has the same index as the operator $\mathcal{K}^{\mathfrak{m}}_{\pi}$.

By Proposition 3.2, the index of $\Pi^{\mathfrak{m}}_{[\mathfrak{G}]}(\beta)$, which equals $2l + \mathrm{Index}(\mathcal{K}^{\mathfrak{m}}_{\pi})$, is at most 0. If $\mathcal{N}^{\mathfrak{m}}_{[\mathfrak{G}]}(\beta, J) = \Pi^{\mathfrak{m}}_{[\mathfrak{G}]}(\beta)^{-1}(J)$ is non-empty and $Y \in \mathcal{N}^{\mathfrak{m}}_{[\mathfrak{G}]}(\beta, J)$, then for every $0 \neq \lambda \in \mathbb{R}$, λY is also in $\mathcal{N}^{\mathfrak{m}}_{[\mathfrak{G}]}(\beta, J)$. This contradicts the fact that this moduli space is at most zero dimensional. This contradiction proves that every curve $C \in \mathcal{M}_h(\alpha, J)$ is 4-rigid. □

5. Smooth complex structures

Fix an identification of $\mathrm{H}_2(M, \mathbb{Q})$ with $\mathbb{Q}^{b_2(M)}$ once for all. For a homology class $\beta \in \mathrm{H}_2(M, \mathbb{Z})$, let $\|\beta\|$ denote its Euclidean norm under this identification. Let $|\beta|$ denote the divisibility of β as before.

For every pair of genera $h \leq g$ and every degree $d > 0$ such that $2g - 2 > d(2h-2)$ fix an exhaustion $\{U_i\}_{i=1}^{\infty}$ of

$$\mathcal{M}_g(h, d) = \left\{\pi : \Sigma \to C \;\middle|\; \Sigma \in \mathcal{M}_g,\ C \in \mathcal{M}_h,\ \deg(\pi) = d\right\}$$

by the open subsets U_i so that $\overline{U_i} \subset U_{i+1}$ is compact and $\mathcal{M}_g(h, d) = \bigcup_{i=1}^{\infty} U_i$. For every smooth Riemann surface $(C, j_C) \in \mathcal{M}_h$ this gives an exhaustion of $\mathcal{M}_g(C, d[C])$ by the open sets $U_i(C, g, d)$ such that the closure $\overline{U_i(C, g, d)} \subset U_{i+1}(C, g, d)$ is compact and

$$\mathcal{M}_g(C, d[C]) = \bigcup_{i=1}^{\infty} U_i(C, g, d).$$

When $2g - 2 = d(2h - 2)$ the moduli space $M_g(C, d[C])$ if finite for every $C \in \mathcal{M}_h$ and we set $U_i(C, g, d) = M_g(C, d[C])$ for every positive integer i.

Let $\mathcal{J}^{\ell}(M, \omega; n, K, N)$ denote the set of all $J \in \mathcal{J}^{\ell}(M, \omega)$ such that every somewhere injective J-holomorphic map $f : C \to M$ with $c_1(M) f_*[C] = 0$ satisfying

- C does not have any components which are collapsed under f,
- the total genus h of C satisfies $h \leq K$,
- $|f_*[C]| \leq n$, $\|f_*[C]\| < K$ and $K^{-1} < \|df(x)\| < K$ for all $x \in C$,

is a 1-rigid embedding, and for every $h \leq g \leq h + K$, every $1 < d \leq n$ and every $\pi \in \overline{U_N(C, g, d)}$, $\mathrm{Ker}(\mathcal{K}_\pi)$ is trivial.

Thus, $\mathcal{J}_4^{\ell}(M, \omega)$ is precisely the intersection

$$\mathcal{J}_4^{\ell}(M, \omega) = \bigcap_{K, N \in \mathbb{Z}^+} \mathcal{J}^{\ell}(M, \omega; 4, K, N),$$

which is of the second category in $\mathcal{J}^{\ell}(M, \omega)$ by Theorem 4.6 for $\ell \neq \infty$.

