ON THE KAPPA RING OF $\overline{\mathcal{M}}_{g,n}$

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ABSTRACT. Let $\kappa_e(\overline{\mathcal{M}}_{g,n})$ denote the kappa ring of $\overline{\mathcal{M}}_{g,n}$ in dimension e (equivalently, in degree $d = 3g-3+n-e$). For $g, e \geq 0$ fixed, as the number n of the markings grows large we show that the rank of $\kappa_e(\overline{\mathcal{M}}_{g,n})$ is asymptotic to

$$\frac{\binom{n+e}{e}\binom{g+e}{e}}{(e+1)!} \simeq \frac{\binom{g+e}{e}n^e}{e!(e+1)!}.$$

When $g \leq 2$ we show that an element $\kappa \in \kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$ is trivial if and only if the integral of κ against all boundary strata is trivial. For $g = 1$ we further show that the rank of $\kappa_{n-d}(\overline{\mathcal{M}}_{1,n})$ is equal to $|\mathrm{P}_1(d, n-d)|$, where $\mathrm{P}_i(d,k)$ denotes the set of partitions $\mathbf{p} = (p_1, ..., p_\ell)$ of d such that at most k of the numbers $p_1, ..., p_\ell$ are greater than i.

1. INTRODUCTION

Let $\epsilon : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ denote the universal curve over the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable genus g, n-pointed curves. Let $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n+1}$ denote the cotangent line bundle over $\overline{\mathcal{M}}_{g,n+1}$ with fiber over a point equal to the cotangent line at the i^{th} marking. Define

$$\psi_i = c_1(\mathbb{L}_i) \in A^1(\overline{\mathcal{M}}_{g,n+1}) \quad \text{and} \quad \kappa_i = \epsilon_*(\psi_{n+1}^{i+1}) \in A^i(\overline{\mathcal{M}}_{g,n}).$$

The push forwards of the κ and ψ classes from the boundary strata generate the tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$ [3, 6]. The kappa ring $\kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$ is the subring of $R^*(\overline{\mathcal{M}}_{g,n})$ generated by $\kappa_1, \kappa_2, ...$ over $\mathbb{Q}$. Let $\kappa^d(\overline{\mathcal{M}}_{g,n})$ denote the $\mathbb{Q}$-module generated by the kappa monomials of degree d and set $\kappa_e(\overline{\mathcal{M}}_{g,n}) = \kappa^{3g-3+n-e}(\overline{\mathcal{M}}_{g,n})$.

Applying the localization formula of [5] to the action of $\mathbb{C}^*$ on the moduli space of stable maps from curves of genus g to $\mathbb{P}^1$ we prove the following theorem.

Theorem 1. *Fix the genus g and the dimension e. As the number n of the marked points grows large, the rank of $\kappa_e(\overline{\mathcal{M}}_{g,n})$ is asymptotic to*

$$\frac{\binom{g+e}{e}\binom{n+e}{e}}{(e+1)!} \simeq \frac{\binom{g+e}{e}n^e}{e!(e+1)!}.$$

Let G be a connected graph which is decorated by assigning a genus to each one of its vertices and let the markings $1, 2, ..., n$ get distributed among the vertices of G. For a vertex $v \in V(G)$ let g_v denote the genus associated with v, d_v denote the degree of v, and n_v denote the number of markings assigned to v. If $2g_v + n_v + d_v > 2$ for every $v \in V(G)$, G is called a *stable weighted graph*. Every stable weighted graph G describes a *combinatorial cycle* $[G]$ in $\overline{\mathcal{M}}_{g,n}$ where

$$g = |E(G)| - |V(G)| + 1 + \sum_{v \in V(G)} g_v.$$

The tautological ring of $[G]$ is denoted by $R^*([G])$. An element $\kappa \in \kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$ is called *combinatorially* trivial if for all relevant stable weighted graphs G as above $\int_{[G]} \kappa = 0$. Let $\kappa_0^*(\overline{\mathcal{M}}_{g,n}) \subset \kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$ denote the set of combinatorially trivial classes and $\kappa_c^*(\overline{\mathcal{M}}_{g,n})$ denote the quotient $\kappa^*(\overline{\mathcal{M}}_{g,n})/\kappa_0^*(\overline{\mathcal{M}}_{g,n})$, which sits in the short exact sequence

$$0 \longrightarrow \kappa_0^*\left(\overline{\mathcal{M}}_{g,n}\right) \longrightarrow \kappa^*\left(\overline{\mathcal{M}}_{g,n}\right) \xrightarrow{\pi_{g,n}} \kappa_c^*\left(\overline{\mathcal{M}}_{g,n}\right) \longrightarrow 0.$$

The quotient $\kappa_c^*(\overline{\mathcal{M}}_{g,n})$ will be called the *combinatorial kappa quotient*. A more careful examination of the localization terms in the argument used to obtain Theorem 1 proves the following theorem.

Theorem 2. *The map $\pi_{g,n}$ is an isomorphism of graded algebras for $g \leq 2$.*

Theorem 2 is a consequence of Keel's Theorem [7] when $g = 0$ and follows from Petersen's work [9] on the structure of the tautological ring for $g = 1$. Our argument in genus one is, however, different from Petersen's argument.

Let $\mathrm{P}(d)$ denote the set of partitions of d and $\mathrm{P}_i(d, k)$ denote the set of $\mathbf{p} = (p_1, ..., p_\ell) \in \mathrm{P}(d)$ such that at most k of the numbers $p_1, ..., p_\ell$ are greater than i. Combining Theorem 2 with combinatorial arguments, the following theorem is also proved in this paper.

Theorem 3. *The rank of $\kappa^d(\overline{\mathcal{M}}_{1,n})$ is equal to $|\mathrm{P}_1(d, n-d)|$.*

Let us now describe our strategy for bounding the rank of the kappa ring. A stable weighted graph G is called a comb graph if G is a tree, contains a distinguished vertex v_∞, and every vertex $v \in V(G) \setminus \{v_\infty\}$ is connected with an edge e_v to v_∞. Furthermore, the markings $1, ..., n$ are all assigned to v_∞. If the sum of the genera associated with the vertices of G is g, we get an embedding

$$\imath^G : \frac{[G]}{\mathrm{Aut}(G)} \longrightarrow \overline{\mathcal{M}}_{g,n},$$

of the quotient of $[G]$ by its group of automorphisms in $\overline{\mathcal{M}}_{g,n}$. The genus associated with v_∞ is denoted by $g_\infty(G)$. For every $\kappa \in \kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$, the localization argument of [10] gives a presentation

$$\kappa = \sum_{\substack{G:\text{comb} \\ g_\infty(G) < g}} \imath_*^G(\psi_G(\kappa)), \qquad \psi_G(\kappa) \in R^*([G]). \tag{1}$$

In particular, for $g = 1$, there is only one comb graph G with $g_\infty(G) < 1$. For this comb graph $[G] \simeq \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,n+1}$. The above argument implies that every element $\kappa \in \kappa_0(\overline{\mathcal{M}}_{1,n})$ is of the form

$$\kappa = \imath_*^G\big[\pi_1^*(\lambda_1)\pi_2^*(\psi)\big], \quad \text{for some } \psi \in R^*(\overline{\mathcal{M}}_{0,n+1}).$$

If a combinatorially trivial class in the tautological ring of $\overline{\mathcal{M}}_{1,n}$ takes the form of the right-hand-side of the above equation one can quickly conclude that $\psi = 0$, and thus $\kappa = 0$.

For arbitrary genus g, let $G_{g,n}$ denote the stable weighted graph whose underlying graph is illustrated in Figure 1. Thus $V(G_{g,n}) = \{v_0, ..., v_g\}$, $G_{g,n}$ has $2g$ edges and all the markings $1, ..., n$ are assigned to v_0. The stable weighted graph $G_{g,n}$ determines an embedding

$$\imath^{g,n} : \frac{\overline{\mathcal{M}}_{0,n+g}}{S_g} \simeq \frac{\overline{\mathcal{M}}_{0,n+g} \times \overbrace{\overline{\mathcal{M}}_{0,3} \times ... \times \overline{\mathcal{M}}_{0,3}}^{g \text{ copies}}}{S_g} = \frac{[G_{g,n}]}{\mathrm{Aut}(G_{g,n})} \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

An inductive use of localization and the above reduction scheme (using the presentation of Equation 1) shows that the rank of

$$\frac{\kappa_0^d\left(\overline{\mathcal{M}}_{g,n}\right)}{\kappa_0^d\left(\overline{\mathcal{M}}_{g,n}\right) \cap \imath_*^{g,n}\left(R^{d-2g}\left(\overline{\mathcal{M}}_{0,n+g}\right)\right)}$$

is small, compared to the rank of $\kappa_c^d(\overline{\mathcal{M}}_{g,n})$. Moreover, one may use Keel's Theorem [7] to show that

$$\kappa_0^d\left(\overline{\mathcal{M}}_{g,n}\right) \cap \imath_*^{g,n}\left(R^{d-2g}\left(\overline{\mathcal{M}}_{0,n+g}\right)\right) = 0.$$

The authors started an investigation of the structure of $\kappa_c^*(\overline{\mathcal{M}}_{g,n})$ in [2] and proved that its rank in dimension e, as the number n of the marked points grows large, is asymptotic to $\binom{g+e}{e}\binom{n+e}{e}/(e+1)!$. Together with the

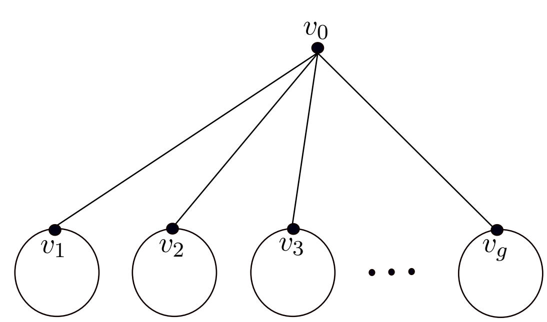

Figure 1. The combinatorial cycle isomorphic to $\overline{\mathcal{M}}_{0,n+g}$ in $\overline{\mathcal{M}}_{g,n}$. The genus associated with all vertices is 0 and all the n markings are assigned to v_0.

above method for estimating the rank of $\kappa_0^d(\overline{\mathcal{M}}_{g,n})$, this gives a proof of Theorem 1. The above scheme describes the heart of our argument in this paper.

Motivated by Theorem 2 we ask the following question.

Question 1. *For which triples (g,n,d) is $\kappa_0^d(\overline{\mathcal{M}}_{g,n})$ trivial?*

Theorem 2 shows that for all (g,n,d) with $g \leq 2$, $\kappa_0^d(\overline{\mathcal{M}}_{g,n})$ is trivial. The authors have recently been able to prove that for $g \geq 3$

$$\mathrm{rank}\left(\kappa^{3g-4+n}(\overline{\mathcal{M}}_{g,n})\right) = \left\lceil \frac{(n+1)(g+1)}{2} \right\rceil + 1 = \mathrm{rank}\left(\kappa_c^{3g-4+n}(\overline{\mathcal{M}}_{g,n})\right) + 2,$$

which shows that $\kappa_0^d(\overline{\mathcal{M}}_{g,n})$ is not always trivial. The proof of the aforementioned fact appears in [2].

The paper is organized as follows. After setting up the notation in Section 2, we describe our localization argument in Section 3. In Section 4 the asymptotic behaviour of the rank of $\kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$ is studied using the localization method of Section 3, and a proof of Theorem 2 is presented. In Section 5 we prove that the quotient map from $\kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$ to its combinatorial quotient is an isomorphism for $g = 1, 2$, and prove Theorem 2. Finally, in Section 6 the rank of $\kappa^d(\overline{\mathcal{M}}_{1,n})$ is computed for all n, d, giving a proof of Theorem 3.

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2. Combinatorial cycles and the ψ classes

It is sometimes more convenient to use alternative bases for the kappa ring of $\overline{\mathcal{M}}_{g,n}$, instead of the kappa classes. Let

$$\pi_{g,n,k}^m : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n+k}$$

denote the forgetful map which forgets the last $m-k$ markings.

Definition 2.1. *For every multi-set $\mathbf{p} = (p_1 \geq p_2 \geq ... \geq p_m)$ of positive integers with $p_k > 1$ and $p_i = 1$ for $k < i \leq m$ define*

- $\ell(\mathbf{p}) := m$ *and* $|\mathbf{p}| := \sum_{i=1}^m p_i$
- $\mathbf{p}^- := (p_1 - 1 \geq ... \geq p_k - 1) \in \mathrm{P}(|\mathbf{p}| - \ell(\mathbf{p}))$
- $\psi(\mathbf{p}) := \psi(p_1, ..., p_m) := \left(\pi_{g,n,0}^m\right)_* \left(\prod_{i=1}^m \psi_{n+i}^{p_i+1}\right) \in \kappa^{|\mathbf{p}|}(\overline{\mathcal{M}}_{g,n})$
- $\kappa(\mathbf{p}) := \kappa(p_1, ..., p_m) := \prod_{i=1}^m \kappa_{p_i} \in \kappa^{|\mathbf{p}|}(\overline{\mathcal{M}}_{g,n})$
- $\langle \mathbf{p} \rangle_{g,n;k} := \left(\pi_{g,n,k}^m\right)_* \left(\frac{1}{m!} \sum_{\sigma \in S_m} \prod_{i=1}^m \frac{1}{1 - p_{\sigma(i)}\psi_{n+i}}\right) \in R^*(\overline{\mathcal{M}}_{g,n+k})$.

