ON THE STRUCTURE OF THE KAPPA-RING

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ABSTRACT. We obtain lower bounds on the rank of the kappa ring $\kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$ of the Deligne-Mumford compactification of the moduli space of curves in different degrees by studying the combinatorial kappa quotient $\kappa^*_c(\overline{\mathcal{M}}_{g,n})$. Let d denote the degree and $e = 3g-3+n-d$ denote the co-degree. We show that if $n > \min\{g-2, e\}$ the rank of $\kappa^d(\overline{\mathcal{M}}_{g,n})$ is bounded below by $|\mathrm{P}(d, e+1)|$ where $\mathrm{P}(d, r)$ denotes the set of partitions of the positive integer d into at most r parts. We compute the rank of $\kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$ in co-degree 1 and prove, in particular, that for $g > 2$ and $n > 0$ this rank is equal to

$$\left\lceil \frac{(n+1)(g+1)}{2} \right\rceil + 1.$$

Furthermore, as g and e remain fixed and n grows large, we prove that the rank of $\kappa^*_c(\overline{\mathcal{M}}_{g,n})$ in co-degree e is asymptotic to

$$\frac{\binom{n+e}{e}\binom{g+e}{e}}{(e+1)!}.$$

Together with the results of [3] this last observation shows that the rank of $\kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$ follows the same asymptotic behavior.

1. Introduction

Let $\epsilon = \pi^1_{g,n} : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ denote the universal curve over the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable genus g, n-pointed curves. Throughout this paper, we will assume that $n > 0$. The psi and kappa classes in the Chow ring of $\overline{\mathcal{M}}_{g,n}$ are defined as follows [8]. Let $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n+1}$ denote the cotangent line bundle over $\overline{\mathcal{M}}_{g,n+1}$ whose fiber over a given point (which is an $(n+1)$-pointed genus g curve) is the cotangent space over the i^{th} marked point. The i^{th} psi-class ψ_i over $\overline{\mathcal{M}}_{g,n+1}$ is then defined by

$$\psi_i = c_1(\mathbb{L}_i) \in \mathcal{A}^1(\overline{\mathcal{M}}_{g,n+1}).$$

Correspondingly, the i-th kappa-class κ_i is defined via

$$\kappa_i = \epsilon_*(\psi^{i+1}_{n+1}) \in \mathcal{A}^i(\overline{\mathcal{M}}_{g,n}).$$

The push forwards of the monomials in κ and ψ classes from the boundary strata span the tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$ [5, 6].

The κ, ψ and tautological classes over an open subset $\mathcal{U}$ of the moduli space $\overline{\mathcal{M}}_{g,n}$, and in particular over the smooth part $\mathcal{M}_{g,n}$ and the moduli $\mathcal{M}^c_{g,n}$ of curves of compact type, are defined by restricting the respective classes from $\overline{\mathcal{M}}_{g,n}$ to the corresponding open subset $\mathcal{U} \subset \overline{\mathcal{M}}_{g,n}$. The kappa ring $\kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$ of the moduli space of curves is the subring of the tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$ generated by the kappa classes $\kappa_1, \kappa_2, ...$ over $\mathbb{Q}$. One may define the kappa ring $\kappa^*(\mathcal{U})$ in a similar way. It was observed by Pandharipande [10] that if one restricts attention to the moduli space $\mathcal{M}^c_{g,n}$, the structure of the kappa ring may be largely understood using a combination of combinatorial arguments and localization ideas. Pandharipande shows, consequently, that the rank of the ring $\kappa^*(\mathcal{M}^c_{g,n})$ in degree d is at most $|\mathrm{P}(d, 2g-2+n-d)|$, where $\mathrm{P}(d,r)$ denotes the set of partitions of d into at most r parts.

We use combinatorial observations to obtain a number of lower bounds on the rank of $\kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$. For this purpose, associated with every stable weighted graph (see the second section for the definition) one may define a natural cycle in the Chow ring of $\overline{\mathcal{M}}_{g,n}$. We refer to such cycles as *combinatorial cycles* in this paper. Combinatorial cycles are usually called the *boundary strata* of $\overline{\mathcal{M}}_{g,n}$ in the literature. Correspondingly, we may define the *combinatorial tautological quotient* $R^*_c(\overline{\mathcal{M}}_{g,n})$ to be the vector space obtained as the quotient of the tautological ring by setting trivial the classes which integrate trivially over all such combinatorial cycles. It also makes sense to talk about the combinatorial kappa quotients, denoted by $\kappa^*_c(\overline{\mathcal{M}}_{g,n})$. Since $\kappa^*_c(\overline{\mathcal{M}}_{g,n})$ is naturally a quotient of $\kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$, the rank of the former gives a lower bound on the rank of the latter. It may be useful to note that the image of $\kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$ in the Gorenstein quotient of the tautological ring is (potentially) larger than $\kappa^*_c(\overline{\mathcal{M}}_{g,n})$. In this paper, we prove a number of theorems about the rank of $\kappa^*_c(\overline{\mathcal{M}}_{g,n})$ in different degrees. In particular we show

Theorem 1. *Let $e = 3g-3+n-d \geq 0$. If $n > \min\{g-2, e\}$ the rank of the combinatorial kappa quotient $\kappa^*_c(\overline{\mathcal{M}}_{g,n})$ in degree d, and thus the rank of $\kappa^d_c(\overline{\mathcal{M}}_{g,n})$, is bounded below by $|\mathrm{P}(d, e+1)|$.*

In co-degree 1, $\kappa^*_c(\overline{\mathcal{M}}_{g,n})$ is relatively easy to describe, as presented in the following theorem.

Theorem 2. *For $d = 3g-4+n$ and $g > 1$ the rank of $\kappa^d_c(\overline{\mathcal{M}}_{g,n})$ is equal to*

$$\left\lceil \frac{(n+1)(g+1)}{2} \right\rceil - 1,$$

while for $g = 1$ the rank is equal to $n-1$.

The reason for the difference between genus 1 and higher genus in the statement of Theorem 2 is that there is a relation between the combinatorial divisors of $\overline{\mathcal{M}}_{1,n}$ as far as the kappa classes are concerned. More precisely, let $D_i \simeq \overline{\mathcal{M}}_{1,n-i-1} \times \overline{\mathcal{M}}_{0,i+3}$, $i = 0, 1, ..., n-2$ denote the divisor in $\overline{\mathcal{M}}_{1,n}$ which corresponds to a degeneration of an n-pointed genus 1 curve to an

$(n-i-1)$-pointed genus 1 curve and an $(i+3)$-pointed genus 0 curve. The labeling of the points on either of the two components is not important, although we fix one such labeling. Let $D_{n-1} \simeq \overline{\mathcal{M}}_{0,n+2}$ denote the divisor which corresponds to a degeneration of an n-pointed genus 1 curve to an $(n+2)$-pointed genus 0 curve.

Theorem 3. *For every element $\kappa \in \kappa^{n-1}(\overline{\mathcal{M}}_{1,n})$*

$$\frac{1}{12}\int_{[D_{n-1}]} \kappa = \sum_{i=0}^{n-2} \binom{n-2}{i} \int_{[D_i]} \kappa.$$

A small modification in the proof of Theorem 2, together with the result of Al-Aidroos [1], implies the following theorem about the kappa ring in co-degree 1:

Theorem 4. *If $n > 0$, the rank of $\kappa^{3g+n-4}(\overline{\mathcal{M}}_{g,n})$ is given by*

$$\mathrm{rank}\left(\kappa^{3g+n-4}(\overline{\mathcal{M}}_{g,n})\right) = \begin{cases} n-1 & \text{if } g=1 \\ \left\lceil \frac{3n+1}{2} \right\rceil & \text{if } g=2 \\ \left\lceil \frac{(n+1)(g+1)}{2} \right\rceil + 1 & \text{if } g>2. \end{cases}$$

In particular, Theorem 2 and Theorem 4 imply that the quotient map from the kappa ring to its combinatorial quotient is not an isomorphism for $g > 2$. This should perhaps be compared with the implications of Theorem 1.5 and Theorem 1.8 from [10], where Pandharipande describes an isomorphism

$$\imath_{g,n} : \kappa^d\left(\mathcal{M}^c_{0,2g+n}\right) \to \kappa^d\left(\mathcal{M}^c_{g,n}\right)$$

for $n > 0$. The combinatorial quotient of $\kappa(\mathcal{M}^c_{g,n})$ may be defined by taking its quotient by the vector space spanned by the kappa classes $\kappa \in \kappa(\mathcal{M}^c_{g,n})$ such that

$$\int_{[D]} \lambda_g \kappa = 0 \quad \forall\, [D].$$

The isomorphism $\imath_{g,n}$ shows in particular, that the quotient map from $\kappa(\mathcal{M}^c_{g,n})$ to its combinatorial quotient is an isomorphism. The authors were initially motivated by this result to explore if the map from the kappa ring of the campactified moduli space of pointed curves to its combinatorial quotient is an isomorphism, and the above discussion gives a negative answer to this question.