Lemma 5.1. *For every $n, K, N \in \mathbb{Z}^+$, and every $\ell \in \{1, 2, 3, ..., \infty\}$*

$$\mathcal{J}^{\ell}(M, \omega; n, K, N) \subset \mathcal{J}^{\ell}(M, \omega)$$

is open.

Proof. Suppose otherwise that $J \in \mathcal{J}^{\ell}(M, \omega; n, K, N)$ and

$$J_i \in \mathcal{J}^{\ell}(M, \omega) - \mathcal{J}^{\ell}(M, \omega; n, K, N)$$

is sequence of almost complex structures converging to J. For every $i = 1, 2, ...$ there is thus a somewhere-injective J_i-holomorphic map $f_i : C_i \to M$ satisfying

- C_i does not have any components which are collapsed under f,
- the total genus h_i of C_i satisfies $h_i \leq K$,
- $|f_{i*}[C_i]| \leq n$, $\|f_{i*}[C_i]\| < K$, and $K^{-1} < \|df_i(x)\| < K$ for all $x \in C_i$,

which is either not a 1-rigid embedding, or there is some $\pi_i \in \overline{U_N(C_i, d_i, g_i)}$ with $\mathrm{Ker}(\mathcal{K}_{\pi_i}) \neq 0$, $g_i < h_i + K$ and $d_i < n$.

After passing to a subsequence, we may assume that all h_i are equal to the same value h, and all the homology classes $f_{i*}[C_i]$ are the same homology class $\alpha \in \mathrm{H}_2(M, \mathbb{Z})$. Gromov compactness theorem implies that a subsequence of f_i converges to a J-holomorphic map $f : C \to M$, which may be decomposed as $f = f' \circ \pi'$ with $\pi' : C \to C'$ a holomorphic map and $f' : C' \to M$ a somewhere injective J-holomorphic map which does not collapse any components. Since $J \in \mathcal{J}^\ell(M, \omega; n, K, N)$, C' is smooth of some genus h' and f' is an embedding representing a homology class $\alpha' \in \mathrm{H}_2(M, \mathbb{Z})$. Let $d' = \deg(\pi')$. The point-wise energy bound $K^{-1} < \|df_i(x)\| < K$ implies that π' is a covering map and $\mathcal{M}_h(C', d'[C'])$ is finite. Furthermore,

$$d' \leq |d'\alpha'| = |\alpha| \leq n.$$

The regularity of J implies that for i sufficiently large there are J_i-holomorphic curves C_i' representing α' which converge to C'. One can also choose the covering map $\pi_i' : \tilde{C}_i \to C_i' \subset M$ from the finite set $\mathcal{M}_h(C_i', d'[C_i'])$ so that π_i' converges to π' as i goes to infinity. Since $f_i : C_i \to M$ is close to $\pi_i' : \tilde{C}_i \to M$ and both are J_i-holomorphic, Theorem 3.3.4 and Proposition 3.3.5 from [8] imply that for sufficiently large i either C_i corresponds to a nontrivial section in $\mathrm{Ker}(\mathcal{K}_{\pi_i'})$ or the image of f_i is in C_i'. From the n-rigidity of C' the former can not happen for large values of i. The latter implies that the degree d' of π' is one and $f : C \to M$ is an embedding. For sufficiently large i, f_i is thus an embedding and since the dimension of $\mathrm{Ker}(\mathcal{K}_f)$ is upper semi-continuous, for i sufficiently large the kernel of $\mathcal{K}_{f_i}$ is trivial as well. Thus, $\{f_i\}_i$ are 1-rigid embeddings (for i is sufficiently large) which converge to the J-holomorphic embedding $f : C \to M$.