Let $\langle \mathbf{p} \rangle = \langle \mathbf{p} \rangle_{g,n;0}$ *and let* $\langle \mathbf{p} \rangle^j$ *denote the degree* j *part of* $\langle \mathbf{p} \rangle$. *Similarly, let* $\langle \mathbf{p} \rangle^j_{g,n;k}$ *denote the degree* j *part of* $\langle \mathbf{p} \rangle_{g,n;k}$ *and set*

$$\langle \mathbf{p} \rangle_j^{g,n;k} := \langle \mathbf{p} \rangle_{g,n;k}^{3g-3+n+k-j}.$$

Lemma 2.2. *The subsets of* $\mathcal{A}^d(\overline{\mathcal{M}}_{g,n})$ *defined by*

$$\Big\{\psi(\mathbf{p}) \;\big|\; \mathbf{p} \in \mathrm{P}(d)\Big\}, \quad \Big\{\kappa(\mathbf{p}) \;\big|\; \mathbf{p} \in \mathrm{P}(d)\Big\} \quad \textit{and} \quad \Big\{\langle \mathbf{p} \rangle^d \big|\; \mathbf{p} \in \mathrm{P}(d)\Big\}$$

are related by invertible linear transformations.

Proof. The transformation relating the first two sets (which is independent of g and n) is due to Faber and is discussed in [1]. The transformation relating the first set to the third set is discussed in Proposition 3 from [4]. This later transformation only depends on $2g-2+n$. □

Let $\Psi(d)$ denote the formal vector space over $\mathbb{Q}$ which is freely generated by the partitions $\mathbf{p} \in \mathrm{P}(d)$. There are surjections

$$\psi_{g,n}, \kappa_{g,n}, \langle\rangle_{g,n} : \Psi(d) \longrightarrow \kappa^d(\overline{\mathcal{M}}_{g,n})$$

which are defined by

$$\psi_{g,n}\Big(\sum_{\mathbf{p} \in \mathrm{P}(d)} a_{\mathbf{p}}.\mathbf{p}\Big) := \sum_{\mathbf{p} \in \mathrm{P}(d)} a_{\mathbf{p}} \psi(\mathbf{p}), \qquad \kappa_{g,n}\Big(\sum_{\mathbf{p} \in \mathrm{P}(d)} a_{\mathbf{p}}.\mathbf{p}\Big) := \sum_{\mathbf{p} \in \mathrm{P}(d)} a_{\mathbf{p}} \kappa(\mathbf{p})$$

$$\text{and} \quad \Big\langle \sum_{\mathbf{p} \in \mathrm{P}(d)} a_{\mathbf{p}}.\mathbf{p} \Big\rangle_{g,n} := \sum_{\mathbf{p} \in \mathrm{P}(d)} a_{\mathbf{p}} \langle \mathbf{p} \rangle.$$

Lemma 2.2 implies that there are invertible matrices $P_d : \Psi(d) \to \Psi(d)$ and $Q_{d,m} : \Psi(d) \to \Psi(d)$ for $m \in \mathbb{Z}^+$ such that

$$\psi_{g,n} = \kappa_{g,n} \circ P_d \quad \text{and} \quad \langle\rangle_{g,n} = \kappa_{g,n} \circ Q_{d,n+2g}.$$

In [10] Pandharipande shows that associated with every pair of partitions

$$\mathbf{m} \in \mathrm{P}(d) \setminus \mathrm{P}(d, 2g-2+n-d) \quad \text{and} \quad \mathbf{p} \in \mathrm{P}(d)$$

there is a rational number $C^{\mathbf{p}}_{\mathbf{m}^-}$ with the property that

- The matrix $\big(C^{\mathbf{p}}_{\mathbf{m}^-}\big)_{\mathbf{m},\mathbf{p}}$ is of full-rank, with rank equal to
$$|\mathrm{P}(d)| - |\mathrm{P}(d, 2g-2+n-d)|.$$
- If we set
$$j(\mathbf{m}^-) := \sum_{\mathbf{p} \in \mathrm{P}(d)} C^{\mathbf{p}}_{\mathbf{m}^-}.\mathbf{p} \in \Psi(d)$$
the restriction of $\langle j(\mathbf{m}^-) \rangle_{g,n} \in \kappa^*\big(\overline{\mathcal{M}}_{g,n}\big)$ to $\mathcal{M}^c_{g,n}$ is trivial.

For $g = 0$, the kernel of the map

$$\langle\rangle_{0,n} : \Psi(d) \longrightarrow \kappa^d(\overline{\mathcal{M}}_{0,n})$$

is generated by $\{j(\mathfrak{m}^-)\}_{\mathfrak{m}}$. This observation gives a surjection

$$T^c_{g,n} : \kappa(\overline{\mathcal{M}}_{0,n+2g}) \to \kappa(\mathcal{M}^c_{g,n}).$$

This map *translates* every kappa class over $\overline{\mathcal{M}}_{0,n+2g}$ to a kappa class over $\mathcal{M}^c_{g,n}$, which comes from the same formal expression. Nevertheless, the aforementioned map does not have a clear geometric meaning (at least to the authors). We will encounter such homomorphisms again when we try to obtain relations among the kappa classes in this paper.

Definition 2.3. *A* weighted graph G *is a finite connected graph with the set* $V(G)$ *of vertices, the set* $E(G)$ *of edges and a weight function*

$$\epsilon = \epsilon_G : V(G) \to \mathbb{Z}^{\geq 0} \times 2^{\{1,...,n\}},$$

where $2^{\{1,...,n\}}$ *denotes the set of subsets of* $\{1, ..., n\}$*. For* $i \in V(G)$ *denote the degree of* i *by* $d_i = d(i)$ *and let* $\epsilon(i) = (g_i, I_i)$*.* G *is called a* stable weighted graph *if* $\{I_i\}_{i\in V(G)}$ *is a partition of* $\{1, ..., n\}$ *and for every vertex* $i \in V(G)$*,* $2g_i + |I_i| + d_i > 2$*. Define* $n(G) = n$ *and*

$$g(G) := \Big(\sum_{i\in V(G)} g_i\Big) + |E(G)| - |V(G)| + 1.$$

Associated with a stable weighted graph G there is a natural map

$$\imath_G : \mathcal{C}(G) = \prod_{i\in V(G)} \overline{\mathcal{M}}_{g_i,|I_i|+d_i} \longrightarrow \overline{\mathcal{M}}_{g(G),n(G)}$$

which is an embedding after we mod out the source by its automorphisms. Thus, a stable weighted graph G determines a *combinatorial cycle*

$$[G] := (\imath_G)_* [\mathcal{C}(G)] \in A_d\left(\overline{\mathcal{M}}_{g(G),n(G)}\right),$$

where $d = 3g(G) - 3 + n(G) - |E(G)|$.

Let $H = H_{g,n}$ be a stable weighted graph with a single vertex v, g self edges from v to itself, and with $\epsilon(v) = (0, \{1, ..., n\})$. H determines a homomorphism $\imath^H : \overline{\mathcal{M}}_{0,n+2g} \to \overline{\mathcal{M}}_{g,n}$. If G is a stable weighted graph with $g(G) = 0$ and $n(G) = n + 2g$ then

$$\int_{\imath^H_*[G]} \langle\mathbf{q}\rangle_{g,n} = \frac{1}{24^g \times g!} \int_{[G]} \langle\mathbf{q}\rangle_{0,n+2g} \quad \forall\ \mathbf{q} \in \mathrm{P}(d).$$

Remark 2.4. *Let* $\kappa = \langle\mathbf{a}\rangle_{g,n}$ *for some* $\mathbf{a} \in \Psi(d)$ *is such that* $\int_{\imath^H_*[G]} \kappa = 0$ *for all stable weighted graph* G *with* $g(G) = 0$ *and* $n(G) = n + 2g$*. By the above observation,* $\langle\mathbf{a}\rangle_{0,n+2g}$ *has trivial integral over all combinatorial cycles, and* *is thus trivial by Keel's Theorem* [7]*. By Pandharipande's result this implies that* $\mathbf{a} = \sum_{\mathfrak{m}} a_{\mathfrak{m}} j(\mathfrak{m}^-)$*, and thus*

$$\kappa = \sum_{\mathfrak{m}} a_{\mathfrak{m}} \langle j(\mathfrak{m}^-)\rangle_{g,n}.$$

Fix the stable weighted graph G and $\psi(\mathbf{p}) = \psi(p_1, ..., p_k)$ with $n = n(G)$, $g = g(G)$, $|\mathbf{p}| + |E(G)| = 3g - 3 + n$ and $V(G) = \{1, ..., m\}$. Let

$$Q = \left\{(h, r) \in \mathbb{Z}^{\geq 0} \times \mathbb{Z}^{>0} \;\middle|\; 2h + r > 2\right\}.$$

The *modified weight* multi-set associated with G is the multi-set

$$\mathbf{q}_G := \left(\theta_G(i) \in Q \;\middle|\; i \in \{1, ..., m\}\right), \quad \text{where}$$
$$\theta_G(i) := (g_i, m_i = |I_i| + d_i), \quad \forall\ 1 \leq i \leq m.$$

The integral

$$\left\langle \psi(\mathbf{p})\ ,\ [G]\right\rangle = \int_{[G]} \psi(\mathbf{p}) = \int_{(\pi^m_{g,n,k})^*[G]} \prod_{j=1}^{k} \psi_{n+j}^{p_j+1} \in \mathbb{Q}$$

only depends on the multi-set $\mathbf{q}_G$ [2]. We denote the value of the above integral by $\langle \psi(\mathbf{p}), \mathbf{q}_G\rangle_{g,n}$, or just $\langle \psi(\mathbf{p}), \mathbf{q}_G\rangle$ if there is no confusion. We denote by $\mathrm{Q}(d; g, n)$ the set of all multi-sets $\mathbf{q} = (\theta_i)_{i=1}^m$ such that $\mathbf{q} = \mathbf{q}_G$ for some stable weighted graph G with $g = g(G)$, $n = n(G)$ and $d = 3g-3+n-|E(G)|$. A kappa class $\kappa \in \kappa^d(\overline{\mathcal{M}}_{g,n})$ is combinatorially trivial if $\langle \kappa, \mathbf{q}\rangle = 0$ for all $\mathbf{q} \in \mathrm{Q}(d; g, n)$.

3. Localization and moduli space of stable maps to $\mathbb{P}^1$

In this section, we closely follow the notation and strategy of Section 8 from [10].

3.1. The vanishing cycles. Fix the integers $m \geq k \geq 0$ and let

$$\overline{\mathcal{M}}_{g,n+m}(\mathbb{P}^1, d)$$

denote the moduli space of stable maps of degree d from curves of genus g with $n + m$ marked points to $\mathbb{P}^1$ and denote by

$$\epsilon : \overline{\mathcal{M}}_{g,n+m}(\mathbb{P}^1, d) \longrightarrow \overline{\mathcal{M}}_{g,n+k}$$

the homomorphism which forgets the map and the last $m - k$ markings.

Let $\mathbb{C}^*$ act on $V = \mathbb{C} \oplus \mathbb{C}$ by

$$\zeta.(z_1, z_2) = (z_1, \zeta z_2)\ \ \forall\ \zeta \in \mathbb{C}^*,\ (z_1, z_2) \in \mathbb{C} \oplus \mathbb{C}.$$

Let $\mathrm{p}_0 = [0 : 1]$ and $\mathrm{p}_\infty = [1 : 0]$ denote the fixed points of the corresponding action on $\mathbb{P}^1 = \mathbb{P}(V)$. For every line bundle $L \to \mathbb{P}(V)$, an equivariant lifting of the $\mathbb{C}^*$ action to L is determined by the weights l_0 and l_∞ of the fiber representations $L_0 = L|_{\mathrm{p}_0}$ and $L_\infty = L|_{\mathrm{p}_\infty}$. The canonical lift of the action for $T_{\mathbb{P}^1}$ has weights $[l_0, l_\infty] = [1, -1]$.

Let $\pi : \mathcal{U}_{g,n+m}(\mathbb{P}(V), d) \to \overline{\mathcal{M}}_{g,n+m}(\mathbb{P}(V), d)$ denote the universal curve and $\mu : \mathcal{U}_{g,n+m}(\mathbb{P}(V), d) \to \mathbb{P}(V)$ denote the universal map. The action of $\mathbb{C}^*$ on $\mathbb{P}(V)$ induces $\mathbb{C}^*$ actions on $\mathcal{U}_{g,n+m}(\mathbb{P}(V), d)$ and $\overline{\mathcal{M}}_{g,n+m}(\mathbb{P}(V), d)$ compatible with π and μ. Let

$$\left[\overline{\mathcal{M}}_{g,n+m}(\mathbb{P}(V), d)\right]^{vir} \in A^{\mathbb{C}^*}_{2g+2d-2+n+m}\left(\overline{\mathcal{M}}_{g,n+m}(\mathbb{P}(V), d)\right)$$

denote the $\mathbb{C}^*$-equivariant virtual fundamental class of $\overline{\mathcal{M}}_{g,n+m}(\mathbb{P}(V), d)$ [5].

We consider three types of equivariant Chow classes over the moduli space $\overline{\mathcal{M}}_{g,n+m}(\mathbb{P}(V), d)$:

- The linearization $[0, 1]$ on $\mathcal{O}_{\mathbb{P}(V)}(-1)$ defines the $\mathbb{C}^*$ action on the rank $d + g - 1$ bundle
$$\mathbb{R} = R^1\pi_*\left(\mu^*\mathcal{O}_{\mathbb{P}(V)}(-1)\right) \longrightarrow \overline{\mathcal{M}}_{g,n+m}(\mathbb{P}(V), d).$$
We denote the top Chern class of this bundle by
$$c_{top}(\mathbb{R}) \in A^{d+g-1}_{\mathbb{C}^*}\left(\overline{\mathcal{M}}_{g,n+m}(\mathbb{P}(V), d)\right).$$
- For each marking i, let $\psi_i \in A^1_{\mathbb{C}^*}\left(\overline{\mathcal{M}}_{g,n+m}(\mathbb{P}(V), d)\right)$ denote the first Chern class of the canonically linearized cotangant line corresponding to the i^{th} marking.