The permutation group S_n acts on $\overline{\mathcal{M}}_{g,n}$, and thus on the tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$. Theorem 4, together with the work of Al-Aidroos, implies that every tautological class in co-degree 1 which is invariant under the action of S_n on $R^*(\overline{\mathcal{M}}_{g,n})$ belongs to the kappa ring.

When the co-degree $e = 3g-3+n-d$ is arbitrary, a theorem similar to Theorem 2 may be proved for the asymptotic behavior of the rank of the combinatorial kappa quotient $\kappa_c^d(\overline{\mathcal{M}}_{g,n})$, as the number n of the marked points grows large:

Theorem 5. *The rank of the combinatorial kappa quotient $\kappa_c(\overline{\mathcal{M}}_{g,n})$ in co-degree e, as the number n of the marked points becomes large, is asymptotic to*

$$\frac{\binom{n+e}{e}.\binom{g+e}{e}}{(e+1)!}.$$

Note that the quantity $\binom{n+e}{e}.\binom{g+e}{e}/(e+1)!$ describes the asymptotics of the number of boundary strata of codimension e in $\overline{\mathcal{M}}_{g,n}/S_n$.

In [3] we show that the quotient map

$$\kappa^*\left(\overline{\mathcal{M}}_{g,n}\right) \longrightarrow \kappa_c^*(\overline{\mathcal{M}}_{g,n})$$

is an isomorphism for $g \leq 2$. This makes our results on the structure of the combinatorial kappa quotient more relevant when the genus is small. Moreover, localization is used in [3] to bound the rank of $\kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$ from above. Together with Theorem 5 this implies that the rank of $\kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$ in co-degree e, as the number n of the marked points becomes large is asymptotic to $\binom{n+e}{e}.\binom{g+e}{e}/(e+1)!$.

The paper is organized as follows. In Section 2 we introduce the combinatorial cycles and discuss the integration of the κ and ψ classes over them. Each d-dimensional combinatorial cycle $\mathcal{C} \subset \overline{\mathcal{M}}_{g,n}$ gives a linear map $\int_{\mathcal{C}} : \kappa^d(\overline{\mathcal{M}}_{g,n}) \to \mathbb{Q}$. Section 3 is (naively speaking) devoted to finding a list of combinatorial cycles $\mathcal{C}_1, ..., \mathcal{C}_N$ such that there are no *trivial* relations among the functionals $\int_{\mathcal{C}_i}$. This gives a $N \times |\mathrm{P}(d)|$ matrix $R(d;g,n)$, and the rank $r(d;g,n)$ of this matrix gives the rank of the combinatorial kappa quotient in degree d. In Section 4 a strategy for estimating $r(d;g,n)$ from below is described and a quick corollary of this strategy is Theorem 1.

In Section 5 we study the kappa ring in co-degree 1, i.e. $d = 3g-4+n$. Explicit combinatorial formulas for the integration of κ and ψ classes over $\overline{\mathcal{M}}_{1,n}$ are used to prove that there is a non-trivial relation among the functionals $\int_{\mathcal{C}_i}$ when $g = 1$ and $d = n-1$, given by Theorem 3. Furthermore, we show that the rank of $\kappa_c^{n-1}(\overline{\mathcal{M}}_{1,n})$ is $n-1$. For $g > 1$ the computation of the rank of $\kappa_c^{3g-4+n}(\overline{\mathcal{M}}_{g,n})$ is reduced, by the results of Section 4, to the computation of the rank of certain $(g+1) \times (g+1)$ matrices with entries consisting of the integrals of ψ classes. We study the aforementioned matrices, compute their determinant using the KdV, String and Dilation equations, conclude that all of them are full rank and prove Theorem 2. The same matrices re-appear as we study the asymptotic behavior of the rank of $\kappa_c^{3g-3+n-e}(\overline{\mathcal{M}}_{g,n})$ as n goes to infinity in Section 6. Once again, the results of Section 5 are used to prove Theorem 5.

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2. Combinatorial cycles and the κ classes

Let us first consider an alternative basis for the kappa ring of $\overline{\mathcal{M}}_{g,n}$, instead of the kappa classes. Let

$$\pi^k_{g,n} : \overline{\mathcal{M}}_{g,n+k} \to \overline{\mathcal{M}}_{g,n}$$

denote the forgetful map which forgets the last k marked points.

Definition 2.1. For every multi-set $\mathbf{p} = (a_1 \geq a_2 \geq ... \geq a_k)$ of positive integers define

- $|\mathbf{p}| := k$ and $\sigma(\mathbf{p}) := \sum_{i=1}^k a_i$
- $\kappa_{\mathbf{p}} := \kappa_{a_1,\dots,a_k} := \left(\pi^k_{g,n}\right)_* \left(\prod_{i=1}^k \psi_{n+i}^{a_i+1}\right) \in \kappa^* \left(\overline{\mathcal{M}}_{g,n}\right)$
- $\kappa(\mathbf{p}) = \kappa(a_1, ..., a_k) := \prod_{i=1}^k \kappa_{a_i} \in \kappa^* \left(\overline{\mathcal{M}}_{g,n}\right)$.

The classes $\kappa_{\mathbf{p}}$, for $\mathbf{p}$ a multi-set corresponding to a partition of $d = \sigma(\mathbf{p})$ in $\mathrm{P}(d)$, span the kappa ring $\kappa^* \left(\overline{\mathcal{M}}_{g,n}\right)$ in degree d as a $\mathbb{Q}$-vector space. In particular, for every positive integer d, $\kappa_d = \kappa(d)$ is the κ class corresponding to the multi-set consisting of the single element d.

Let $\mathbf{p} = (a_1, a_2, ..., a_k)$ be a multi-set. For $\tau \in S_k$ a permutation over k elements let

$$\tau = \tau_1\tau_2...\tau_r,$$

be the canonical cycle decomposition for τ, including the 1-cycles. Let $\tau(\mathbf{p})_i$, for $i = 1, 2, ..., r$, denote the sum of the elements of $\mathbf{p}$ whose indices correspond to the i-th cycle τ_i. Define

$$\kappa(\tau, \mathbf{p}) := \prod_{i=1}^r \kappa_{\tau(\mathbf{p})_i}.$$

The following general formula is due to Faber and is discussed in [2].

Lemma 2.2. *For every partition* $\mathbf{p} \in \mathrm{P}(d)$ *as above we have*

$$\kappa_{\mathbf{p}} = \sum_{\tau \in S_k} \kappa(\tau, \mathbf{p}).$$

An immediate consequence of the above formula is the following lemma, c.f. lemma 1 in [11].

Lemma 2.3. *There is an invertible* $|\mathrm{P}(d)|\times|\mathrm{P}(d)|$ *matrix* $A_d = (A_{\mathbf{p}}^{\mathbf{q}})_{\mathbf{p},\mathbf{q}\in\mathrm{P}(d)}$, *independent of* g *and* n, *such that*

$$\kappa_{\mathbf{p}} = \sum_{\mathbf{q}\in\mathrm{P}(d)} A_{\mathbf{p}}^{\mathbf{q}}\kappa(\mathbf{q})$$

is satisfied in $\mathcal{A}^d(\overline{\mathcal{M}}_{g,n})$ *for all* g *and* n.

Proof. The transformation is given by Faber's formula from lemma 2.2 above. In the partial ordering of $\mathrm{P}(d)$ by length the transformation is triangular, with 1's on the diagonal, hence invertible. □

Our main tool for proving the independence of the generators in the kappa ring is the integration against the combinatorial cycles.

Definition 2.4. A *weighted graph* G is a finite connected graph with a set $V(G)$ of vertices and a set $E(G)$ of edges, and a weight function

$$\epsilon = \epsilon_G : V(G) \to \mathbb{Z}^{\geq 0} \times 2^{\{1,\dots,n\}},$$

where $2^{\{1,\dots,n\}}$ denotes the set of subsets of $\{1,...,n\}$. For $i \in V(G)$ denote the degree of $i$by $d_i = d(i)$ and let $\epsilon(i) = (g_i, I_i)$. A *stable weighted graph* is a weighted graph G with the property that $\{I_i \mid i \in V(G)\}$ is a partition of $\{1,...,n\}$ and for every vertex $i \in V(G)$, $2g_i + |I_i| + d_i > 2$. If G is a stable weighted graph define $n(G) = n$ and

$$g(G) := \left(\sum_{i\in V(G)} g_i\right) + |E(G)| - |V(G)| + 1.$$

Suppose that G is a stable weighted graph as above. An automorphism ϕ of G consists of a pair of bijective maps $\phi_V : V(G) \to V(G)$ and $\phi_E : E(G) \to E(G)$ and a subset $E_\phi \subset E(G)$ of *twists* such that the following are satisfied.

- For every edge $e \in E(G)$ connecting the vertices $i, j \in V(G)$, $\phi_E(e)$ is an edge connecting $\phi_V(i)$ and $\phi_V(j)$.
- For every $i \in V(G)$, $\epsilon(i) = \epsilon(\phi(i))$.
- Every $e \in E_\phi$ connects a vertex to itself.