For the above subsequence of f_i, there are thus $\pi_i \in \overline{U_N(C_i, d_i, g_i)}$ with $\mathrm{Ker}(\mathcal{K}_{\pi_i}) \neq 0$, $g_i < h + K$ and $d_i \leq n$. By passing to a further subsequence, we may assume that $d_i = d$ and $g_i = g$ for every i. The sequence $\pi_i \in \overline{U_N(C_i, d_i, g_i)}$ thus converges to some $\pi \in \overline{U_N(C, d, g)}$. Since $J \in \mathcal{J}^\ell(M, \omega; n, K, N)$, $\mathrm{Ker}(\mathcal{K}_\pi) = 0$, and since the dimension of $\mathrm{Ker}(\mathcal{K}_\pi)$ is upper semi-continuous, for i sufficiently large the kernel of $\mathcal{K}_{\pi_i}$ is trivial as well. This contradiction completes the proof. $\square$

Lemma 5.2. *The subspace $\mathcal{J}^\infty(M,\omega;4,K,N) \subset \mathcal{J}^\infty(M,\omega)$ is dense with respect to C^ℓ topology for every $\ell > 1$.*

Proof. For $\ell \neq \infty$ the lemma follows from Theorem 4.6. Since

$$\mathcal{J}^\infty(M,\omega;4,K,N) = \mathcal{J}^\infty(M,\omega) \cap \mathcal{J}^\ell(M,\omega;4,K,N),$$

and $\mathcal{J}^\ell(M,\omega;4,K,N)$ is an open dense subset of $\mathcal{J}^\ell(M,\omega)$ the claim of the lemma is implied for smooth almost complex structures as well. □

The above two lemmas imply the following theorem.

Theorem 5.3. *The subset $\mathcal{J}_4^\infty(M,\omega) \subset \mathcal{J}^\infty(M,\omega)$ is of the second category.*

Proof. Since Lemma 5.2 is true for all ℓ, $\mathcal{J}^\infty(M,\omega;4,K,N)$ is dense in $\mathcal{J}^\infty(M,\omega)$ in C^∞ topology and

$$\mathcal{J}_4^\infty(M,\omega) = \bigcap_{K,N\in\mathbb{Z}^{>0}} \mathcal{J}^\infty(M,\omega;4,K,N)$$

is the intersection of a countable collection of open dense subsets of $\mathcal{J}^\infty(M,\omega)$.
□

References

[1] Bryan, J., Pandharipande, R., BPS-states of curves in Calabi-Yau 3-folds, *Geom. Topol.* 5 (2001), 287–318.

[2] Bryan, J., Pandharipande, R., On rigidity of stable maps to Calabi-Yau threefolds, *The interaction of finite-type and Gromov-Witten invariants (BIRS 2003)*, 97–104, Geom. Topol. Monogr., 8, Geom. Topol. Publ., Coventry, 2006.

[3] Cox, D., Katz, S., *Mirror Symmetry and Algebraic Geometry*, Math. Surveys and Mono. , vol 68, AMS, Providence RI, 1999.

[4] Fulton, W., Harris, J., *Representation theory, a first course*, Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991.

[5] Gopakumar, R., Vafa, C., M theory and topological strings II, (1998), arxiv:hep-th/9812127.

[6] Griffiths, P., Harris, J., *Pripnciples of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994.

[7] Ionel, E. N., Parker, T., Relative Gromov invariants, *Ann. of Math.* 157 (2003), 45–96.

[8] McDuff, D., Salamon, D., *J-holomorphic curves and quantum cohomology*, University Lecture Series, 6. American Mathematical Society, Providence, RI, 1994.

[9] McDuff, D., Salamon, D., *J-holomorphic curves and symplectic topology*, American Mathematical Society Colloquium Publications, 52. American Mathematical Society, Providence, RI, 2004.

[10] Miranda, R., *Algebric curves and Riemann surfaces*, Graduate Studies in Mathematics, 5. American Mathematical Society, Providence, RI, 1995.

[11] Pandharipande, R. , Hodge integrals and degenerate contributions, *Comm. Math. Phys.*, 208 (1999), no. 2, 489–506.

[12] Rose, H. E., *A course on finite groups* Universitext, Springer-Verlag London, Ltd., London, 2009.

[13] Smale, S., An infinite dimensional version of Sard's theorem, *Amer. J. Math.*, 87 (1965), 861–866.

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P. O. Box 19395-5746, Tehran, Iran

E-mail address: `eaman@ipm.ir`