- With $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n+m}(\mathbb{P}(V), d) \to \mathbb{P}(V)$ denoting the i-th evaluation map and with the $\mathbb{C}^*$-linearization $[1, 0]$ on $\mathcal{O}_{\mathbb{P}(V)}(1)$, let
$$\rho_i = c_1\left(\mathrm{ev}_i^*\mathcal{O}_{\mathbb{P}(V)}(1)\right) \in A^1_{\mathbb{C}^*}\left(\overline{\mathcal{M}}_{g,n+m}(\mathbb{P}(V), d)\right),$$
while with the $\mathbb{C}^*$ linearization $[0, -1]$ on $\mathcal{O}_{\mathbb{P}(V)}(1)$ we let
$$\tilde{\rho}_i = c_1\left(\mathrm{ev}_i^*\mathcal{O}_{\mathbb{P}(V)}(1)\right) \in A^1_{\mathbb{C}^*}\left(\overline{\mathcal{M}}_{g,n+m}(\mathbb{P}(V), d)\right).$$

Note that in the non-equivariant limit $\rho_i^2 = 0$, and that ϵ is equivariant with respect to the trivial action on $\overline{\mathcal{M}}_{g,n+k}$.

Fix the cycle dimension e and the sequence $\mathbf{n} = (n_1, ..., n_m)$ with

$$\sum_{i=1}^{m} n_i = d + g - 1 - e - l, \quad l > 0,$$

which determines a partition in $\mathrm{P}(d + g - 1 - e - l; m)$, denoted by $\mathbf{n}$ by slight abuse of the notation. The partition $\mathbf{n}$ is of the form $\mathbf{n} = \mathbf{m}^-$, where

$$\mathbf{m} \in \mathrm{P}(d) \setminus \mathrm{P}(d, \max\{e - g + 1, k - 1\}).$$

Let $I(\mathbf{n}) = I(\mathbf{n}; d, g, n, k)$ denote the $\mathbb{C}^*$-equivariant push-forward

$$\epsilon_*\left(\rho^l_{n+1}\prod_{i=1}^{m}\rho_{i+n}\psi^{n_i}_{i+n}\prod_{j=1}^{n}\tilde{\rho}_j\ c_{top}(\mathbb{R}) \cap \left[\overline{\mathcal{M}}_{g,n+m}(\mathbb{P}(V), d)\right]^{vir}\right).$$

Since the degree of

$$\rho_{n+1}^{l}\left(\prod_{i=1}^{m}\rho_{i+n}\psi_{i+n}^{n_i}\right)\left(\prod_{j=1}^{n}\tilde{\rho}_j\right)c_{top}(\mathbb{R})$$

is $2d+2g-e-2+n+m$ and the cycle dimension of the virtual fundamental class is $2d+2g-2+n+m$, the cycle dimension of the class $I(\mathbf{n})$ is

$$e=(2d+2g-2+n+m)-(2d+2g-e-2+n+m).$$

In other words, $I(\mathbf{n})\in A_{\mathbb{C}^*}^{3g-3+n+k-e}\left(\overline{\mathcal{M}}_{g,n+k}\right)$. Since the exponent of ρ_{n+1} is at least 2, $I(\mathbf{n})$ vanishes in the non-equivariant limit.

3.2. **The localization terms.** The virtual localization formula of [5] may be used to calculate $I(\mathbf{n})$ in terms of the tautological classes on $\overline{\mathcal{M}}_{g,n+k}$. The sum in the localization formula is over connected decorated graphs Γ (indexing the $\mathbb{C}^*$-fixed loci of $\overline{\mathcal{M}}_{g,n+m}(\mathbb{P}(V),d)$). Every vertex of Γ either lies over p_0 or over p_∞, and is labelled by a genus. The edges of the graph lie over $\mathbb{P}^1$ and are labelled with degrees (of the maps corresponding to the edges). The total sum of these degrees is equal to d. The graphs carry $n+m$ markings over their vertices. For a vertex v of Γ let $d(v)$ denote the degree of v.

If a graph Γ has a vertex v over p_0 with $d(v)>1$, v yields a trivial Chern root of the bundle $\mathbb{R}$ with trivial weight 0 in the numerator of the localization formula, by our choice of linearization on the bundle $\mathbb{R}$. Hence the contribution of such graphs to the sum in the localization formula is trivial. Thus, only comb graphs Γ contribute to $I(\mathbf{n})$. Every comb graph contains a set $V_0=V_0(\Gamma)$ of vertices which lie over p_0, and each $v\in V_0$ is connected by an edge to a unique vertex v_∞ which lies over p_∞. The linearization of the classes $\rho_{n+1},...,\rho_{n+m}$ and $\tilde{\rho}_1,...,\tilde{\rho}_n$ implies that the first n markings lie on v_∞ and the last m markings are placed on the vertices in V_0. For every $v\in V_0$ let g_v denote the genus associated with v and let $I_v\subset\{1,...,m\}$ determine the subset of the last m markings which is associated with v. Note that the genus associated with v_∞ is $g_\infty=g-\sum_{v\in V_0}g_v$. Denote the degree associated with the edge connecting v to v_∞ by p_v. The fixed locus associated with the decorated comb graph Γ is thus determined by a multi-set $\{(g_v,p_v,I_v)\}_{v\in V_0}$ such that $\sum g_v\leq g$, $\sum_v p_v=d$ and $\{I_v\}_v$ is a partition of $\{1,...,m\}$. We abuse the notation and use Γ to refer to this associated multi-set. The partition $(p_v)_{v\in V_0(\Gamma)}$ of d is denoted by $\mathbf{p}_\Gamma$.

The group S_Γ of permutations $\sigma:V_0\to V_0$ of the vertices in $V_0=V_0(\Gamma)$ acts on the multi-set associated with Γ by sending $\{(g_v,p_v,I_v)\}_{v\in V_0}$ to $\{(g_v,p_{\sigma(v)},I_v)\}_{v\in V_0}$. We denote the image of Γ under the action of $\sigma\in S_\Gamma$ by $\sigma(\Gamma)$. The automorphism group of Γ consists of the permutations σ of the vertices such that for every vertex $v\in V_0$ either $I_v=I_{\sigma(v)}=\emptyset$ and

$g_{\sigma(v)} = g_v$, or $\sigma(v) = v$. We denote the group of automorphisms of Γ by $\mathrm{Aut}(\Gamma)$.

If $I_v = \{i_1, ..., i_{k_v}\}$ the fixed locus corresponding to Γ contains a product factor $\overline{\mathcal{M}}^{v,\Gamma} \simeq \overline{\mathcal{M}}_{g_v,k_v+1}$, provided that $2g_v + k_v > 1$. The subset I_v labels k_v of the markings on $\overline{\mathcal{M}}^{v,\Gamma}$ and we use the vertex v itself to label the last marking on this moduli space. The classes $\psi_{i_j+n}^{n_{i_j}}$ carry trivial $\mathbb{C}^*$ weight. Moreover, the integrand term $c_{top}(\mathbb{R})$ yields a factor λ_{g_v} on $\overline{\mathcal{M}}^{v,\Gamma}$. Thus, we obtain the class $\lambda_{g_v}\psi_{v,\Gamma}(\mathbf{n}) \in A^*(\overline{\mathcal{M}}^{v,\Gamma})$ where

$$\psi_{v,\Gamma}(\mathbf{n}) := \prod_{i=1}^{k_v} \psi_{i_j+n}^{n_{i_j}} \in A^{n_{i_1}+\ldots+n_{i_{k_v}}}\left(\overline{\mathcal{M}}^{v,\Gamma}\right)$$

over this product factor, which is trivial unless

$$\sum_{j=1}^{k_v}(n_{i_j} - 1) \le 2g_v - 2.$$

In particular, $(g_v, k_v) \neq (0, i)$ with $i > 1$. In other words, if $g_v = 0$ the vertex v can accommodate at most one of the markings from $\{n + 1, ..., n + m\}$.

Let $\overline{\mathcal{M}}^{\infty,\Gamma} := \overline{\mathcal{M}}_{g_\infty,n+|V_0(\Gamma)|}$, where the last $|V_0(\Gamma)|$ markings are again labelled by the vertices in $V_0(\Gamma)$. Denote the subset of genus zero vertices in V_0 with no markings on them by $V^0 = V^0(\Gamma)$, the subset of genus zero vertices v with one marking by $V^1 = V^1(\Gamma)$ and set $V^2 = V^2(\Gamma) = V_0 \setminus (V^0 \cup V^1)$. For $v \in V^1$, if I_v consists of the single element $i \in \{1, ..., m\}$ we set $n_v = i$.

The contribution of the fixed locus corresponding to Γ to $I(\mathbf{n})$ may be computed following [10]. The only difference is that in this case

- The contribution from the deformation of the source (i.e. smoothing the nodes) adds an extra product factor
$$\prod_{v\in V^2(\Gamma)} \frac{1}{\left(\frac{t}{p_v}\right) + \psi_v},$$
A power $\psi_v^{m_v}$ of ψ_v thus appears in the contribution of Γ to $I(\mathbf{n})$ for smoothing the node corresponding to the vertex v.
- The deformation of the map contributes a factor of $e(\mathbb{E}^* \otimes \mathbf{1})$ over each one of the components in the fixed locus which are mapped to p_0 (i.e. over each $\overline{\mathcal{M}}^{v,\Gamma}$ with $v \in V^2(\Gamma)$). The Euler class of $\mathbb{E}^* \otimes \mathbf{1}$ over the product factor corresponding to a vertex $v \in V^2(\Gamma)$ can contribute via a lambda-class $(-1)^{g_v-h_v}\lambda_{h_v}$ for some integer $0 \le h_v \le g_v$. Since $\lambda_{g_v}^2 = 0$ over $\overline{\mathcal{M}}^{v,\Gamma}$ we may further assume that $h_v < g_v$.

The terms corresponding to Γ in $I(\mathbf{n})$ are thus indexed by the set $c(\Gamma)$ of the multi-sets $c = (h_v, m_v)_{v\in V^2(\Gamma)}$ with such that

- $0 \le h_v < g_v$.
- $0 \le m_v \le 2g_v - h_v - 2 - \sum_{i\in I_v} n_i$.

Let $\overline{\mathcal{M}}^{\Gamma}$ denote the fixed locus corresponding to Γ and $\pi_v : \overline{\mathcal{M}}^{\Gamma} \to \overline{\mathcal{M}}^{v,\Gamma}$ denote the projection map over the product factor corresponding to the vertex v. Denote the projection from $\overline{\mathcal{M}}^{\Gamma}$ to $\overline{\mathcal{M}}^{\infty,\Gamma}$ by π_∞. The restriction of ϵ to $\overline{\mathcal{M}}^{\Gamma}$ gives a map from $\overline{\mathcal{M}}^{\Gamma}$ to $\overline{\mathcal{M}}_{g,n+k}$. The contribution corresponding to Γ and $c = (h_v, m_v)_{v\in V^2(\Gamma)} \in c(\Gamma)$ takes the form

$$I(\mathbf{n},\Gamma,c) = B(\mathbf{n},\Gamma,c)\ \epsilon_*\left(\psi(\mathbf{n},\Gamma,c) \cap \left[\overline{\mathcal{M}}_\Gamma\right]^{vir}\right)$$

where

$$\psi(\mathbf{n},\Gamma,c) := \left(\prod_{v\in V^2(\Gamma)} \pi_v^*\left(\lambda_{g_v}\lambda_{h_v}\psi_v^{m_v}\psi_{v,\Gamma}(\mathbf{n})\right)\right)\pi_\infty^*\left(\prod_{v\in V_0(\Gamma)}\frac{1}{1-p_v\psi_v}\right)_{e(\mathbf{n},\Gamma,c)}$$

$$e(\mathbf{n},\Gamma,c) = e - \sum_{v\in V^2(\Gamma)}\Big(|I_v| + 2g_v - 2 - h_v - m_v - \deg(\psi_{v,\Gamma}(\mathbf{n}))\Big)$$

and the coefficient $B(\mathbf{n},\Gamma,c)$ is defined by

$$\frac{1}{\mathrm{Aut}(\mathbf{p}_\Gamma)}\left(\prod_{v\in V^0(\Gamma)}\frac{p_v^{p_v-1}}{p_v!}\right)\left(\prod_{v\in V^1(\Gamma)}\frac{p_v^{p_v-m_v}}{p_v!}\right)\left(\prod_{v\in V^2(\Gamma)}\frac{p_v^{p_v+m_v+1}}{p_v!}\right)$$

Here, for a tautological class ψ, $(\psi)_l$ denotes the part of ψ corresponding to the cycle dimension l.

For Γ as above and $v \in V^2(\Gamma)$, set $J_v = I_v \cap \{1,...,k\}$ and define

$$k(\Gamma) = \Big|\left\{n_v \mid v\in V^1(\Gamma)\right\}\cap\{1,...,k\}\Big| + \left|V^2(\Gamma)\right|.$$

Correspondingly, set

$$\overline{\mathcal{N}}^{v,\Gamma} = \overline{\mathcal{M}}_{g_v,1+|J_v|},\quad \forall v\in V^2(\Gamma),\qquad \overline{\mathcal{N}}^{0,\Gamma} = \prod_{v\in V^2(\Gamma)}\overline{\mathcal{N}}^{v,\Gamma}$$

$$\overline{\mathcal{N}}^{\infty,\Gamma} = \overline{\mathcal{M}}_{g_\infty,n+k(\Gamma)}\qquad\text{and}\qquad \overline{\mathcal{N}}^{\Gamma} = \overline{\mathcal{N}}^{0,\Gamma}\times\overline{\mathcal{N}}^{\infty,\Gamma}.$$

Let $\pi^{\infty,\Gamma} : \overline{\mathcal{M}}^{\infty,\Gamma} \to \overline{\mathcal{N}}^{\infty,\Gamma}$ denote the forgetful map which forgets the markings $n+k+1,...,n+m$. Denote the projection maps from $\overline{\mathcal{N}}^{\Gamma}$ to $\overline{\mathcal{N}}^{0,\Gamma}$ and $\overline{\mathcal{N}}^{\infty,\Gamma}$ by q_0 and q_∞, respectively, and define $q_1^* = q_\infty^* \circ \pi_*^{\infty,\Gamma}$. The map $\epsilon : \overline{\mathcal{M}}^{\Gamma} \to \overline{\mathcal{M}}_{g,n+k}$ factors through an embedding of $\overline{\mathcal{N}}^{\Gamma}$ in $\overline{\mathcal{M}}_{g,n+k}$. Thus, there are Chow classes $\eta_i(\mathbf{n},\Gamma,c)\in R_i(\overline{\mathcal{N}}^{0,\Gamma})$ with the property that

$$I(\mathbf{n},\Gamma,c)=\jmath^\Gamma_*\left(\sum_{i=0}^{e} q_0^*\Big(\eta_i(\mathbf{n},\Gamma,c)\Big)q_1^*\left(\prod_{v\in V_0(\Gamma)}\frac{1}{1-p_v\psi_v}\right)_{e-i}\right), \tag{2}$$

where $\jmath^\Gamma:\overline{\mathcal{N}}^\Gamma\to\overline{\mathcal{M}}_{g,n+k}$ is the embedding of the quotient of $\overline{\mathcal{N}}^\Gamma$ by its automorphisms in $\overline{\mathcal{M}}_{g,n+k}$.