If $\phi = (\phi_V, \phi_E, E_\phi)$ and $\psi = (\psi_V, \psi_E, E_\psi)$ are a pair of automorphisms of G, define

$$\psi \circ \phi = (\psi_V \circ \phi_V, \psi_E \circ \phi_E, E_{\psi\circ\phi}),$$

where the subset $E_{\psi\circ\phi} \subset E(G)$ is defined by

$$E_{\psi\circ\phi} := \Big\{e \in E(G) \;\Big|\; \big(e \in E_\phi \text{ and } \phi(e) \notin E_\psi\big) \text{ or } \big(e \notin E_\phi \text{ and } \phi(e) \in E_\psi\big)\Big\}$$

The automorphisms of G form a group which is denoted by $\mathrm{Aut}(G)$.

Associated with a stable weighted graph G we may construct a cycle in the Chow ring of $\overline{\mathcal{M}}_{g(G),n(G)}$ as follows. Associated with a vertex $i \in V(G)$, let $\overline{\mathcal{M}}(i)$ denote the moduli space $\overline{\mathcal{M}}_{g_i,|I_i|+d_i}$ where the labels of the first $|I_i|$ marked points are chosen from I_i. The labels of the last d_i marked points correspond to those edges in $E(G)$ which are adjacent to i. If e is an edge which connects i to a different vertex j of G, one of the last d_i marked points is labeled with e. If e is an edge connecting i to itself, two of the last d_i marked points will be labeled with e^+ and e^-. Let $\mathcal{C}(G)$ denote the product

$$\mathcal{C}(G) = \prod_{i\in V(G)} \overline{\mathcal{M}}(i) = \prod_{i\in V(G)} \overline{\mathcal{M}}_{g_i,|I_i|+d_i}.$$

Any automorphism ϕ of of the stable weighted graph G gives an automorphism of the space $\mathcal{C}(G)$. The automorphism $\phi = (\phi_V, \phi_E, E_\phi)$ takes $\overline{\mathcal{M}}(i)$ to $\overline{\mathcal{M}}(\phi_V(i))$, by taking a marking corresponding to an edge e connecting i to j to a marking corresponding to the edge $\phi_E(e)$, and taking the pair of markings (e^-, e^+) to $(\phi_E(e)^-, \phi_E(e)^+)$ (respectively to $(\phi_E(e)^+, \phi_E(e)^-)$) if $e \notin E_\phi$ (respectively if $e \in E_\phi$). Note that if $\phi_V(i) \neq i$ then $I_i = I_{\phi_V(i)}$ implies that $I_i = I_{\phi_V(i)} = \emptyset$. The group $\mathrm{Aut}(G)$ of automorphisms of G thus acts on $\mathcal{C}(G)$. There is a map $\imath_G$ from the product $\mathcal{C}(G)$ to the moduli space $\overline{\mathcal{M}}_{g(G),n(G)}$, which is defined as follows. Choose a point $(\Sigma_i, \mathbf{z}_i \cup \mathbf{w}_i)_{i\in V(G)}$ of $\mathcal{C}(G)$, where $\mathbf{z}_i$ and $\mathbf{w}_i$ are sets of $|I_i|$ and d_i marked points on the curve Σ_i of genus g_i respectively. For an edge $e \in E(G)$ connecting $i, j \in V(G)$ glue the marked points in $\mathbf{w}_i$ and $\mathbf{w}_j$ corresponding to e to each other. For an edge $e \in E(G)$ connecting $i \in V(G)$ to itself glue the marked points in $\mathbf{w}_i$ which are labeled with e^- and e^+ to each other. The result is a stable curve

$$\imath_G\left((\Sigma_i, \mathbf{z}_i \cup \mathbf{w}_i)_{i\in V(G)}\right) \in \overline{\mathcal{M}}_{g(G),n(G)}.$$

The map $\imath_G$ respects the action of $\mathrm{Aut}(G)$, and gives an embedding of $\mathcal{C}(G)/\mathrm{Aut}(G)$ in $\overline{\mathcal{M}}_{g(G),n(G)}$. Thus, a stable weighted graph G determines a cycle

$$(\imath_G)_*[\mathcal{C}(G)] = |\mathrm{Aut}(G)|.(\imath_G)_*\left[\frac{\mathcal{C}(G)}{\mathrm{Aut}(G)}\right] \in \mathcal{A}_d\left(\overline{\mathcal{M}}_{g(G),n(G)}\right),$$

where $d = 3g(G) - 3 + n(G) - |E(G)|$. This cycle is denoted by $[G]$, by slight abuse of notation.

Example 2.5. *Let G be the stable weighted graph which consists of a vertex v and two edges e_1 and e_2 which connect v to itself, while $\epsilon(v) = (0,0)$. In this case $\mathcal{C}(G) = \overline{\mathcal{M}}_{0,4}$ where the marked points are labeled with e_1^-, e_1^+, e_2^- and e_2^+. We may define an automorphism ϕ by setting*

$$\phi_V(v) = v, \phi_E(e_1) = e_2, \phi_E(e_2) = e_1 \quad \textit{and} \quad E_\phi = \{e_1\}.$$

The action of ϕ on $\overline{\mathcal{M}}_{0,4}$ is then given by

$$\phi(e_1^+) = e_2^-,\ \phi(e_1^-) = e_2^+,\ \phi(e_2^+) = e_1^+ \ \textit{and}\ \phi(e_2^-) = e_1^-.$$

Note that ϕ^2 fixes v, e_1 and e_2, while $E_{\phi^2} = \{e_1, e_2\}$. Moreover, ϕ is an element of order 4, and that the automorphism group consists of 8 elements.

For a stable weighted graph G let us assume that $n = n(G)$ and $g = g(G)$. Let $\kappa_{\mathbf{p}} = \kappa_{a_1,\dots,a_k}$ be a κ class. If

$$\sigma(\mathbf{p}) + |E(G)| = 3g - 3 + n$$

we may integrate $\kappa_{\mathbf{p}}$ over $[G] \in \mathcal{A}_{3g-3+n-|\mathbf{p}|}(\overline{\mathcal{M}}_{g,n})$. The integral

$$\Big\langle \kappa_{\mathbf{p}}\ ,\ [G] \Big\rangle = \int_{[G]} \kappa_{\mathbf{p}} = \int_{(\pi^k_{g,n})^*[G]} \left(\prod_{j=1}^{k} \psi_{n+j}^{a_j+1} \right) \in \mathbb{Q}$$

may be computed in terms of the integrals of the top degree ψ classes over the moduli spaces $\overline{\mathcal{M}}_{g,n}$ as follows. The components of $(\pi^k_{g,n})^*[G]$ are indexed by the functions

$$j : \big\{1, 2, ..., k\big\} \longrightarrow V(G).$$

Here $j(p)$ corresponds to the component $\overline{\mathcal{M}}(j(p))$ of $\mathcal{C}(G)$ which contains the image of the $(n+p)$-th marked point under the forgetful map. Let us denote the space of all such maps by $[k, G]$. For $j \in [k, G]$ let us denote the corresponding component of $(\pi^k_{g,n})^*[G]$ by $[G, j]$. Define

$$\psi_i(\mathbf{p}, j) := \prod_{p \in j^{-1}(i)} \psi_{n+p}^{a_p+1}, \quad \forall\ j \in [k, G],\ i \in V(G).$$

We may then compute

(1)

$$\begin{aligned} \int_{(\pi^k_{g,n})^*[G]} \left(\prod_{j=1}^{k} \psi_{n+j}^{a_j+1} \right) &= \sum_{j \in [k,G]} \int_{[G,j]} \left(\prod_{j=1}^{k} \psi_{n+j}^{a_j+1} \right) \\ &= \sum_{j \in [k,G]} \left(\prod_{i \in V(G)} \int_{\left[\overline{\mathcal{M}}_{g_i, |I_i| + d_i + |j^{-1}(i)|}\right]} \psi_i(\mathbf{p}, j) \right). \end{aligned}$$

The degree of $\psi_i(\mathbf{p}, j)$ may be computed as

$$\deg\left(\psi_i(\mathbf{p}, j)\right) = \sum_{p \in j^{-1}(i)} (a_p + 1) = \left|j^{-1}(i)\right| + \sum_{p \in j^{-1}(i)} a_p.$$

The integrals in the last line of (1) are trivial unless the degree of $\psi_i(\mathbf{p}, j)$ matches the dimension of $\overline{\mathcal{M}}_{g_i, |I_i| + d_i + |j^{-1}(i)|}$, i.e. if and only if

$$\sum_{p \in j^{-1}(i)} a_p = 3g_i - 3 + |I_i| + d_i.$$

Let

$$Q = \Big\{ (g, n) \in \mathbb{Z}^{\geq 0} \times \mathbb{Z}^+ \ \big|\ 2g + n > 2 \Big\}.$$

If $V(G) = \{1, ..., M\}$, the modified weight multi-set associated with G is the multi-set

$$\mathbf{q}_G := \Big(\theta_G(i) \in Q \;\Big|\; i \in \{1, ..., M\}\Big), \quad \text{where}$$
$$\theta_G(i) := (g_i, m_i = |I_i| + d_i), \quad \forall\, 1 \le i \le M.$$

In other words, $\theta_G(i)$ records the genus and the total number of the marked points (i.e. marked points and nodes) on the component corresponding to the vertex i, and $\mathbf{q}_G$ records all the pairs corresponding to the vertices of G (in a sense) regardless of the structure of G as a graph. We may assume that $3g_i - 3 + m_i > 0$ for $i = 1, ..., k$ and $(g_i, m_i) = (0, 3)$ for $i = k+1, ..., M$.