Associated with the sequence $\mathbf{n}$, we obtain the following relation in the tautological ring of $\overline{\mathcal{M}}_{g,n+m}$

$$\sum_{\Gamma}\sum_{c\in c(\Gamma)} I(\mathbf{n},\Gamma,c)=0 \tag{3}$$

where $I(\mathbf{n},\Gamma,c)$ is given as in (2).

3.3. **The leading term.** Among all Γ, we distinguish the decorated comb graphs with $V^2(\Gamma)=\emptyset$, and denote the set of such graphs by $\mathcal{J}=\mathcal{J}(g,n,m,d)$. For $\Gamma\in\mathcal{J}$ the set $c(\Gamma)$ is trivial, and the corresponding coefficient is

$$B(\mathbf{n},\Gamma)=\frac{1}{\mathrm{Aut}(\mathbf{p}_\Gamma)}\left(\prod_{v\in V^0(\Gamma)}\frac{p_v^{p_v-1}}{p_v!}\right)\left(\prod_{v\in V^1(\Gamma)}\frac{p_v^{p_v-n_v}}{p_v!}\right)$$

We may thus compute

$$I(\mathbf{n},\Gamma)=B(\mathbf{n},\Gamma)\ \jmath^\Gamma_*\left(\prod_{v\in V_0(\Gamma)}\frac{1}{1-p_v\psi_v}\right)_e$$

We call the expression

$$I^{lead}(\mathbf{n})=\sum_{\Gamma\in\mathcal{J}} I(\mathbf{n},\Gamma)$$

the leading term in the relation of Equation (3).

Associated with a partition in $\mathrm{P}(d+g-1-e-l;m)$ which is represented by the sequence $\mathbf{n}$ there are $m!/\mathrm{Aut}(\mathbf{n})$ different sequences which may be constructed from $\mathbf{n}$. For $\sigma\in S_m$, let $\sigma(\mathbf{n})$ denote the sequence obtained by applying the permutation σ to $\mathbf{n}$. Let

$$J(\mathbf{n},\Gamma,c):=\frac{1}{m!}\sum_{\sigma\in S_m} I(\sigma(\mathbf{n}),\Gamma,c)\quad\text{and}$$
$$J(\mathbf{n}):=\frac{1}{m!}\sum_{\sigma\in S_m} I(\sigma(\mathbf{n}))=\sum_{\Gamma}\sum_{c\in c(\Gamma)} J(\mathbf{n},\Gamma,c).$$

Note that $J(\mathbf{n},\Gamma,c)$ and $J(\mathbf{n})$ depend on the partition associated with $\mathbf{n}$, and not the sequence $\mathbf{n}$ itself.

Let $\Psi_e(d;g,n,k)$ denote the $\mathbb{Q}$-module generated by the subset

$$\left\{\langle\mathbf{p}\rangle_e^{g,n;k}\ \big|\ \mathbf{p}\in \mathrm{P}(d)\setminus \mathrm{P}(d,k-1)\right\}\subset R_e(\overline{\mathcal{M}}_{g,n+k}),$$

and set $\Psi_e(d;g,n)=\Psi_e(d;g,n,0)$. The expressions

$$J^{lead}(\mathbf{n})=\frac{1}{m!}\sum_{\sigma\in S_m} I^{lead}(\sigma(\mathbf{n}))$$

belong to $\Psi_e(d;g,n,k)$.

Suppose that Γ is a decorated comb graph which contributes to $I^{lead}(\mathbf{n})$. Associated with Γ is the partition $\mathbf{p}_\Gamma=(p_v)_{v\in V_0(\Gamma)}$ in $\mathrm{P}(d)$. The partition associated with $\sigma(\Gamma)$ is $\mathbf{p}_\Gamma$ as well. Given $\mathbf{p}=(p_1,...,p_\ell)\in \mathrm{P}(d)$, in order to determine the decorated comb graph Γ which contributes to $I^{lead}(\mathbf{n})$ and satisfies $\mathbf{p}_\Gamma=\mathbf{p}$ we need to specify an injection $\phi:\{1,...,m\}\to\{1,...,\ell\}$, which determines the m vertices in $V_0(\Gamma)=\{1,...,\ell\}$ which carry the markings $\{n+1,...,n+m\}$. Let

$$J(\mathbf{n},\mathbf{p})=\sum_{\Gamma\in\mathcal{J}:\ \mathbf{p}_\Gamma=\mathbf{p}} J(\mathbf{n},\Gamma).$$

The above observation implies that

$$\begin{aligned}J(\mathbf{n},\mathbf{p})&=\prod_{i=1}^{\ell}\frac{p_i^{p_i-1}}{p_i!}\sum_{\phi}\prod_{j=1}^{m}p_{\phi(i)}^{1-n_i}\left(\pi_{g,n,k}^{\ell}\right)_*\left(\frac{1}{\ell!}\sum_{\sigma\in S_\ell}\prod_{i=1}^{\ell}\frac{1}{1-p_{\sigma(i)}\psi_{n+i}}\right)_e\\&=\prod_{i=1}^{\ell}\frac{p_i^{p_i-1}}{p_i!}\sum_{\phi}\prod_{j=1}^{m}p_{\phi(i)}^{1-n_i}\Big\langle\mathbf{p}\Big\rangle_e^{g,n;k}=C_{\mathbf{n}}^{\mathbf{p}}\Big\langle\mathbf{p}\Big\rangle_e^{g,n;k}.\end{aligned}$$

Let us define

$$\Phi_e(d;g,n,k):=\Big\langle J^{lead}(\mathbf{m}^-)\ |\ \mathbf{m}\in \mathrm{P}(d)\setminus \mathrm{P}(d,\max\{e-g+1,k-1\})\Big\rangle_{\mathbb{Q}}.$$

Proposition 3.1. *The $\mathbb{Q}$-module $\Psi_e(d;g,n,k)/\Phi_e(d;g,n,k)$ is generated by the classes in*

$$G_e(d;g,n,k)=\left\{\big\langle\mathbf{p}\big\rangle_e^{g,n;k}\ \big|\ \mathbf{p}\in \mathrm{P}(d,e-g+1)\setminus \mathrm{P}(d,k-1)\right\}$$

Proof. The coefficients $C_{\mathbf{n}}^{\mathbf{p}}$ form a matrix C which is a minor of the matrix $M_0(d)$, studied in Proposition 9.1 from [10]. It is shown that there is an upper triangular (with respect to the refinement ordering) square matrix Y whose rows and columns are indexed by the partitions in $\mathrm{P}(d)$ such that $M_0(d)Y$ is lower-triangular (with respect to the same order) with non-zero diagonal entries. The minor C of $M_0(d)$ corresponds to the partitions in

$$\mathrm{P}(d)\setminus \mathrm{P}(d,\max\{e-g+1,k-1\}).$$

The matrix CY is thus lower triangular. Note that CY is not a square matrix, but the number of its rows is less than or equal to the number of its columns, and it makes sense for CY to be lower triangular with non-zero entries on the diagonal. In particular, the classes $\langle \mathbf{p}\rangle_e^{g,n;k}$ with $\mathbf{p} \in \mathrm{P}(d)\setminus\mathrm{P}(d, e-g+1)$ may be expressed in terms of the classes in $\Phi_e(d;g,n,k)$, as well as the classes in $G_e(d;g,n,k)$. □

4. The asymptotic behaviour of the rank of the kappa ring

Let $f_1, f_2 : \mathbb{Z}^{\geq n} \to \mathbb{R}$ be real valued functions. If

$$\limsup_{n\to\infty} \frac{f_1(n)}{f_2(n)} \leq 0$$

we write $f_1(n) \in \mathfrak{o}(f_2(n))$.

Theorem 4.1. *Fix the genus g, the integer $k \geq 0$, the cycle dimension e and the difference $n-d \in \mathbb{Z}$. Then*

$$\mathrm{rank}\left(\Psi_e(d;g,n,k)\right) - \frac{\binom{n+e}{e}\binom{g+e}{e}\binom{k+e}{e}}{(e+1)!} \in \mathfrak{o}(n^e).$$

Proof. First of all, for every $\mathbf{n} = \mathbf{m}^-$ as in Section 3, using (2) we have

$$\begin{aligned} J^{lead}(\mathbf{n}) &= -\sum_{\Gamma}\sum_{c\in c(\Gamma)} J(\mathbf{n},\Gamma,c) \\ &= \frac{-1}{m!}\sum_{\substack{\sigma\in S_m\\ 0\leq i\leq e\\ \Gamma,c}} \jmath_*^\Gamma\left(q_0^*\Big(\eta_i(\sigma(\mathbf{n}),\Gamma,c)\Big)q_1^*\left(\prod_{v\in V_0(\Gamma)}\frac{1}{1-p_v\psi_v}\right)_{e-i} \cap \left[\overline{\mathcal{N}}^\Gamma\right]^{vir}\right) \\ &\in \sum_\Gamma\sum_{i=0}^{e} \jmath_*^\Gamma\left(R_i(\overline{\mathcal{N}}^{0,\Gamma})\otimes\Psi_{e-i}(d;g_\infty(\Gamma),n,k(\Gamma))\right). \end{aligned}$$

Here, the summation notation is used for summing the subspaces of the tautological ring of $\overline{\mathcal{M}}_{g,n+k}$ and the sums are over all graphs Γ with $g_\infty(\Gamma) < g$.

Proposition 3.1 implies that

$$\mathrm{rank}\left(\Psi_e(d;g,n,k)\right) - \mathrm{rank}\left(\Phi_e(d;g,n,k)\right) \in \mathfrak{o}(n^e).$$

We now use induction on the genus g to prove Theorem 4.1. The induction hypothesis implies that the rank of

$$R_i\left(\overline{\mathcal{N}}^{0,\Gamma}\right)\otimes\Psi_{e-i}(d;g_\infty(\Gamma),n,k(\Gamma))$$

belongs to $\mathfrak{o}(n^e)$ unless $i=0$, since the rank of the ring $R_i(\overline{\mathcal{N}}^{0,\Gamma})$ does not grow with n. In other words,

$$\mathrm{rank}\left(\frac{\Psi_e(d;g,n,k)}{\Psi_e(d;g,n,k)\cap\sum_\Gamma \jmath_*^\Gamma\left(\Psi_e(d;g_\infty(\Gamma),n,k(\Gamma))\right)}\right) \in \mathfrak{o}(n^e), \tag{4}$$

where we abuse the notation by setting

$$\jmath_*^\Gamma\left(\Psi_e(d;g_\infty(\Gamma),n,k(\Gamma))\right) := \jmath_*^\Gamma\left(R_0(\overline{\mathcal{N}}^{0,\Gamma})\otimes\Psi_e(d;g_\infty(\Gamma),n,k(\Gamma))\right).$$

In order to compute the asymptotic rank of $\Psi_e(d;g,n,k)$, we thus restrict ourselves to its subspace which consists of the push-forwards from the strata corresponding to the comb graphs Γ, and over the corresponding cycle $\overline{\mathcal{M}}^\Gamma = \overline{\mathcal{M}}^{0,\Gamma}\times\overline{\mathcal{M}}^{\infty,\Gamma}$ we assume that the tautological class is the product of the point class from $\overline{\mathcal{M}}^{0,\Gamma}$ and a class in $\Psi_e(d;g_\infty(\Gamma),n,k(\Gamma))$.

We represent the point class corresponding to the factor $\overline{\mathcal{M}}^{v,\Gamma}$ of $\overline{\mathcal{M}}^{0,\Gamma}$ by a nodal pointed curve C_v which is obtained as follows. The curve C_v corresponds to the weighted graph illustrated in Figure 2. Applying the above inductive scheme to the subspaces $\Psi_e(d;g_\infty(\Gamma),n,k(\Gamma))$ we may reduce the genus g_∞ to zero.

Let $\mathcal{G}$ denote the set of all stable weighted graphs with a distinguished vertex v_∞ such that $\theta(v_\infty)=(0,n+k_\infty)$ for some $0\le k_\infty=k_\infty(G)\le k$, and for all $v\in V(G)\setminus\{v_\infty\}$, $\theta(v)=(0,k_v)$ with $k_v+d_v=3$ (where d_v denotes the degree of the vertex v). Moreover, we assume that $k_\infty+\sum_v k_v=k$, that G has g self-edges and that by deleting these self-edges G becomes a connected tree. For $G\in\mathcal{G}$ we get an embedding

$$\imath^G:\mathcal{C}_G\simeq\overline{\mathcal{M}}_{0,k_\infty+d_\infty}\to\overline{\mathcal{M}}_{g,n+k}.$$

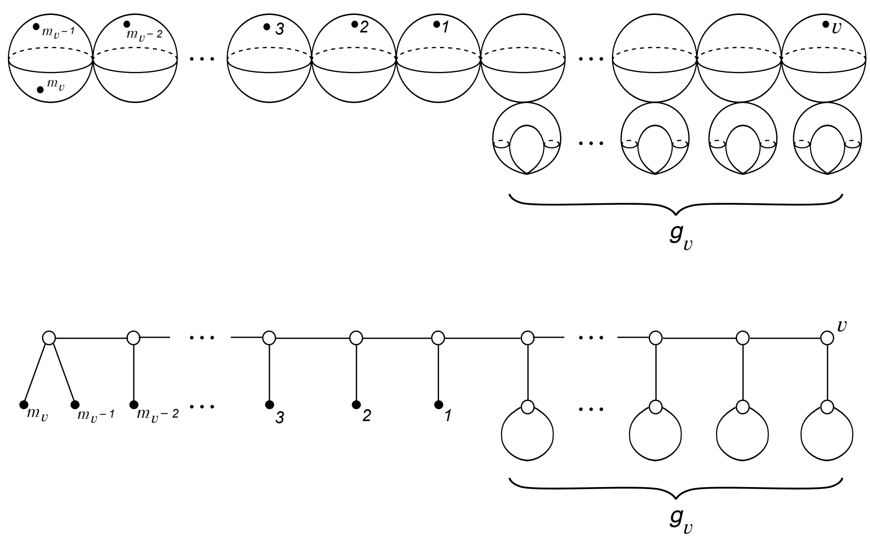

FIGURE 2. The curve C_v of genus g_v with m_v+1 marked points, and the decorated graph representing it. The total number of vertices is $2g_v+2m_v-1$. The marking v is placed on the special vertex on the upper right corner.

An inductive use of (4) implies that

$$\text{rank}\left(\frac{\Psi_e(d;g,n,k)}{\Psi_e(d;g,n,k)\cap\sum_{G\in\mathcal{G}}\imath_*^G\left(\Psi_e(d;0,n,k_\infty(G))\right)}\right)\in\mathfrak{o}(n^e). \tag{5}$$

Finally, note that all embeddings $\imath^G$ factor through the embedding

$$\imath^{g,k}:\mathcal{C}_{g,k}=\mathcal{C}_{G_{g,k}}\to\overline{\mathcal{M}}_{g,n+k}$$

which corresponds to the stable weighted graph $G_{g,k}$ with vertices $v_\infty,1,2,...,g$, such that for every $i=1,...,g$, the graph $G_{g,k}$ contains an edge connecting the vertex i to v_∞ together with a self edge from i to itself and $\theta(i)=(0,0)$. Moreover, $\theta(v_\infty)=(0,n+k)$. As a result of this observation, from (5) we obtain

$$\text{rank}\left(\Psi_e(d;g,n,k)/\left(\Psi_e(d;g,n,k)\cap\imath_*^{g,k}\left(R_e(\mathcal{C}_{g,k})\right)\right)\right)\in\mathfrak{o}(n^e). \tag{6}$$

It is thus enough to prove that

$$\text{rank}\left(\Psi_e(d;g,n,k)\cap\imath_*^{g,k}\left(R_e(\mathcal{C}_{g,k})\right)\right)-\frac{\binom{n+e}{e}\binom{g+k+e}{e}}{(e+1)!}\in\mathfrak{o}(n^e)$$

The tautological ring of $\mathcal{C}_{g,k}\simeq\overline{\mathcal{M}}_{0,n+g+k}$ is generated by combinatorial cycles. Let $S_{n,g,k}$ denote the set of permutations in S_{n+g+k} which preserve the sets $\{1,...,n\},\{n+1,...,n+g\}$ and $\{n+g+1,...,n+g+k\}$. If $D\subset\overline{\mathcal{M}}_{0,g+n+k}$ is a combinatorial cycle (i.e. one of the boundary strata in $\overline{\mathcal{M}}_{0,g+n+k}$), every $\sigma\in S_{n,g,k}$ acts on D by permuting the markings on the curves in D to give a corresponding combinatorial cycle $\sigma(D)$.

Suppose that the push-forward $\beta=\imath_*^{g,k}(\alpha)$ belongs to $\Psi_e(d;g,n,k)$ for $\alpha=\sum_{i=1}^N D_i$ with $D_i\in R_e(\mathcal{C}_{g,k})$. Since $\mathcal{C}_{g,k}\simeq\overline{\mathcal{M}}_{0,n+g+k}$, we may set

$$\sigma(\alpha)=\sum_{i=1}^N\sigma(D_i)\quad\forall\ \sigma\in S_{n+g+k}.$$

Note that

$$\begin{aligned}\beta=\imath_*^{g,k}(\alpha)=\sigma(\beta)=\imath_*^{g,k}(\sigma(\alpha))\quad&\forall\ \ \sigma\in S_{n,g,k}\\ \Rightarrow\ \beta=\imath_*^{g,k}\left(\frac{1}{n!\times g!\times k!}\sum_{\sigma\in S_{n,g,k}}\sigma(\alpha)\right)&=:\imath_*^{g,k}(\overline{\alpha}).\end{aligned}$$

The class $\overline{\alpha}\in R_e(\overline{\mathcal{M}}_{0,n+g+k})$ is determined by its integrals over combinatorial cycles. The $\mathbb{Q}$-module generated by the combinatorial cycles in dimension e is the same as $R^e=R_{n+g+k-3-e}(\overline{\mathcal{M}}_{0,n+g+k})$. If D is a combinatorial cycle of dimension e in R^e we have

$$\langle\overline{\alpha},\sigma(D)\rangle=\langle\sigma^{-1}(\overline{\alpha}),D\rangle=\langle\overline{\alpha},D\rangle\quad\forall\ \ \sigma\in S_{n,g,k}.$$

Integration against $\overline{\alpha}$ thus gives a map

$$\int_{\overline{\alpha}} : \frac{R_{n+g+k-3-e}(\overline{\mathcal{M}}_{0,n+g+k})}{S_{n,g,k}} \longrightarrow \mathbb{Q}$$

which determines $\overline{\alpha}$.

Every combinatorial cycle D as above determines a combinatorial cycle in $R_{n+3g+k-3-e}(\overline{\mathcal{M}}_{g,n+k})$ as follows. Suppose that D is associated with a stable weighted graph G and that

$$\epsilon_G : V(G) \to \mathbb{Z}^{\geq 0} \times 2^{\{1,\dots,n+g+k\}}$$

is the corresponding weight function. For every vertex $v \in V(G)$, $\epsilon_G(v) = (0, I_v)$ with I_v disjoint subsets of $\{1,...,n+g+k\}$ which give a partition of it. Let $\pi(G)$ denote the stable weighted graph with the same underlying graph G and the weight function defined by

$$\epsilon_{\pi(G)}(v) := (|I_v \cap \{n+1,...,n+g\}|\,, I_v \setminus \{n+1,...,n+g\}) \quad \forall\ v \in V(G).$$

Let $\pi(D)$ denote the combinatorial cycle associated with $\pi(G)$. If $\pi(D) = \pi(D')$ then $\imath_*^{g,k}(D) = \imath_*^{g,k}(D')$. Moreover, the intersection of $\pi(D)$ with $\mathcal{C}_{g,k}$ is transverse and

$$\pi(D) \cap \imath_*^{g,k}\left(\overline{\mathcal{M}}_{0,n+g+k}\right) = \#\{D' \mid \pi(D') = \pi(D)\}\imath_*^{g,k}(D).$$

From here we obtain

$$\langle \imath_*^{g,k}(\overline{\alpha}), \pi(D)\rangle = \#\left\{D' \mid \pi(D') = \pi(D)\right\}\langle \overline{\alpha}, D\rangle.$$

In other words, the map $\int_{\overline{\alpha}}$ is determined by the evaluation

$$\int_{\beta} : \frac{R_{n+g+k-3-e}(\overline{\mathcal{M}}_{0,n+g+k})}{S_{n,g,k}} \longrightarrow \mathbb{Q}$$
$$\int_{\beta}(D) = \frac{1}{\#\{D' \mid \pi(D') = \pi(D)\}}\langle \beta, \pi(D)\rangle.$$

Since $\beta \in \Psi_e(d; g, n, k)$ the evaluation $\int_\beta(D)$ only depends on the modified weight function associated with D, and not the underlying graph G. In other words, in order to determine $\int_\beta(D)$ one needs to specify

- The dimensions $(d_0, d_1, ..., d_e)$ of each one of the $e+1$ components of D, with the property that $\sum_i d_i = n+g+k-3-e$.
- The number of markings from $\{n+1, ..., n+g\}$ on each one of the $e+1$ components of D.
- The number of markings from $\{n+g+1, ..., n+g+k\}$ on each one of the $e+1$ components of D.

Asymptotically, the number of ways this can be done is equal to

$$\frac{\binom{n+e}{e}\binom{g+e}{e}\binom{k+e}{e}}{(e+1)!}.$$

This completes the proof of the theorem. □

Corollary 4.2. *Fix the dimension e and the genus g and let the number n of the markings grow large. Then the rank of $\kappa_e(\overline{\mathcal{M}}_{g,n})$ as a module over $\mathbb{Q}$ is asymptotic to*

$$\frac{\binom{n+e}{e}\binom{g+e}{e}}{(e+1)!}.$$

Proof. By Theorem 4.1 we know that

$$\operatorname{rank}\left(\kappa_e(\overline{\mathcal{M}}_{g,n})\right) - \frac{\binom{n+e}{e}\binom{g+e}{e}}{(e+1)!} \in \mathfrak{o}(n^e).$$

Theorem 5 from [2] implies that

$$\frac{\binom{n+e}{e}\binom{g+e}{e}}{(e+1)!} - \operatorname{rank}\left(\kappa_e(\overline{\mathcal{M}}_{g,n})\right) \in \mathfrak{o}(n^e).$$

These two observations complete the proof of the corollary. □

5. The kappa ring versus the combinatorial kappa quotient

In this section, we study the quotient map from $\kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$ to $\kappa^*_c(\overline{\mathcal{M}}_{g,n})$ and show that for $g \leq 2$ this quotient map is an isomorphism.

5.1. **The kappa ring and its combinatorial quotient in genus 1.** We would first like to focus on $g = 1$. As before, let

$$\Phi_{n-d}(d;1,n) = \left\langle J^{lead}(\mathbf{m}^-) \mid \mathbf{m} \in \mathrm{P}(d) \setminus \mathrm{P}(d, n-d) \right\rangle.$$

Other than the comb graphs in $\mathcal{J} = \mathcal{J}(1, n, m, d)$, the only possible comb graphs are the comb graphs Γ with a distinguished vertex $v_0 \in V_0(\Gamma)$ with associated genus $g_0 = 1$, and with $g_v = 0$ for all other vertices of Γ. The image of the corresponding components of the fixed locus under the forgetful map

$$\epsilon : \overline{\mathcal{M}}_{1,n+m}(\mathbb{P}(V), d) \to \overline{\mathcal{M}}_{1,n}$$

coincides with the image of

$$\imath = \imath^{1,0} : \overline{\mathcal{M}}_{0,n+1} \simeq [\mathrm{pt}] \times \overline{\mathcal{M}}_{0,n+1} \subset \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,n+1} \longrightarrow \overline{\mathcal{M}}_{1,n}.$$

In particular, for every $\mathbf{m} \in \mathrm{P}(d) \setminus \mathrm{P}(d, n-d)$ we have

$$J^{lead}(\mathbf{m}^-) = \imath_*(\kappa(\mathbf{m})) \qquad \kappa(\mathbf{m}) \in \Psi_{n-d}(d; 0, n, 1).$$

Theorem 5.1. *The quotient map from $\kappa^*\left(\overline{\mathcal{M}}_{1,n}\right)$ to $\kappa^*_c\left(\overline{\mathcal{M}}_{1,n}\right)$ is an isomorphism.*

Proof. It is enough to show that if a kappa class $\kappa \in \kappa_e(\overline{\mathcal{M}}_{g,n})$ is combinatorially trivial then it is zero. Suppose that $\kappa = \langle \mathbf{a} \rangle_{1,n}$ for some $\mathbf{a} \in \Psi(d)$ (we refer to Section 2 for the definitions). There is a homomorphism

$$\jmath : \overline{\mathcal{M}}_{0,n+2} \longrightarrow \overline{\mathcal{M}}_{1,n},$$

which gives an embedding of $\overline{\mathcal{M}}_{0,n+2}/(\mathbb{Z}/2\mathbb{Z})$ into $\overline{\mathcal{M}}_{1,n}$. The integral of κ over all combinatorial cycles of the form $\jmath_*(D)$ is trivial, since κ is combinatorially trivial. However, this implies that

$$\int_D \langle \mathbf{a}\rangle_{0,n+2} = \int_{\jmath_*(D)} \langle \mathbf{a}\rangle_{1,n} = 0 \qquad \forall \ D,$$

i.e. $\langle \mathbf{a}\rangle_{0,n+2} = 0$. By Remark 2.4

$$\langle \mathbf{a}\rangle_{0,n+2} = 0 \Rightarrow \mathbf{a} = \sum_{\mathbf{m}\in \mathrm{P}(d)\setminus \mathrm{P}(d,n-d)} a_{\mathbf{m}} \left(\sum_{\mathbf{p}\in \mathrm{P}(d)} C^{\mathbf{p}}_{\mathbf{m}^-}.\mathbf{p}\right),$$

for some rational coefficients $a_{\mathbf{m}}$, $\mathbf{m} \in \mathrm{P}(d) \setminus \mathrm{P}(d, n-d)$, i.e. κ is a linear combination of the kappa classes $J^{lead}(\mathbf{m}^-)$. Thus, there is a tautological class $\psi \in R_{d-2}(\overline{\mathcal{M}}_{0,n+1})$ such that $\kappa = \imath_*(\psi)$. In particular

$$\langle \psi, D\rangle = \frac{1}{\#\{D' \mid \pi(D) = \pi(D')\}} \langle \kappa, \pi(D)\rangle = 0 \quad \forall \ D,$$

$$\Rightarrow \ \psi = 0 \ \Rightarrow \ \kappa = 0.$$

This completes the proof. □

5.2. **The case of $\overline{\mathcal{M}}_{2,n}$.** Let us now consider the case $g = 2$. Fix a partition

$$\mathbf{n} \in \mathrm{P}(d, 2d-2-n-l) \qquad l > 0.$$

If the contribution of Γ to $J(\mathbf{n})$ is non-trivial $\overline{\mathcal{N}}^{0,\Gamma}$ is either trivial, or one of

$$\overline{\mathcal{M}}_{1,1}, \quad \overline{\mathcal{M}}_{2,1} \quad \text{or} \quad \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,1}.$$