Moreover, the value computed in equation 1 does not depend on the graph G, and only depends on the modified weight multi-set $\mathbf{q}_G$. We define $\langle \mathbf{p}, \mathbf{q}_G \rangle := \langle \kappa_{\mathbf{p}}, [G] \rangle$. In fact, the quantity $\langle \mathbf{p}, \mathbf{q} \rangle$ may be defined for every multi-set $\mathbf{q}$ of the elements of Q.

Let $\mathbf{p}(\mathbf{q})$ denote the multi-set $(3g_i - 3 + m_i)_{i=1}^k$, which corresponds to a partition of $d = \dim([G])$. For a partition $\mathbf{p} \in \mathrm{P}(d)$, let $\mathrm{Q}(\mathbf{p}; g, n)$ denote the set of all multi-sets $\mathbf{q} = \{(g_i, m_i)\}_{i=1}^M$ of elements of Q such that $\mathbf{p}(\mathbf{q}) = \mathbf{p}$ and there is a stable weighted graph G with

$$\mathbf{q} = \mathbf{q}_G, \quad g = g(G), \quad \text{and} \quad n = n(G).$$

If $\mathbf{p} = (a_1 \ge a_2 \ge ... \ge a_k > 0)$ is a partition of d and $\mathbf{q} \in \mathrm{Q}(\mathbf{p}; g, n)$, after possible re-arrangement of the indices we have

- $M \ge k, \quad (g_i, m_i) = (0, 3)$ for $k < i \le M$.
- $0 \le g_i \le \left\lfloor \frac{a_i+2}{3} \right\rfloor$ for $1 \le i \le k$.
- $(k + d + 2) - (2g + n) \le \sum_{i=1}^k g_i \le g$.

The last inequality follows since

$$g = \left(\sum_{i=1}^k g_i\right) + |E(G)| - |V(G)| + 1 = \left(\sum_{i=1}^k g_i\right) + (3g - 3 + n - d) - M + 1,$$

and $M \ge k$.

Lemma 2.6. *For every $\mathbf{p} = (a_1 \ge a_2 \ge ... \ge a_k > 0) \in P(d)$ and every multi-set $\mathbf{q}$ of M elements $\{\theta_i = (g_i, m_i)\}_{i=1}^M$ in Q the following are true.*

(1) *The elements of $\mathrm{Q}(\mathbf{p}; g, n)$ are in one-to-one correspondence with the choice of genera $(g_1, ..., g_k)$ satisfying the following two relations.*

$$0 \le g_i \le \left\lfloor \frac{a_i + 2}{3} \right\rfloor \quad \forall \;\; i = 1, ..., k,$$
$$(d + k + 2) - (2g + n) \le \sum_{i=1}^k g_i \le g.$$

(2) *For every integer $e \geq M-1$ such that*

$$\sum_{i=1}^{M} m_i \geq 2e$$

there is a connected stable weighted graph $G = G(\mathbf{q}, e)$ with e edges such that $\mathbf{q} = \mathbf{q}_G$.

Proof. We have already seen that for $\mathbf{q} \in \mathrm{Q}(\mathbf{p}; g, n)$ the properties stated before Lemma 2.6 are satisfied. This immediately gives the genera $(g_1, ..., g_k)$. On the other hand, if $(g_1, ..., g_k)$ are given as above, one may set $m_i := a_i - 3g_i + 3$, for $i = 1, ..., k$, and $(g_i, m_i) = (0, 3)$ for $k < i \leq M$, where

$$M = 2g - 2 + n + \sum_{i=1}^{k} (g_i - a_i) \geq k.$$

Consider the multi-set $\mathbf{q} = ((g_i, m_i))_{i=1}^{M}$. The desired stable weighted graph G corresponding to $\mathbf{q}$ should have $e = 3g - 3 + n - d$ edges. Since

$$\sum_{i=1}^{M} m_i = 3M + d - 3\sum_{i=1}^{k} g_i = n + 2e \geq 2e,$$

the second part of the lemma implies that there is a stable weighted graph G which corresponds to $\mathbf{q}$. It thus suffices to prove the second claim of the lemma.

Suppose that a multi-set $\mathbf{q} = (g_i, m_i)_{i=1}^{M}$ is given as above, and that $\sum_{i=1}^{M} m_i \geq 2e \geq (2M-2)$. Assume that m_1 is the smallest of all m_i. If $m_1 \geq 2$ let A be a set of $\sum_{i=1}^{M} m_i$ elements

$$A = \{(i, j) | i = 1, ..., M, j = 1, ..., m_i\},$$

and B denote a set of e (disjoint) pairs of elements of A, where $M - 1$ of the pairs are the following

$$((1, 2), (2, 1)), ((2, 2), (3, 1)), ..., ((M-1, 2), (M, 1)).$$

This is possible since the total number of points in A is at least $2e$, $e \geq M-1$, and each m_i is at least 2. Let G be the graph with vertices $V(G) = \{1, ..., M\}$ and $E(G)$ be a set of e edges, where for every pair $((i, p_i), (j, p_j)) \in B$ we draw an edge between i and j in $E(G)$. Finally, let $\epsilon_G(i) = (g_i, I_i)$ where I_i is a set of $m_i - d_i$ elements. It is straightforward to check that G, together with the weight function ϵ_G is a stable weighted graph .

If $m_1 = 1$, let

$$\mathbf{q}' = (g_i, m_i)_{i=2}^{M}, \quad e' = e - 1.$$

Clearly, $e' \geq M-2$, and using an induction on the number of vertices, there is a corresponding stable weighted graph G' with e' edges and $\mathbf{q}' = \mathbf{q}_{G'}$.

Suppose that the vertex $i \in \{2, ..., M\}$ has degree d_i in G' and assume that $\epsilon_{G'}(i) = (g_i, I_i)$ where I_i contains n_i elements. Thus we have $m_i = d_i + n_i$. Note that

$$\begin{aligned} & 2e-2 = \left(\sum_{i=1}^{M} m_i\right) - \left(\sum_{i=2}^{M} n_i\right) - 1 \geq 2e - \left(\sum_{i=2}^{M} n_i\right) - 1 \\ & \Rightarrow \quad \sum_{i=2}^{M} n_i \geq 1. \end{aligned} \tag{2}$$

Let G be the stable weighted graph obtained by adding a vertex 1 to G', assigning the genus g_1 to 1, and letting $I_1 = \emptyset$. We attach one edge connecting 1 to one of the vertices corresponding to the non-zero values of n_i, $i = 2, ..., M$. The last inequality in (2) implies that this is always possible. The stable weighted graph G will then have the required properties. This completes the argument in the second case (i.e. $m_1 = 1$) by induction on the number M of vertices.

□

3. Preliminaries on combinatorial cycles

Let us assume that the genus g, the number n of the marked points and the degree d are given as before. Set $e = 3g-3+n-d$, and let $\mathrm{Q}(d; g, n)$ denote the union of $\mathrm{Q}(\mathbf{p}; g, n)$ for $\mathbf{p} \in \mathrm{P}(d)$, and $\mathrm{P}(d; g, n)$ denote the subset of $\mathrm{P}(d)$ consisting of $\mathbf{p} \in \mathrm{P}(d)$ which are of the form $\mathbf{p}(\mathbf{q})$ for some $\mathbf{q} \in \mathrm{Q}(d; g, n)$.

Proposition 3.1. *Fix the co-degree $e = (3g-3+n) - d \geq 0$. If $n > \min\{g-2, e\}$ every partition of d into at most $e+1$ elements can be realized as the partition $\mathbf{p}(\mathbf{q}_G)$ for some stable weighted graph G, i.e.*

$$\mathrm{P}(d; g, n) = \mathrm{P}(d, e+1).$$

Proof. Let us assume that $n \geq \min\{e+1, g-1\}$. Since the stable weighted graphs corresponding to $\mathrm{P}(d; g, n)$ have e edges and are connected, they can have at most $e+1$ vertices, implying the inclusion $\mathrm{P}(d; g, n) \subset \mathrm{P}(d, e+1)$.