Correspondingly, the map $\jmath^\Gamma : \overline{\mathcal{N}}^\Gamma \to \overline{\mathcal{M}}_{2,n}$ is one of the four maps

$$\jmath^0 = Id : \overline{\mathcal{M}}_{2,n} \to \overline{\mathcal{M}}_{2,n} \qquad \jmath^1 : \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,n+2} \to \overline{\mathcal{M}}_{2,n}$$
$$\jmath^{2,n} = \jmath^2 : \overline{\mathcal{M}}_{2,1} \times \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{2,n} \qquad \jmath^3 : \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,n+1} \to \overline{\mathcal{M}}_{2,n}.$$

The comb graphs which correspond to $\jmath^0$ form the leading term $J^{lead}(\mathbf{n})$ as their contribution to $J(\mathbf{n})$. Since a factor λ_1 appears over either of the two $\overline{\mathcal{M}}_{1,1}$ components in the domain of $\jmath^1$, the contribution of the comb graphs which correspond to $\jmath^1$ is a class of the form $\imath^{2,n}_*(\kappa(\mathbf{n}))$ for some tautological class $\kappa(\mathbf{n}) \in R^{d-4}(\overline{\mathcal{M}}_{0,n+2})$. With a similar reasoning, the comb graphs corresponding to $\jmath^3$ contribute via a class of the form

$$\jmath^3_* \left(\pi_1^*(\lambda_1)\pi_2^*(\psi(\mathbf{n}))\right), \quad \psi(\mathbf{n}) \in \Psi_{n+3-d}(d; 1, n, 1),$$

where (abusing the notation) π_i denotes the projection map from the domain of either of $\jmath^j$ over the i-th product factor, for $j = 0, 1, 2, 3$ and $i = 1, 2, 3$.

Let δ denote the divisor

$$\overline{\mathcal{M}}_{2,1} \setminus \mathcal{M}_{2,1} = \frac{[\overline{\mathcal{M}}_{1,3}]}{\mathbb{Z}/2\mathbb{Z}} + [\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,2}] = \delta_0 + \delta_1.$$

By the argument of Section 8 from [8] over $\overline{\mathcal{M}}_{2,1}$ we get

$$\begin{aligned} &\kappa_1 = 2\lambda_1 + \delta_1 + \psi_1 \quad \text{ and } \quad \lambda_1 = \frac{1}{12}(\kappa_1 + \delta - \psi_1) \\ \Rightarrow \quad &\lambda_2\kappa_1 = 7\lambda_2\lambda_1 + \lambda_2\psi_1 \quad \text{ and } \quad \lambda_2\delta_1 = 5\lambda_2\lambda_1. \end{aligned}$$

Thus, the Chow factor over the component $\overline{\mathcal{M}}_{2,1}$ in the domain of $\jmath^2$ is a linear combination of λ_2, $\lambda_2\psi_1$, $\lambda_2\lambda_1$ and the point class.

For every tautological class $\psi \in R^*(\overline{\mathcal{M}}_{0,n+1})$ note that

$$\jmath^2_* \left(\pi_1^*[\mathrm{pt}]\pi_2^*(\psi)\right) \in \mathrm{Im}\left(\imath_*^{2,n}\right).$$

Consequently, the total contribution corresponding to the comb graphs Γ with $\jmath^\Gamma = \jmath^2$ and $c \in c(\Gamma)$ corresponding to a multiple of the point class over $\overline{\mathcal{M}}_{2,1}$ is of the form $\imath_*^{2,n}(\kappa'(\mathbf{n}))$ for some $\kappa'(\mathbf{n}) \in R^{d-4}(\overline{\mathcal{M}}_{0,n+2})$. We thus obtain relations of the form

$$\begin{aligned} J^{lead}(\mathbf{n}) =&\imath_*^{2,n}\big[\kappa(\mathbf{n})+\kappa'(\mathbf{n})\big] + \jmath^3_*\big[\pi_1^*(\lambda_1)\pi_2^*(\psi(\mathbf{n}))\big] \\ &+ \jmath^2_*\Big[\pi_1^*(\lambda_2)\pi_2^*(\beta(\mathbf{n})) + \pi_1^*(\lambda_2\lambda_1)\pi_2^*(\gamma_1(\mathbf{n})) + \pi_1^*(\lambda_2\psi_1)\pi_2^*(\gamma_2(\mathbf{n}))\Big] \end{aligned}$$

where $\quad \beta(\mathbf{n}) \in R^{d-3}(\overline{\mathcal{M}}_{0,n+1}) \quad$ and $\quad \gamma_i(\mathbf{n}) \in R^{d-4}(\overline{\mathcal{M}}_{0,n+1}), \quad i = 1, 2.$

Let us now assume that $\kappa \in \kappa^d(\overline{\mathcal{M}}_{2,n})$ is combinatorially trivial. For every stable weighted graph H with the property that the combinatorial cycle associated with H is of dimension d and lives in $\overline{\mathcal{M}}_{2,n}$ we get $\int_{[H]}\kappa = 0$. Applying the above assumption to the stable weighted graphs with the zero genus associated with all vertices we find that κ is a linear combination of the classes $J^{lead}(\mathbf{n})$ by Remark 2.4. The above observation implies that there are classes

$$\begin{aligned} &\alpha \in R^{d-4}(\overline{\mathcal{M}}_{0,n+2}), \qquad &&\gamma_i \in R^{d-4}(\overline{\mathcal{M}}_{0,n+1}) \quad i = 1, 2 \\ &\beta \in R^{d-3}(\overline{\mathcal{M}}_{0,n+1}) \quad \text{ and } &&\psi \in \Psi_{n+3-d}(d; 1, n, 1) \end{aligned}$$

such that

$$\begin{aligned} \kappa =&\jmath^3_*\big[\pi_1^*(\lambda_1)\pi_2^*(\psi)\big] + \lambda_2\jmath^1_*\big[\pi_3^*(\alpha)\big] \\ &+ \lambda_2\jmath^2_*\Big[\pi_2^*(\beta) + \pi_1^*(\lambda_1)\pi_2^*(\gamma_1) + \pi_1^*(\psi_1)\pi_2^*(\gamma_2)\Big] \end{aligned}$$

In getting rid of $\imath_*^{2,n}$ and replacing it with $\jmath^1_*$ we are using the fact that

$$\frac{1}{24^2}\imath_*^{2,n}(a) = \jmath^1_*\big(\pi_1^*(\lambda_1)\pi_2^*(\lambda_1)\pi_3^*(a)\big) \quad \forall\ a \in R^*(\overline{\mathcal{M}}_{0,n+2}).$$

Moreover, since the restriction of λ_2 to

$$\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,n+2} \subset \overline{\mathcal{M}}_{2,n}$$

gives a factor of λ_1 over either of the product factors $\overline{\mathcal{M}}_{1,1}$,

$$\jmath^1_*(\pi_1^*(\lambda_1)\pi_2^*(\lambda_1)\pi_3^*(a)) = \lambda_2\jmath^1_*(\pi_3^*(a)).$$

Consider a stable weighted graph G which determines a combinatorial cycle over $\overline{\mathcal{M}}_{1,n+1}$ with cycle dimension $d-2$ (i.e. of codimension $n+3-d$). Let v denote the vertex of G which carries the *special marking* $n+1$. As in Section 4 let $\pi(G)$ denote the stable weighted graph obtained from G by removing the special marking from v and increasing the genus g_v by 1. Note that $\pi(G)$ determines a combinatorial cycle of dimension d in $\overline{\mathcal{M}}_{2,n}$. Let us assume that the genus associated with all vertices of G is zero. Since $\pi(G)$ contains a loop, the restriction of λ_2 to $[\pi(G)]$ is trivial. We thus find

$$0 = \int_{[\pi(G)]} \kappa = \int_{[\pi(G)]} \jmath^3_* \left(\pi_1^*(\lambda_1)\pi_2^*(\psi)\right) = \frac{1}{24}\int_{[G]} \psi.$$

In particular, the integral of $\psi \in \Psi_{n+3-d}(d;1,n,1)$ over all combinatorial cycles consisting only of genus zero components is trivial.

By Proposition 3.1 ψ may be represented as the sum of an element

$$\psi' \in \Phi_{n+3-d}(d;1,n,1)$$

and a linear combination of the classes in $G_{n+3-d}(d;1,n,1)$. Every linear combination of the classes in $G_{n+3-d}(d;1,n,1)$ is a linear combination of the classes of the form $\psi_{n+1}^{p-1}\psi(\mathbf{p})$ with $\mathbf{p}$ a partition in $\mathrm{P}(d-2-p,n+2-d)$. Let $\widehat{\mathrm{P}}(d)$ denote the set of marked partitions $(p;\mathbf{p})$ of d, i.e. the set of pairs $(p;\mathbf{p})$ such that $p \le d$ is a positive integer and $\mathbf{p} \in \mathrm{P}(d-p)$. Let $\widehat{\mathrm{P}}(d,k)$ denote the subset of $\widehat{\mathrm{P}}(d)$ which consists of the marked partitions $(p;\mathbf{p})$ with $\ell(\mathbf{p}) < k$. The above observation implies that associated with every $(p;\mathbf{p}) \in \widehat{\mathrm{P}}(d-2,n+3-d)$ there is a rational coefficient $A_{(p;\mathbf{p})}$ such that

$$\psi = \psi' + \sum_{(p;\mathbf{p})\in\widehat{\mathrm{P}}(d-2,n+3-d)} A_{(p;\mathbf{p})}\psi_{n+1}^{p-1}\psi(\mathbf{p}).$$

Since the integral of both ψ and ψ' over all combinatorial cycles which only consist of genus zero components is zero, we conclude that for every such combinatorial cycle D we have

$$\sum_{(p;\mathbf{p})\in\widehat{\mathrm{P}}(d-2,n+3-d)} A_{(p;\mathbf{p})}\int_D \psi_{n+1}^{p-1}\psi(\mathbf{p}) = 0. \tag{7}$$

For a combinatorial cycle D as above which consists only of genus zero components, let p_D-1 denote the dimension of the component containing the $(n+1)$-th marking. Let $\mathbf{p}_D$ denote the partition of $d-2-p_D$ determined by the dimensions of the rest of the components. Note that the marked partition $(p_D,\mathbf{p}_D)$ of D consists of at most $n-(d-3)$ components. The integral of $\psi_{n+1}^{p-1}\psi(\mathbf{p})$ against the combinatorial cycle D only depends on

$$(p_D;\mathbf{p}_D) \in \mathrm{P}(d-2,n+3-d).$$

The $|\widehat{\mathrm{P}}(d-2,n+3-d)| \times |\widehat{\mathrm{P}}(d-2,n+3-d)|$ matrix containing all possible integrals $\langle \psi_{n+1}^{p-1}\psi(\mathbf{p}),(p_D;\mathbf{p}_D)\rangle$ is triangular with respect to the refinement ordering with non-zero entries on the diagonal, (7) implies that $A_{(p;\mathbf{p})}=0$ for all $(p;\mathbf{p})\in\widehat{\mathrm{P}}(d-2,n+3-d)$. In particular, $\psi=\psi'$ belongs to $\Phi_{n+3-d}(d;1,n,1)$.

Since $\psi\in\Phi_{n+3-d}(d;1,n,1)$ it may be expressed in terms of the tautological classes pushed forward using

$$\imath^1:\overline{\mathcal{M}}_{1,2}\times\overline{\mathcal{M}}_{0,n+1}\to\overline{\mathcal{M}}_{1,n+1}\quad\text{and}\quad \imath^2:\overline{\mathcal{M}}_{1,1}\times\overline{\mathcal{M}}_{0,n+2}\to\overline{\mathcal{M}}_{1,n+1}.$$

Repeating the argument which was employed at the beginning of this subsection we may write

$$\begin{aligned}
&\psi=\imath^1_*\big[\pi_1^*(\lambda_1)\pi_2^*(\gamma_3)\big]+\imath^2_*\big[\pi_1^*(\lambda_1)\pi_2^*(\alpha')\big]\\
\Rightarrow\ &\jmath^3_*\big[\pi_1^*(\lambda_1)\pi_2^*(\psi)\big]=\lambda_2\Big[\jmath^1_*\big(\pi_3^*(\alpha')\big)+\jmath^2_*\big(\pi_1^*(\delta)\pi_2^*(\gamma_3)\big)\Big]\\
\Rightarrow\ &\kappa=\lambda_2\Big[\jmath^1_*\big(\pi_3^*(\alpha+\alpha')\big)+\jmath^2_*\big(\pi_2^*(\beta)\big)\Big]\\
&\qquad+\lambda_2\jmath^2_*\Big[\pi_1^*(\lambda_1)\pi_2^*(\gamma_1)+\pi_1^*(\psi_1)\pi_2^*(\gamma_2)+\pi_1^*(\delta_1)\pi_2^*(\gamma_3)\Big].
\end{aligned}$$

The second equality follows since the restriction of λ_2 to either of

$$\overline{\mathcal{M}}_{1,1}\times\overline{\mathcal{M}}_{1,2}\times\overline{\mathcal{M}}_{0,n+1},\overline{\mathcal{M}}_{1,1}\times\overline{\mathcal{M}}_{1,1}\overline{\mathcal{M}}_{0,n+2}\subset\overline{\mathcal{M}}_{2,n}$$

gives a factor of $1=\lambda_0$ over every product factor $\overline{\mathcal{M}}_{0,\star}$, and a factor of λ_1 over every product factor $\overline{\mathcal{M}}_{1,\star}$.