For the reverse inclusion, first suppose that $d \leq n + 2g - 2$, and that a partition $\mathbf{p} = (a_1, ..., a_k)$ of d with $k \leq 3g - 2 + n - d$ is given. Define

$$r := \min\{g, k\}, \quad M := (2g-2+n-d) + \min\{g, k\}$$

$$g_i := \begin{cases} 1 & \text{if} \quad 1 \leq i \leq r \\ 0 & \text{if} \quad r < i \leq M \end{cases} \quad \text{and} \quad m_i := \begin{cases} a_i & \text{if} \quad 1 \leq i \leq r \\ a_i + 3 & \text{if} \quad r < i \leq k \\ 3 & \text{if} \quad k < i \leq M \end{cases},$$

$$\mathbf{q} := \big((g_i, m_i) \bigm| i \in \{1, ..., M\}\big).$$

The assumptions on d and k imply that $r \geq 0$, while $\mathbf{p}(\mathbf{q}) = \mathbf{p}$. Moreover,

$$\sum_{i=1}^{M} m_i = d + 3(M - r) = n + 2(3g - 3 + n - d) \geq 2(3g - 3 + n - d).$$

Then Lemma 2.6 implies that $\mathbf{q} \in \mathrm{Q}(d; g, n)$, and consequently, $\mathbf{p} \in \mathrm{P}(d; g, n)$.

Thus, it suffices to prove the inclusion for $d = 2g - 2 + n + r$, where $0 < r < g$, i.e. to show that

$$\mathrm{P}(d, g - r) \subset \mathrm{P}(d; g, n).$$

Let $\mathbf{p} = (a_1 \geq ... \geq a_k > 0)$ be an element of $\mathrm{P}(d, g - r)$. We first claim that

$$\sum_{i=1}^{k} \left\lfloor \frac{a_i - 1}{3} \right\rfloor \geq r.$$

If the above inequality is not satisfied

$$\begin{aligned} & r - 1 \geq \sum_{i=1}^{k} \left\lfloor \frac{a_i - 1}{3} \right\rfloor \geq \sum_{i=1}^{k} \left(\frac{a_i}{3} - 1\right) = \frac{d}{3} - k \\ \Rightarrow \quad & 2g - 2 + n + r = d \leq 3(k + r) - 3 \leq 3g - 3 \\ \Rightarrow \quad & \begin{cases} g - 2 \geq n + r - 1 \geq n & \text{and} \\ e = g - r - 1 \geq n. \end{cases} \end{aligned}$$

Thus $n \leq \min\{e, g - 2\}$. This contradiction proves the claim.

Choose the integers $0 \leq \epsilon_i \leq \lfloor \frac{a_i - 1}{3} \rfloor$, $i = 1, ..., k$ so that

$$\left(\sum_{i=1}^{k} \left\lfloor \frac{a_i - 1}{3} \right\rfloor\right) - \left(\sum_{i=1}^{k} \epsilon_i\right) = r.$$

Set

$$\begin{aligned} & g_i := \left\lfloor \frac{a_i - 1}{3} \right\rfloor + 1 - \epsilon_i \quad \text{and} \\ & m_i := a_i - 3 \left\lfloor \frac{a_i - 1}{3} \right\rfloor + 3\epsilon_i > 0, \\ \Rightarrow \quad & 3(g_i - 1) + m_i = a_i, \quad i = 1, ..., k. \end{aligned}$$

Note that $(g_i, m_i) \in Q$ and $\mathbf{q} = (g_i, m_i)_{i=1}^{k}$ is a multi-set with $\mathbf{p} = \mathbf{p}(\mathbf{q})$. We have $n = n(\mathbf{q}, e)$ and $g = g(\mathbf{q}, e)$. Moreover,

$$\sum_{i=1}^{k} m_i = d - 3r = 2(g - 1 - r) + n \geq 2(g - 1 - r) = 2e,$$

and Lemma 2.6 implies that $\mathbf{q} \in \mathrm{Q}(d; g, n)$. Thus $\mathbf{p} \in \mathrm{P}(d; g, n)$, and the proof is complete. □

4. The general strategy for obtaining lower bounds

Let us define a partial order on $\mathrm{P}(d)$ by setting $\mathbf{p}_1 \lhd \mathbf{p}_2$ if $\mathbf{p}_1$ refines $\mathbf{p}_2$. Thus, $\langle \mathbf{p}, \mathbf{q} \rangle$ is non-zero only if $\mathbf{p}$ refines $\mathbf{p}(\mathbf{q})$. For $\mathbf{q} = ((g_i, m_i))_{i=1}^{M}$

$$\Lambda(\mathbf{q}) := \frac{1}{|\mathrm{Aut}(\mathbf{p}(\mathbf{q}))|} \langle \mathbf{p}(\mathbf{q}), \mathbf{q} \rangle = \prod_{i=1}^{M} \frac{1}{24^{g_i} \times g_i!} \neq 0,$$

where $\mathrm{Aut}(\mathbf{p}(\mathbf{q}))$ denotes the set of automorphisms of $\mathbf{p}(\mathbf{q})$ as a multi-set. The rank $r(d; g, n)$ of the matrix

$$R(d; g, n) := \left(\frac{1}{\Lambda(\mathbf{q})} \Big\langle \mathbf{p}, \mathbf{q} \Big\rangle \right)_{\substack{\mathbf{p} \in \mathrm{P}(d) \\ \mathbf{q} \in \mathrm{Q}(d; g, n)}}$$

is a lower bound for the rank of $\kappa^d(\overline{\mathcal{M}}_{g,n})$. Let $\langle \kappa_1, ..., \kappa_{3g-3+n} \rangle_{\mathbb{Q}}^d$ denote the $\mathbb{Q}$-vector space of degree d polynomials in $\kappa_1, \kappa_2, \ldots$. The matrix $R(d; g, n)$ gives a surjective linear map

$$R(d; g, n) : \Big\langle \kappa_1, ..., \kappa_{3g-3+n} \Big\rangle_{\mathbb{Q}}^d \longrightarrow \mathbb{Q}^{r(d; g, n)},$$

and thus a surjection

$$\jmath_{g,n}^d : \kappa^d \left(\overline{\mathcal{M}}_{g,n} \right) \longrightarrow \frac{\Big\langle \kappa_1, ..., \kappa_{3g-3+n} \Big\rangle_{\mathbb{Q}}^d}{\mathrm{Ker}\left(R(d; g, n) \right)}.$$

Definition 4.1. We call a class $\alpha \in R^*(\overline{\mathcal{M}}_{g,n})$ *combinatorially trivial* if for every stable weighted graph G

$$\int_{[G]} \alpha = 0.$$

We denote the quotients of $R^*(\overline{\mathcal{M}}_{g,n})$ and $\kappa^* \left(\overline{\mathcal{M}}_{g,n} \right)$ by combinatorially trivial tautological classes by $R_c^*(\overline{\mathcal{M}}_{g,n})$ and $\kappa_c^* \left(\overline{\mathcal{M}}_{g,n} \right)$ respectively.

In other words, there is an induced isomorphism of vector spaces

$$\jmath_{g,n}^d : \kappa_c^d \left(\overline{\mathcal{M}}_{g,n} \right) \longrightarrow \frac{\Big\langle \kappa_1, ..., \kappa_{3g-3+n} \Big\rangle_{\mathbb{Q}}^d}{\mathrm{Ker}\left(R(d; g, n) \right)}.$$

We would now like to describe a general strategy for achieving lower bounds for the rank of $\kappa^d(\overline{\mathcal{M}}_{g,n})$. The strategy will be implemented in a few cases in the following sections.

Definition 4.2. Fix the genus g and the number n of the marked points for $\overline{\mathcal{M}}_{g,n}$, as well as the degree d and set $e = 3g - 3 + n - d$. Let $<$ denote a total ordering on $\mathrm{P}(d, e+1)$ which refines the partial order $\lhd$. A *fine assignment* (with respect to a subset P of $\mathrm{P}(d)$ and the total order $<$) is a function $f : P \to \mathrm{P}(d, e+1)$ satisfying the following properties:

$$\begin{aligned}&(1)\ \mathbf{p} \lhd f(\mathbf{p})\\ &(2)\ \mathbf{p} \lhd \mathbf{p}',\quad \mathbf{p}' \in \mathrm{P}(d, e+1) \quad \Rightarrow \quad f(\mathbf{p}) < \mathbf{p}'.\end{aligned}$$

Fix a fine assignment $f : P \to \mathrm{P}(d, e+1)$. Consider a block decomposition of $R(d; g, n)$ where, for every $\mathbf{p}_0 \in \mathrm{P}(d, e+1)$, the rows corresponding to all $\mathbf{q} \in \mathrm{Q}(\mathbf{p}_0; g, n)$ belong to the same block, while the columns corresponding to all $\mathbf{p} \in f^{-1}(\mathbf{p}_0)$ belong to the same block as well. In particular, associated with every such $\mathbf{p}_0$ we may introduce the matrix

$$R_f(\mathbf{p}_0; g, n) = \left(\frac{1}{\Lambda(\mathbf{q})}\langle \mathbf{p}, \mathbf{q}\rangle\right)_{\substack{\mathbf{q}\in \mathrm{Q}(\mathbf{p}_0;g,n)\\ \mathbf{p}\in f^{-1}(\mathbf{p}_0)}},$$

and will denote its rank by $r_f(\mathbf{p}_0; g, n)$.