The above considerations imply the following lemma.

Lemma 5.2. *If the integral of $\kappa\in\kappa^d(\overline{\mathcal{M}}_{2,n})$ over all combinatorial cycles in $\overline{\mathcal{M}}_{2,n}$ with the sum of the genera of the components less than 2 is trivial then there are tautological classes $a\in R^{d-4}(\overline{\mathcal{M}}_{0,n+2}),b\in R^{d-3}(\overline{\mathcal{M}}_{0,n+1})$ and $c,c'\in R^{d-4}(\overline{\mathcal{M}}_{0,n+1})$ such that*

$$\kappa=\imath^{2,n}_*(a)+\lambda_2\jmath^{2,n}_*\Big(\pi_2^*(b)+\pi_1^*(\lambda_1)\pi_2^*(c)+\pi_1^*(\psi_1)\pi_2^*(c')\Big).\tag{8}$$

Proof. Set $a=\alpha+\alpha'$, $b=\beta$, $c=\gamma_1+\frac{20}{3}\gamma_3$ and $c'=\gamma_2$. □

Theorem 5.3. *The quotient map from $\kappa^d(\overline{\mathcal{M}}_{2,n})\to\kappa^d_c(\overline{\mathcal{M}}_{2,n})$ is an isomorphism.*

Proof. Every combinatorially trivial kappa class κ has a representation of the form (8). Consider a stable weighted graph G which corresponds to a combinatorial cycle in $\overline{\mathcal{M}}_{0,n+2}$. Treating the last two markings in $\overline{\mathcal{M}}_{0,n+2}$ as the special markings, $\pi(G)$ may be defined as a stable weighted graph determining a combinatorial cycle in $\overline{\mathcal{M}}_{2,n}$. If the intersection of $\pi(G)$ with the image of $\jmath^{2,n}$ is non-empty then the markings $p=n+1$ and $q=n+2$

lie over the same vertex of G. In this latter case, the intersection of $[\pi(G)]$ with the image of $\jmath^{2,n}$ is transverse, unless the vertex of G containing p, q is a vertex v with $d(v) = 1$ and $\epsilon(v) = (0, \{p, q\})$. However, if the intersection of $\jmath^{2,n}$ with $[\pi(G)]$ is transverse then

$$\int_{[\pi(G)]} \lambda_2 \jmath_*^{1,n}\Big[\pi_2^*(b) + \pi_1^*(\lambda_1)\pi_2^*(c) + \pi_1^*(\psi_1)\pi_2^*(c')\Big]$$
$$= \int_{[\mathrm{Im}(\jmath^{2,n})]\cap\pi(G)} \lambda_2\Big[\pi_2^*(b) + \pi_1^*(\lambda_1)\pi_2^*(c) + \pi_1^*(\psi_1)\pi_2^*(c')\Big].$$

In this case, the intersection includes a factor $\overline{\mathcal{M}}_{2,1}$, and since the degree of the Chow class over this factor is at most 3 the above integral is trivial. We conclude that unless G contains a vertex v with $\epsilon(v) = (0, \{p, q\})$ and $d(v) = 1$

$$\int_{[G]} a = 2 \times 24^2 \times \#\{D \mid \pi(D) = [\pi(G)]\} \times \int_{\pi(G)} \kappa = 0.$$

Let us now assume that the stable weighted graph G has a vertex v with $d(v) = 1$ and $\epsilon(v) = (0, \{p, q\})$. Let w be the vertex of G which is connected to v by an edge e. The vertex v corresponds to a product factor $\overline{\mathcal{M}}_{0,3}$ where the three markings are labelled by $\{p, q, e\}$. The vertex w corresponds to a factor $\overline{\mathcal{M}}_{0,k+1}$ where the markings are denoted by $\{e, p_1, ..., p_k\}$, and with e denoting the marking which corresponds to the edge e. For every subset $A \subset \{p_1, ..., p_k, p, q\} = B$ with $2 \le |A| \le k$, let G_A denote the stable weighted graph obtained as follows. Delete the edges of G which are adjacent to either of v and w, except for e, to obtain a sub-graph H of G with $V(H) = V(G)$. If e' denotes a deleted edge of G which connects some vertex u of G to w then e' corresponds to one of the markings $p_i \in \{p_1, ..., p_k\}$. If $p_i \in A$ then add an edge to H which connects u to v. Otherwise, add an edge to H which connects u to w. This gives a graph G_A. Let $\epsilon(w) = (0, I_w)$. Define the weight function over the vertices of G_A by

$$\epsilon_A(u) = \begin{cases} (0, A \cap (I_w \cup \{p, q\})) & \text{if } u = v \\ (0, (B \setminus A) \cap (I_w \cup \{p, q\})) & \text{if } u = w \\ \epsilon(u) & \text{otherwise} \end{cases}.$$

In particular, $G_{\{p,q\}} = G$ as stable weighted graphs.

Keel's Theorem [7] implies that

$$\sum_{\substack{A:\ p,q\in A \\ p_1,p_2\in B\setminus A}} [G_A] = \sum_{\substack{A:\ p,p_1\in A \\ q,p_2\in B\setminus A}} [G_A]$$

In particular, we obtain

$$\int_{[G]} a = \sum_{\substack{A:\ p,p_1\in A\\ q,p_2\in B\setminus A}} \int_{[G_A]} a - \sum_{\substack{A:\ p,q\in A\\ p_1,p_1\in B\setminus A\\ A\neq\{p,q\}}} \int_{[G_A]} a = 0.$$

The above discussion implies that the integral of a over all combinatorial cycles is trivial, and thus $a = 0$.

On the other hand, if G is a stable weighted graph containing a vertex v with $d(v) = 1$ and $\epsilon(v) = (0, \{p.q\})$, the combinatorial cycle $[\pi(G)]$ is included in the image of the map $\jmath^{2,n}$. Every such stable weighted graph G corresponds to another stable weighted graph G^* obtained from G by removing the vertex v from G. If w is the unique vertex adjacent to v by the edge e, we define

$$\epsilon_{G^*}(u) = \begin{cases} (0, I_w \cup \{e\}) & \text{if } u = w \\ \epsilon(u) & \text{otherwise} \end{cases}.$$

The stable weighted graph G^* determines a combinatorial cycle in $\overline{\mathcal{M}}_{0,n+1}$, where e corresponds to the last marking. Conversely, every combinatorial cycle in $\overline{\mathcal{M}}_{0,n+1}$ is of the form G^*. For every $H = G^*$ we obtain

$$\begin{aligned} 0 = \int_{[\pi(G)]} \kappa &= \frac{1}{2\times 24^2}\int_{[G]} a + \left(\int_{[\overline{\mathcal{M}}_{2,1}]} \lambda_2\lambda_1\psi_1\right)\int_{[H]} c + \left(\int_{[\overline{\mathcal{M}}_{2,1}]} \lambda_2\psi_1^2\right)\int_{[H]} c' \\ &= \frac{1}{2\times 24^2}\int_{[H]} (c + \frac{9}{4}c'). \end{aligned}$$

Thus $4c+9c' = 0$. Next, let G be a stable weighted graph which corresponds to a cycle of dimension $d-4$ in $\overline{\mathcal{M}}_{0,n+1}$. Let v denote the vertex of G which contains the marking $n+1$, and let $\epsilon(v) = (0, I_v)$. Let $\tilde{G}$ denote the stable weighted graph obtained from G as follows. We add a vertex w to G and connect it to v by a single edge. Then we set

$$\epsilon_{\tilde{G}}(u) = \begin{cases} (1, \emptyset) & \text{if } u = w \\ (1, I_v \setminus \{n+1\}) & \text{if } u = v \\ \epsilon(u) & \text{otherwise} \end{cases}.$$

The intersection of the combinatorial cycle $[\tilde{G}]$ with the image of $\jmath^{2,n}$ is always transverse, and they cut each other in

$$[\delta_1] \times [G] \subset \overline{\mathcal{M}}_{2,1} \times \overline{\mathcal{M}}_{0,n+1}.$$

For every stable weighted graph G as above we thus find

$$0 = \int_{[\tilde{G}]} \kappa = \left(\int_{[\delta_1]} \lambda_2\lambda_1\right)\int_{[G]} c + \left(\int_{[\delta_1]} \lambda_2\psi_1\right)\int_{[G]} c' = \frac{1}{24^2}\int_{[G]} c'.$$

Thus $c = c' = 0$ and $\kappa = \lambda_2 \jmath_*^{2,n}\big(\pi_2^*(b)\big)$.

Finally, let G be a stable weighted graph which corresponds to a cycle of dimension $d-3$ in $\overline{\mathcal{M}}_{0,n+1}$. Let v denote the vertex of G which contains the marking $n+1$, and let $\epsilon(v) = (0, I_v)$. Let $\overline{G}$ denote the stable weighted graph obtained from G as follows. The graph $\overline{G}$ is obtained from G by adding a pair of vertices w_1 and w_2 to G, which are connected by the edges e_1 and e_2 to v. We define the corresponding weight function by

$$\epsilon_{\overline{G}}(u) = \begin{cases} (1, \emptyset) & \text{if } u = w_i, \quad i = 1, 2 \\ (0, I_v \setminus \{n+1\}) & \text{if } u = v \\ \epsilon(u) & \text{otherwise} \end{cases}.$$

The combinatorial cycle $[\overline{G}]$ cuts the image of $\jmath^{2,n}$ transversely in

$$\big(\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,3}\big) \times [G] \subset \overline{\mathcal{M}}_{2,1} \times \overline{\mathcal{M}}_{0,n+1}.$$

For every stable weighted graph G as above we thus obtain

$$0 = \int_{[\overline{G}]} \kappa = \left(\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 \right)^2 \int_{[G]} b = \frac{1}{24^2} \int_{[G]} b.$$

Thus $b = 0$ and κ is trivial. □

6. The kappa ring of $\overline{\mathcal{M}}_{1,n}$

In this section we make an explicit computation of the rank of $\kappa^d(\overline{\mathcal{M}}_{1,n})$.

6.1. **The κ-trivial combinatorial cycles.** Recall that $\mathrm{Q}(d; g, n)$ is the set of all multi-sets $\mathbf{q} = (\theta_i)_{i=1}^m$ consisting of the elements

$$\theta_i \in Q = \left\{ (h, r) \in \mathbb{Z}^{\geq 0} \times \mathbb{Z}^{>0} \;\middle|\; 2h + r > 2 \right\}$$

such that $\mathbf{q} = \mathbf{q}_G$ for some stable weighted graph G with $g = g(G)$, $n = n(G)$ and $d = 3g - 3 + n - |E(G)|$.

For a partition $\mathbf{n} = (n_1, ..., n_k) \in \mathrm{P}(n)$ of length $k = \ell(\mathbf{n})$, let

$$\mathbf{q}_0(\mathbf{n}) = \{(0, n_1 + 2), ..., (0, n_k + 2)\} \in \mathrm{Q}(n - k; 1, n), \quad \text{and}$$
$$\mathbf{q}_l(\mathbf{n}) = \{(1, l), (0, n_1 + 2), ..., (0, n_k + 2)\} \in \mathrm{Q}(n + l - k; 1, n + l), \quad l \geq 1.$$

Every element of $\mathrm{Q}(d; 1, n)$ is of the form $\mathbf{q}_l(\mathbf{n})$ for some non-negative integer l and some $\mathbf{n} \in \mathrm{P}(n - l; n - d)$. Recall that $\mathrm{P}(d; k)$ denotes the set of the partitions of d into precisely k parts.

If $\mathbf{n}, \mathbf{m} \in \mathrm{P}(d)$ define $\mathbf{n} < \mathbf{m}$ if $\mathbf{n}$ refines $\mathbf{m}$. This partial ordering may be extended to a total ordering on $\mathrm{P}(d)$. We fix one such total ordering and will refer to it as the *refinement ordering*. Define

$$\mathfrak{p} : \mathrm{Q}(d; g, n) \longrightarrow \mathrm{P}(d, 3g - 2 + n - d)$$

by sending $\mathbf{q} = \{(g_i, n_i)\}_{i=1}^k$ to $\{3g_i - 3 + n_i\}_{i=1}^k$. If $\langle \psi(\mathbf{n}), \mathbf{q} \rangle$ is non-trivial for some $\mathbf{q} \in \mathrm{Q}(d; g, n)$ then $\mathbf{n}$ refines $\mathfrak{p}(\mathbf{q})$ [2].