Lemma 4.3. *If $f : P \to \mathrm{P}(d, e+1)$ is a fine assignment as above, then*

$$r(d; g, n) \geq \sum_{\mathbf{p}\in \mathrm{P}(d,e+1)} r_f(\mathbf{p}; g, n).$$

Proof. The fine assignment f determines a block decomposition of a sub-matrix of $R(d; g, n)$ which is upper triangular with respect to the order $<$. Since the matrices $R_f(\mathbf{p}; g, n)$ correspond to the diagonal in this block form, the above lemma follows. □

Remark 4.4. More generally, let I be a totally ordered set with the order $<$ and $f_1 : \mathrm{P}(d; g, n) \to I$ and $f_2 : \mathrm{Q}(d; g, n) \to I$ be surjective functions so that

- For $\mathbf{p}, \mathbf{p}' \in \mathrm{P}(d; g, n)$ with $\mathbf{p} \lhd \mathbf{p}'$, $f_1(\mathbf{p}) < f_1(\mathbf{p}')$.
- For $\mathbf{q}, \mathbf{q}' \in \mathrm{Q}(d; g, n)$ with $\mathbf{p}(\mathbf{q}) \lhd \mathbf{p}(\mathbf{q}')$, $f_2(\mathbf{q}) < f_2(\mathbf{q}')$.
- For $\mathbf{p} \in \mathrm{P}(d; g, n)$ and $\mathbf{q} \in \mathrm{Q}(d; g, n)$ with $\mathbf{p} \lhd \mathbf{p}(\mathbf{q})$, $f_1(\mathbf{p}) < f_2(\mathbf{p})$.

Then I determines a block decomposition of $R(d; g, n)$, and $R(d; g, n)$ is upper triangular with respect to this decomposition. Thus

$$r(d; g, n) \leq \sum_{p\in P} \mathrm{rank}\left(R_{f_1,f_2}(p)\right)$$

where $R_{f_1,f_2}(p)$ is the sub-matrix of $R(d; g, n)$ determined by the columns corresponding to $f_1^{-1}(p)$ and the rows corresponding to $f_2^{-1}(p)$.

The restriction of every fine assignment to $P \cap \mathrm{P}(d, e+1)$ is the identity. Theorem 1 is now an immediate consequence of Lemma 4.3.

Corollary 4.5. *If $n > \min\{g-2, e\}$ the rank of $\kappa_c^d(\overline{\mathcal{M}}_{g,n})$ is greater than or equal to $p(d, e+1)$.*

Proof. Take $P = \mathrm{P}(d; g, n) = \mathrm{P}(d, e+1)$, $f : P \to P$ the identity map, and $<$ any refinement of $\lhd$. Since $r_f(\mathbf{p}; g, n) = 1$ for all $\mathbf{p} \in P$, we are done. □

5. The combinatorial kappa quotient in co-degree one

In this section, we apply Lemma 4.3 to the study of the rank of $\kappa_c^{3g-4+n}(\overline{\mathcal{M}}_{g,n})$. We will first handle the case $g=1$ using explicit formulas for the integrals of the ψ classes.

5.1. **The combinatorial kappa quotient of $\overline{\mathcal{M}}_{1,n}$ in co-degree one.** Denote the set of k-element subsets of $N=\{1,...,n\}$ by $\binom{N}{k}$. For every n-tuple of real numbers $a_1,...,a_n$, denote their i-th symmetric product by $\sigma_i(a_1,...,a_n)$. In other words, $\sigma_0(a_1,...,a_n)=1$ and

$$\sigma_i(a_1,...,a_n)=\sum_{J\in\binom{N}{i}}\left(\prod_{j\in J}a_j\right).$$

The following theorem is essentially proved as Theorem 2.3 in [4].

Theorem 5.1. *Suppose that the non-negative integers $a_1,...,a_n$ are given so that $a_1+...+a_n=n$. Then*

$$\int_{\overline{\mathcal{M}}_{1,n}}\prod_{j=1}^{n}\psi_j^{a_j}=\frac{1}{24}\binom{n}{a_1,...,a_n}\left(1-\sum_{i=2}^{n}\frac{\sigma_i(a_1,...,a_n)}{i(i-1)\binom{n}{i}}\right). \tag{3}$$

Consider the graphs shown in Figure 1 together with the illustrated weight functions, which determine stable weighted graphs G_i $i=0,...,n-1$. With the notation of the introduction $D_i=[G_i]$ for $i=0,...,n-2$, while $D_{n-1}=2[G_{n-1}]$, since $|\mathrm{Aut}(G_{n-1})|=2$.

Theorem 5.2. *For every element $\kappa\in\kappa^{n-1}(\overline{\mathcal{M}}_{1,n})$*

$$\frac{1}{24}\int_{[G_{n-1}]}\kappa=\sum_{i=0}^{n-2}\binom{n-2}{i}\int_{[G_i]}\kappa. \tag{4}$$

Proof. Let $\tilde{\mathbf{p}}=(b_1,...,b_k)\in\mathrm{P}(n-1)$ be a partition of $n-1$ of length k and set $\mathbf{p}=(a_1,...,a_k)$ where $a_i=b_i+1$ for $i=1,...,k$. Set

$$F(\mathbf{p})=F(a_1,...,a_k)=\int_{[G_{n-1}]}\kappa_{\tilde{\mathbf{p}}}-24\sum_{i=0}^{n-2}\binom{n-2}{i}\int_{[G_i]}\kappa_{\tilde{\mathbf{p}}}.$$

With $N=\{1,...,k\}$, $I^\circ=N-I$ for every $I\subset N$, and following the notation set in the proof of Theorem 5.1

$$F(\mathbf{p})=\binom{\sigma(N)}{\mathbf{p}(N)}-\sum_{I\subset N}\binom{\sigma(N)-|N|-1}{\sigma(I)-|I|-1}\binom{\sigma(I)}{\mathbf{p}(I)}\binom{\sigma(I^\circ)}{\mathbf{p}(I^\circ)}$$
$$+\sum_{J\subset I\subset N}(|J|-2)!\binom{\sigma(N)-|N|-1}{\sigma(I)-|I|-1}\binom{\sigma_J(I)}{\mathbf{p}_J(I)}\binom{\sigma(I^\circ)}{\mathbf{p}(I^\circ)}.$$

The above equation may be used to define the function F for every partition $\mathbf{p}$ (relaxing the condition $a_i > 1$ for $i = 1, ..., k$). Set

$$\mathbf{p}_i = (a_1, ..., a_{i-1}, a_i - 1, a_{i+1}, ..., a_k).$$

Assuming $a_i > 1$ for $i = 1, ..., k$,

$$\begin{aligned}
F(\mathbf{p}_i) =& \frac{a_i}{\sigma(N)}\binom{\sigma_i(N)}{\mathbf{p}_i(N)} - \sum_{i\in I\subset N} \frac{a_i\lambda(I)}{\sigma(I)\lambda(N)}\binom{\lambda(N)}{\lambda(I)}\binom{\sigma(I)}{\mathbf{p}(I)}\binom{\sigma(I^\circ)}{\mathbf{p}(I^\circ)} \\
& - \sum_{i\in I^\circ\subset N} \frac{a_i(\lambda(I^\circ)+1)}{\sigma(I^\circ)\lambda(N)}\binom{\lambda(N)}{\lambda(I)}\binom{\sigma(I)}{\mathbf{p}(I)}\binom{\sigma(I^\circ)}{\mathbf{p}(I^\circ)} \\
& + \sum_{\substack{J\subset I\subset N \\ i\in J}} \frac{(a_i-1)\lambda(I)}{\sigma_J(I)\lambda(N)}(|J|-2)!\binom{\lambda(N)}{\lambda(I)}\binom{\sigma_J(I)}{\mathbf{p}_J(I)}\binom{\sigma(I^\circ)}{\mathbf{p}(I^\circ)} \\
& + \sum_{\substack{J\subset I\subset N \\ i\in I-J}} \frac{a_i\lambda(I)}{\sigma_J(I)\lambda(N)}(|J|-2)!\binom{\lambda(N)}{\lambda(I)}\binom{\sigma_J(I)}{\mathbf{p}_J(I)}\binom{\sigma(I^\circ)}{\mathbf{p}(I^\circ)} \\
& + \sum_{\substack{J\subset I\subset N \\ i\in I^\circ}} \frac{a_i(\lambda(I^\circ)+1)}{\sigma(I^\circ)\lambda(N)}(|J|-2)!\binom{\lambda(N)}{\lambda(I)}\binom{\sigma_J(I)}{\mathbf{p}_J(I)}\binom{\sigma(I^\circ)}{\mathbf{p}(I^\circ)}.
\end{aligned}$$

Summing over $i = 1, ..., k$ we obtain

$$\begin{aligned}
\sum_{i=1}^{k} F(\mathbf{p}_i) =& \binom{\sigma(N)}{\mathbf{p}(N)} - \sum_{I\subset N} \frac{\lambda(I)+(\lambda(I^\circ)+1)}{\lambda(N)}\binom{\lambda(N)}{\lambda(I)}\binom{\sigma(I)}{\mathbf{p}(I)}\binom{\sigma(I^\circ)}{\mathbf{p}(I^\circ)} \\
& + \sum_{J\subset I\subset N} \frac{\lambda(I)+(\lambda(I^\circ)+1)}{\lambda(N)}(|J|-2)!\binom{\lambda(N)}{\lambda(I)}\binom{\sigma_J(I)}{\mathbf{p}_J(I)}\binom{\sigma(I^\circ)}{\mathbf{p}(I^\circ)}.
\end{aligned}$$

Thus

$$\sum_{i=1}^{k} F(\mathbf{p}_i) = F(\mathbf{p}). \tag{5}$$

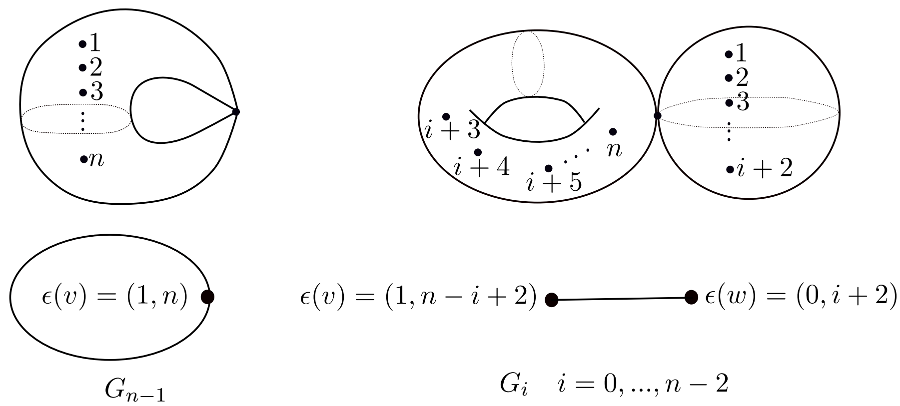

FIGURE 1. The stable weighted graph G_i for $i = 0, ..., n-1$ is illustrated. Each stable weighted graph corresponds to a divisor in $\overline{\mathcal{M}}_{1,n}$

If $\widehat{\mathbf{p}} = (a_1, ..., a_k, 1)$ is obtained by adding a 1 to $\mathbf{p}$,

$$
\begin{aligned}
F(\widehat{\mathbf{p}}) =& (\sigma(N)+1)\binom{\sigma(N)}{\mathbf{p}(N)} - \sum_{\substack{I\subset N\\ \widehat{I}=I\cup\{k+1\}}} (\sigma(I)+1)\binom{\lambda(N)}{\lambda(I)}\binom{\sigma(I)}{\mathbf{p}(I)}\binom{\sigma(I^\circ)}{\mathbf{p}(I^\circ)}\\
& - \sum_{\substack{I\subset N\\ \widehat{I}=I}} (\sigma(I^\circ)+1)\binom{\lambda(N)}{\lambda(I)}\binom{\sigma(I)}{\mathbf{p}(I)}\binom{\sigma(I^\circ)}{\mathbf{p}(I^\circ)}\\
& + \sum_{\substack{J\subset I\subset N\\ \widehat{J}=J\cup\{k+1\},\widehat{I}=I\cup\{k+1\}}} \left((|J|-1)!\binom{\lambda(N)}{\lambda(I)}\binom{\sigma_J(I)}{\mathbf{p}_J(I)}\binom{\sigma(I^\circ)}{\mathbf{p}(I^\circ)}\right)\\
& + \sum_{\substack{J\subset I\subset N\\ \widehat{J}=J,\widehat{I}=I\cup\{k+1\}}} (\sigma_J(I)+1)\left((|J|-2)!\binom{\lambda(N)}{\lambda(I)}\binom{\sigma_J(I)}{\mathbf{p}_J(I)}\binom{\sigma(I^\circ)}{\mathbf{p}(I^\circ)}\right)\\
& + \sum_{\substack{J\subset I\subset N\\ \widehat{J}=J,\widehat{I}=I}} (\sigma(I^\circ)+1)\left((|J|-2)!\binom{\lambda(N)}{\lambda(I)}\binom{\sigma_J(I)}{\mathbf{p}_J(I)}\binom{\sigma(I^\circ)}{\mathbf{p}(I^\circ)}\right)\\
& + \sum_{\substack{J=\{i\}\subset I\subset N\\ \widehat{J}=J\cup\{i,k+1\},\widehat{I}=I\cup\{k+1\}}} \frac{a_i}{\sigma(I)}\binom{\lambda(N)}{\lambda(I)}\binom{\sigma(I)}{\mathbf{p}(I)}\binom{\sigma(I^\circ)}{\mathbf{p}(I^\circ)}\\
& (\sigma(N)+1)F(\mathbf{p}) - \sum_{I\subset N}\binom{\lambda(N)}{\lambda(I)}\binom{\sigma(I)}{\mathbf{p}(I)}\binom{\sigma(I^\circ)}{\mathbf{p}(I^\circ)}\\
& + \sum_{J=\{i\}\subset I\subset N} \frac{a_i}{\sigma(I)}\binom{\lambda(N)}{\lambda(I)}\binom{\sigma(I)}{\mathbf{p}(I)}\binom{\sigma(I^\circ)}{\mathbf{p}(I^\circ)}\\
& \Rightarrow \qquad F(\widehat{\mathbf{p}}) = (\sigma(N)+1)F(\mathbf{p}).
\end{aligned}
$$

The above computation, together with (5), reduce the proof to the case where $k = 1$ and $a_1 = 1$, which is straight forward. □

Remark 5.3. The above proof uses the combinatorial formulas for the terms appearing in $F(\mathbf{p})$ to show that $F(\mathbf{p})$ (in a sense) satisfies the String and the Dilation equations. One can present a purely geometric proof for these two equations, and obtain a proof of Theorem 5.2 which does not use Theorem 5.1.

Theorem 5.4. *The rank of* $\kappa_c^{n-1}(\overline{\mathcal{M}}_{1,n})$ *is equal to* $n-1$.

Proof. One needs to show that the rank of $R(n-1;1,n)$ is $n-1$. The proof for $n \le 4$ may be done by direct computation. We may thus assume that $n > 4$. Since the total number of rows in $R(n-1;1,n)$ is n, Theorem 5.2 implies that $r(n-1;1,n)$ is at most $n-1$. The rows of $R(n-1;1,n)$ are in correspondence with the divisors $D_0, D_1, \dots, D_{n-1}$. If the rank of $R(n-1;1,n)$ is less than $n-1$ there is a divisor $E = \sum_{i=1}^{n-1} d_i D_i$ such that $\int_E \kappa = 0$ for all $\kappa \in \kappa^{n-1}(\overline{\mathcal{M}}_{1,n})$. Note that E is chosen so that the coefficient of D_0 is zero. We may compute the integral of κ_{n-1} over E and obtain

$$0 = \int_E \kappa_{n-1} = d_{n-1} \int_{D_{n-1}} \kappa_{n-1}.$$

Thus $d_{n-1} = 0$. For a κ class of the form $\tilde{\kappa} = \kappa_{i,n-1-i}$ (where $i = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$) the integral of $\tilde{\kappa}$ over D_j is zero unless $j = i$ or $j = n-1-i$. Using Theorem 5.1 we find

$$\begin{aligned} 0 = \int_E \tilde{\kappa} &= d_i \int_{D_i} \tilde{\kappa} + d_{n-1-i} \int_{D_{n-1-i}} \tilde{\kappa} \\ &= \frac{1}{24}(d_i + d_{n-1-i}). \end{aligned}$$

In particular, $d_{n-1-i} = -d_i$. Similarly, for $1 < i \le \lfloor \frac{n-1}{2} \rfloor$ and the κ class $\widehat{\kappa} = \kappa_{1,i-1,n-1-i}$ the integral of $\widehat{\kappa}$ over D_j is zero unless j belongs to the set $\{1, i-1, i, n-1-i, n-i, n-2\}$. Thus, using Theorem 5.1 again, we find

$$\begin{aligned} 0 = \int_E \widehat{\kappa} = d_1 \left(\int_{D_1} \widehat{\kappa} - \int_{D_{n-2}} \widehat{\kappa} \right) &+ d_{i-1} \left(\int_{D_{i-1}} \widehat{\kappa} - \int_{D_{n-i}} \widehat{\kappa} \right) \\ &+ d_i \left(\int_{D_i} \widehat{\kappa} - \int_{D_{n-i-1}} \widehat{\kappa} \right) \\ = \frac{1}{24} \left(-\binom{n-2}{i-1} d_1 - (n-i) d_{i-1} + i d_i \right). \end{aligned}$$

Note that the above equation is also true for $i = 2$. By induction on i we conclude $d_i = \binom{n-2}{i-1} d_1$. We may thus assume that $d_1 = 1$, i.e.

$$E = \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-2}{i-1} (D_i - D_{n-1-i}).$$