We call a formal linear combination

$$a_1\mathbf{q}_1 + a_2\mathbf{q}_2 + ... + a_k\mathbf{q}_k \quad a_i \in \mathbb{Q}, \quad \mathbf{q}_i \in \mathrm{Q}(d; g, n)$$

a κ-trivial cycle if for every kappa class $\kappa \in \kappa^d(\overline{\mathcal{M}}_{g,n})$

$$a_1\langle \kappa, \mathbf{q}_1 \rangle + a_2\langle \kappa, \mathbf{q}_2 \rangle + ... + a_k\langle \kappa, \mathbf{q}_k \rangle = 0.$$

The space of κ-trivial cycles is a subspace of the vector space $\langle \mathrm{Q}(d; g, n) \rangle_{\mathbb{Q}}$ freely generated by the elements of $\mathrm{Q}(d; g, n)$, and we denote its rank by $r(d; g, n)$. The quotient $V(d; g, n)$ of $\langle \mathrm{Q}(d; g, n) \rangle_{\mathbb{Q}}$ by the space of κ-trivial cycles is a vector space isomorphic to $\kappa_c^d(\overline{\mathcal{M}}_{g,n})$. Thus, the rank of the combinatorial kappa quotient $\kappa_c^*(\overline{\mathcal{M}}_{g,n})$ in degree d may be computed as

$$\mathrm{rank}\left(\kappa_c^d(\overline{\mathcal{M}}_{g,n})\right) = |\mathrm{Q}(d; g, n)| - r(d; g, n).$$

Proposition 6.1. *The rank of $\kappa^d(\overline{\mathcal{M}}_{1,n})$ is at most $|\mathrm{P}_1(d, n-d)|$, where $P_i(d, k)$ denotes the set of partitions $\mathbf{p} = (p_1, ..., p_\ell)$ of d such that at most k of the numbers $p_1, ..., p_\ell$ are greater than i.*

Proof. By Theorem 5.1, it suffices to show that the rank of $\kappa_c^d(\overline{\mathcal{M}}_{1,n})$ is at most $|\mathrm{P}_1(d, n-d)|$. Theorem 3 from [2] implies that $\frac{1}{24}\mathbf{q}_0(n) - \sum_{i=1}^{n-1}\binom{n-2}{i-1}\mathbf{q}_i(n-i)$ is κ-trivial. Consequently, for every partition $\mathbf{n} = (n_1, ..., n_k) \in \mathrm{P}(d)$

$$\frac{1}{24}\mathbf{q}_0(n, n_1, ..., n_k) - \sum_{i=1}^{n-1}\binom{n-2}{i-1}\mathbf{q}_i(n-i, n_1, ..., n_k) \tag{9}$$

is κ-trivial in $\langle \mathrm{Q}\,(n + d - k - 1; 1, n + d) \rangle_{\mathbb{Q}}$. Thus, for every $\mathbf{n} \neq (1, 1, ..., 1)$ in $\mathrm{P}(n)$, $\mathbf{q}_0(\mathbf{n}) \in V(n - \ell(\mathbf{n}); 1, n)$ is equal to a linear combination of the cycles $\mathbf{q}_l(\mathbf{m})$ for $l \geq 1$ and $\mathbf{m} \in \mathrm{P}(n - l; \ell(\mathbf{n}))$. In other words, $V(d; 1, n)$ is generated by $\mathbf{q}_l(\mathbf{n})$ for $l \geq 1$ and $\mathbf{n} \in \mathrm{P}(n - l; n - d)$.

For $\mathbf{n} = (a, b) \in \mathrm{P}(n)$ with $a > b$, (9) gives the following two equations in $V(n-2; 1, n)$:

$$\begin{aligned} \frac{1}{24}\mathbf{q}_0(\mathbf{n}) &= \sum_{i=1}^{a-1}\binom{a-2}{i-1}\mathbf{q}_i(a-i, b) \\ &= \sum_{i=1}^{b-1}\binom{b-2}{i-1}\mathbf{q}_i(a, b-i). \end{aligned} \tag{10}$$

Thus,

$$\mathbf{q}_{b-1}(a, 1, n_1, ..., n_k) \in V\left(a + b - 2 - k + \sum_i n_i; 1, a + b + \sum_i n_i\right)$$

may be expressed as a linear combination of

$$\mathbf{q}_i(a-i, b, n_1, ..., n_k), \quad i = 1, ..., a-1 \quad \text{and}$$
$$\mathbf{q}_j(a, b-j, n_1, ..., n_k), \; j = 1, ..., b-2.$$

This observation implies that $V(d; 1, n)$ is generated by the following elements of $\mathrm{Q}(d; 1, n)$ (with $l \geq 1$):

- $\mathbf{q}_l(n_1 \leq ... \leq n_{n-d})$ with $\sum_{i=1}^{n-d} n_i = n - l$, and $n_1 \geq 2$
- $\mathbf{q}_l(1 \leq n_2 \leq ... \leq n_{n-d})$ with $\sum_{i=2}^{n-d} n_i = n - l - 1$ and $n_{n-d} \leq l + 1$.

Denote the above two sets of generators by $A_1(d; 1, n)$ and $A_2(d; 1, n)$ respectively, and set

$$A(d; 1, n) = A_1(d; 1, n) \cup A_2(d; 1, n).$$

Every element of $A_1(d; 1, n)$ corresponds to the partition

$$(n_1 - 1, ..., n_{n-d} - 1) \in \mathrm{P}(d - l; n - d).$$

The size $|A_1(d; 1, n)|$ is thus equal to $\sum_{l=1}^{2d-n} |\mathrm{P}(d-l; n-d)|$. Every partition in $A_2(d; 1, n)$ gives the partition

$$((n_2 - 1) \leq (n_3 - 1) \leq ... \leq (n_{n-d} - 1) \leq l) \in \mathrm{P}(d, n - d).$$

Thus, $V(d; 1, n)$ is generated by a set of size

$$|\mathrm{P}(d, n-d)| + \sum_{l=1}^{2d-n} |\mathrm{P}(d - l; n - d)|.$$

Sending the partition $\mathbf{n} = (n_1 \leq ... \leq n_{n-d}) \in \mathrm{P}(d - l; n - d)$ to

$$\mathbf{n}[l] := (n_1, ..., n_{n-d}, 1, ..., 1) \in \mathrm{P}(d)$$

gives a bijection (extending the inclusion $\mathrm{P}(d, n - d) \subset \mathrm{P}_1(d, n - d)$)

$$\mathrm{P}(d, n-d) \cup \coprod_{l=1}^{2d-n} \mathrm{P}(d - l; n - d) \longrightarrow \mathrm{P}_1(d, n - d).$$

This completes the proof of Proposition 6.1. □

6.2. **Independence of the generators.**

Theorem 6.2. *The rank of* $\kappa^d(\overline{\mathcal{M}}_{1,n})$ *is equal to* $|\mathrm{P}_1(d, n - d)|$.

Proof. By Proposition 6.1, it is enough to show that the elements of $A(d; 1, n)$ are linearly independent in $V(d; 1, n)$.

If $\mathbf{n} \in \mathrm{P}(d, n - d)$, the integral of $\psi(\mathbf{n}) \in \kappa^d(\overline{\mathcal{M}}_{1,n})$ against every $\mathbf{q} \in A_1(d; 1, n)$ is zero, since the length of $\mathfrak{p}(\mathbf{q})$ is $n - d + 1$, while the length of $\mathbf{n}$ is at most $n - d$ (thus $\mathbf{n}$ does not refine $\mathfrak{p}(\mathbf{q})$). Meanwhile, the map $\mathfrak{p} : \mathrm{Q}(d; 1, n) \to \mathrm{P}(d, n - d + 1)$ gives an injection

$$\mathfrak{p} : A_2(d; 1, n) \to \mathrm{P}(d, n - d).$$

With respect to the refinement ordering on $\mathrm{P}(d, n-d)$ the matrix

$$\Big(\big\langle \psi\big(\mathfrak{p}(\mathbf{q})\big), \mathbf{q}' \big\rangle \Big)_{\mathbf{q},\mathbf{q}' \in A_2(d;1,n)}$$

is triangular with non-zero diagonal entries, and is thus full-rank. The above two observations reduce the proof of Theorem 6.2 to showing that the elements of $A_1(d;1,n)$ are linearly independent in $V(d;1,n)$.

For every $\mathbf{n} \in \mathrm{P}(n-l; n-d)$, every $\mathbf{p} \in \mathrm{P}(d)$, and every integer $N \geq 0$

$$\Big\langle \psi(\mathbf{p}), \mathbf{q}_l(\mathbf{n}) \Big\rangle_{1,n} = \Big\langle \psi(\mathbf{p}), \mathbf{q}_l(\mathbf{n}[N]) \Big\rangle_{1,n+N}.$$

In order to prove the independence of the elements of $A_1(d;1,n)$, it is thus enough to prove the independence of the elements of

$$A_1^N(d;1,n) \subset A_1(d;1,n+N)$$

consisting of $\mathbf{q}_l(\mathbf{n}[N])$ with $\mathbf{q}_l(\mathbf{n}) \in A_1(d;1,n)$.

For $\mathbf{n} = (n_1, ..., n_k) \in \mathrm{P}(n)$ and $\mathbf{m} = (m_1, ..., m_p) \in \mathrm{P}(m)$ define

- $\widehat{\mathbf{n}} := (n_1+1, ..., n_k+1, 1, ..., 1) \in \mathrm{P}(2n; n)$ and
- $\mathbf{n} \cup \mathbf{m} := (n_1, ..., n_k, m_1, ..., m_p) \in \mathrm{P}(m+n)$

Lemma 6.3. *For every positive integer l and every $\mathbf{n} \in \mathrm{P}(n)$ the cycle*

$$\mathbf{q}_l(\mathbf{n}[l]) + \frac{1}{24}\sum_{\mathbf{m}\in\mathrm{P}(l)} \left(\frac{(-1)^{\ell(\mathbf{m})}(\ell(\mathbf{m})-1)!}{|\mathrm{Aut}(\mathbf{m})|} \binom{|\mathbf{m}|}{\mathbf{m}} \right) \mathbf{q}_0(\widehat{\mathbf{m}} \cup \mathbf{n}) \tag{11}$$

is κ-trivial.

Proof. We use induction on l. For $l = 1$, Lemma 6.3 follows directly from (9). Suppose now that the claim is proved for $1, 2, ..., l-1$. Using (9), for every $\mathbf{n} \in \mathrm{P}(n)$ we make the following computation in $V(n+l; 1, k+l)$:

$$\begin{aligned}
\mathbf{q}_l(\mathbf{n}[l]) &= \frac{1}{24}\mathbf{q}_0(\{l+1\} \cup \mathbf{n}[l-1]) - \sum_{i=1}^{l-1} \binom{l-1}{i} \mathbf{q}_{l-i}(\{i+1\} \cup \mathbf{n}[l-1]) \\
&= \frac{1}{24}\mathbf{q}_0\left(\widehat{\{l\}} \cup \mathbf{n}\right) +
\end{aligned}$$

$$\frac{1}{24}\sum_{\substack{\mathbf{m}\in\mathrm{P}(l)\\ \mathbf{m}\neq(l)}} \sum_{\substack{i \text{ apears in } \mathbf{m}\\ r:\text{multiplicity of } i}} \binom{|\mathbf{m}|-1}{i}\binom{|\mathbf{m}|-i}{\mathbf{m}\setminus\{i\}} \frac{(-1)^{\ell(\mathbf{m})-1}(\ell(\mathbf{m})-2)!}{\frac{|\mathrm{Aut}(\mathbf{m})|}{r}} \mathbf{q}_0(\widehat{\mathbf{m}} \cup \mathbf{n})$$

$$= -\frac{1}{24}\sum_{\mathbf{m}\in\mathrm{P}(l)} \left(\frac{(-1)^{\ell(\mathbf{m})}(\ell(\mathbf{m})-1)!}{|\mathrm{Aut}(\mathbf{m})|} \binom{|\mathbf{m}|}{\mathbf{m}} \right) \mathbf{q}_0(\widehat{\mathbf{m}} \cup \mathbf{n}),$$

where the last equality follows since

$$\sum_{\substack{i \text{ apears in } \mathbf{m} \\ r:\text{multiplicity of } i}} r.\frac{|\mathbf{m}|-i}{|\mathbf{m}|} = \ell(\mathbf{m})-1.$$

This completes the proof of Lemma 6.3. □

In particular, every element of $A_1^{2d-n}(d;1,n) \subset A(d;1,2d)$ is a linear combination (in $V(d;1,2d)$) of the cycles of the form $\mathbf{q}_0(\mathbf{n})$ with $\mathbf{n} \in \mathrm{P}(2d;d)$ having at least $n-d+1$ terms greater than or equal to 2. Such $\mathbf{n}$'s are determined by

$$\mathbf{m} = \mathbf{n}^- \in \mathrm{P}(d) - \mathrm{P}(d, n-d).$$

Define $\mathfrak{q}(\mathbf{m}) = \mathbf{q}_0(\mathbf{n})$.

Let us denote the matrix expressing the elements of $A_1^{2d-n}(d;1,n)$ in terms of $\mathfrak{q}(\mathbf{m})$ with $\mathbf{m} \in \mathrm{P}(d)-\mathrm{P}(d,n-d)$ by $M(d;1,n)$. The rows of $M(d;1,n)$ are thus indexed by the elements of $\mathrm{P}(d)-\mathrm{P}(d,n-d)$ and its columns are indexed by the elements of $A_1(d;1,n)$. In particular,

$$\mathbf{q}_l(\mathbf{n}) \in A_1(d;1,n) \quad \Rightarrow \quad \mathbf{n}[l] \in \mathrm{P}(d)-\mathrm{P}(d,n-d),$$

and the $(\mathbf{q}_l(\mathbf{n}), \mathbf{n}[l])$ component of $M(d;1,n)$ is equal to $\frac{(-1)^{l-1}(l-1)!}{24}$. Moreover, if $\mathbf{m} \in \mathrm{P}(2d,d)$ corresponds to some non-zero entry of $M(d;1,n)$ in the column corresponding to $\mathbf{q}_l(\mathbf{n})$ then $\mathbf{m}^-$ refines $\mathbf{n}$. In other words, the square sub-matrix of $M(d;1,n)$ corresponding to the rows indexed by $\mathbf{n}[l]$ with $\mathbf{q}_l(\mathbf{n}) \in A_1(d;1,n)$ is triangular with non-zero elements on the diagonal (if we use the refinement ordering on the partitions). Hence $M(d;1,n)$ is a matrix of full rank equal to $|A_1(d;1,n)|$.

In order to finish the proof, it is enough to show that the matrix

$$N(d;1,g) = \big(\langle \psi(\mathbf{p}), \mathfrak{q}(\mathbf{p}')\rangle\big)_{\mathbf{p},\mathbf{p}' \in \mathrm{P}(d)-\mathrm{P}(d,n-d)}$$

is invertible. This is true since the matrix is upper triangular with non-zero diagonal elements with respect to the refinement ordering over the partitions. This completes the proof of Theorem 6.2. □

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