Finally, we integrate the class $\kappa_{1,1,\dots,1}$ over E. We first note that using String and Dilation equations and induction on k one may prove

$$\int_{\overline{\mathcal{M}}_{1,2k}} \psi_1^2 \dots \psi_k^2 = \frac{2^{k-1} k!(k-1)!}{24}.$$

Using the above explicit formula we have

$$\begin{aligned}\int_{D_i} \kappa_{1,\dots,1} &= \binom{n-1}{i} \binom{2i}{2,2,\dots,2} \left(\frac{2^{n-i-2}(n-i-1)!(n-i-2)!}{24} \right) \\ &= \left(\frac{\binom{n-1}{i}(2i)!(2n-2-2i)!}{24 \times 2^{n-1}} \right) \left(\frac{(2n-2i-3)!}{((2n-2i-3)!!)^2} \right)\end{aligned}$$

From here we compute

$$\begin{aligned}0 = \int_E \kappa_{1,\dots,1} &= \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-2}{i-1} \left(\int_{D_i} \kappa_{1,\dots,1} - \int_{D_{n-1-i}} \kappa_{1,\dots,1} \right) \\ &= \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(\frac{\binom{n-2}{i-1}\binom{n-1}{i}(2i)!(2n-2-2i)!}{24 \times 2^{n-1}} \right) (I_{n-i-1} - I_i) \\ &\text{where} \quad I_j = \frac{(2j-1)!}{((2j-1)!!)^2}.\end{aligned}$$

Since $\frac{I_{j+1}}{I_j} = \frac{(2j)}{(2j+1)} < 1$ for all j, we conclude that $I_{n-1-i} - I_i < 0$ for $0 < i < \frac{n-1}{2}$, i.e. the right-hand-side of the above equality is negative. The contradiction completes the proof. □

Combined with the result of Petersen [9] on the structure of the tautological ring of $\overline{\mathcal{M}}_{1,n}$, the above theorem implies that the rank of $\kappa^{n-1}(\overline{\mathcal{M}}_{1,n})$ is $n-1$.

5.2. **The ψ classes of length two.** Fix $g > 1$ and $m > 3g-1$, and consider the square matrix

$$\begin{aligned}M(m; g) &= (n_j(h,m))_{\substack{h=0,1,\dots,g \\ j=0,2,3,\dots,g+1}}, \quad \text{where} \\ n_j(h,m) &:= 24^h \times h! \times \int_{\overline{\mathcal{M}}_{h,m-3h+3}} \psi_1^{m-j} \psi_2^j \quad j = 0,1,\dots,m.\end{aligned}$$

We begin our investigation with the study of the entries of this matrix. Let $P_j(h) = n_j(h, 3h-1)$.

Lemma 5.5. *P_j is a polynomial of degree j, with the leading coefficient*

$$\frac{6^j}{(2j+1)!!}.$$

Proof. The claim is trivial for $j=0$. Note that by definition

$$P_j(h) = \langle \tau_j \tau_{3h-1-j} \rangle.$$

Applying the KdV equation in the case $j \geq 1$ we get:

$$\begin{aligned} P_j(h) = & \left(\frac{1}{2j+1} - 2\right) P_{j-1}(h) + \left(\frac{1}{2j+1} - 1\right) P_{j-2}(h) \\ & + \frac{1}{2j+1}\left(\binom{h}{\frac{j}{3}} + 2\binom{h}{\frac{j-1}{3}}\right) \\ & + \frac{6h}{2j+1}\left(\sum_{k=0}^{4}\binom{4}{k} P_{j-1-k}(h-1)\right). \end{aligned} \tag{6}$$

Both claims are then quick (inductive) implications of (6). □

Let us assume that $P_j(h) = \sum_{k=0}^{j} A_k^j h^{j-k}$.

Lemma 5.6. *With the above notation fixed, $n_j(h,m)$ is a polynomial in the variables h and m of degree j. If*

$$n_j(h,m) = \sum_{\substack{p+q \leq i \\ p,q \geq 0}} A_j(p,q) h^p m^q,$$

then $A_j(p,q) \neq 0$ for $p+q=j$.

Proof. For $j=0$, $n_0(h,m)=1$ and the claim is trivial. Suppose now that

$$n_i(h,m) = \sum_{p+q \leq i} A_i(p,q) h^p m^q,$$

for $0 \leq i < j$. Suppose that $m > 3h-1$ is an arbitrary integer and $j > 0$. From the String equation

$$\int_{\overline{\mathcal{M}}_{h,m-3h+3}} \psi_1^{m-j}\psi_2^{j} = \int_{\overline{\mathcal{M}}_{h,m-3h+2}} \psi_1^{m-j-1}\psi_2^{j} + \int_{\overline{\mathcal{M}}_{h,m-3h+2}} \psi_1^{m-j}\psi_2^{j-1}$$

we obtain

$$\begin{aligned} n_j(h,m) &= n_j(h,m-1) + n_{j-1}(h,m-1) \\ &= \ldots = n_j(h,3h-1) + \sum_{k=3h-1}^{m-1} n_{j-1}(h,k) \\ &= P_j(h) + \sum_{p+q<j} A_{j-1}(p,q) h^p \left(\sum_{k=3h-1}^{m-1} k^q\right) \end{aligned} \tag{7}$$

Equation (7) determines $n_j(h,m)$ as a polynomial of degree j. The degree j part of n_j may be computed from the degree $j-1$ part of n_{j-1}. More precisely, let $a_j(i) := A_j(j-i,i)$ and set $m_j(h,m) = \sum_{i=0}^{j} a_j(i)h^{j-i}m^i$. Then

$$\sum_{i=0}^{j} a_j(i)h^{j-i}m^i = A_0^j h^j + \sum_{i=1}^{j} \frac{a_{j-1}(i-1)(h^{j-i}m^i - 3^i h^j)}{i!}$$
$$= \left(\frac{6^j}{(2j+1)!!} - \sum_{i=0}^{j-1} \frac{a_{j-1}(i)3^{i+1}}{(i+1)!} \right) h^j + \sum_{i=1}^{j} \frac{a_{j-1}(i-1)}{i!} h^{j-i}m^i.$$

From here

$$a_j(i) = \frac{a_{j-i}}{\prod_{k=1}^{i} k!}, \quad \text{where } a_j = \begin{cases} \frac{6^j}{(2j+1)!!} - \sum_{i=1}^{j} \frac{3^i}{\prod_{k=1}^{i} k!} a_{j-i} & \text{if } j > 0 \\ 1 & \text{if } j = 0 \end{cases}.$$

For every rational number x and every prime number p let $\operatorname{ord}_p(x)$ denote the integer k such that there are integers a, b such that $x = p^k(a/b)$ and $p \nmid ab$. Using the above recursive formula and by induction on j

$$\operatorname{ord}_2(a_j) = -\sum_{i=1}^{j} \operatorname{ord}_2(i!) \quad \forall \;\; j = 0, 1, 2, \ldots$$
$$\Rightarrow \;\; 0 = \operatorname{ord}_2(a_0) = \operatorname{ord}_2(a_1) > \operatorname{ord}_2(a_2) > \operatorname{ord}_2(a_3) > \operatorname{ord}_2(a_4) > \ldots.$$

Thus$a_j \neq 0$, and consequently $A_j(p,q) \neq 0$ for $p + q = j$. □

Lemma 5.6 implies that we may formally extend $n_j(h,m)$ and define it for the values of h, m which do not necessarily satisfy $m \geq 3h-1$ or $m \geq j$. For $m \geq 3h-1$ we trivially have $n_j(h,m) = n_{m-j}(h,m)$. It happens that the aforementioned symmetry extends to a slightly larger range of values.

Lemma 5.7. *For every j satisfying $j \geq 2h+1$, $n_j(h, j-1) = 0$.*

Proof. If $j \geq 3h$, by the String equation

$$n_j(h, j-1) = n_j(h,j) - n_{j-1}(h, j-1) = 1 - 1 = 0.$$

The KdV equation implies that for every $i, j \neq 0$

$$(2i+1)\langle \tau_i \tau_j \tau_0^{n+2} \rangle_h = (2n+1)\langle \tau_{i-1}\tau_j\tau_0^{n+1}\rangle_h + \frac{1}{4}\langle \tau_{i-1}\tau_j\tau_0^{n+4}\rangle_{h-1}$$
$$+ \frac{1}{24^h \times h!} \sum_{p=0}^{n+1} \binom{n+1}{p} \binom{g}{\frac{i-p}{3}} \tag{8}$$

Setting $i = m - j$ and $n = m - 3h - 1$, (8) implies

$$\begin{aligned} n_j(h,m) &= \frac{2m-6h-1}{2m-2j+1} n_{j-1}(h,m-1) \\ &\quad + \frac{6h}{2m-2j+1} n_{j-1}(h-1,m-1) + \frac{1}{2m-2j+1} q_j(h,m), \\ q_j(h,m) &:= \sum_{p=0}^{m-3h} \binom{m-3h}{p} \binom{h}{\frac{j-p}{3}} = \sum_{p=0}^{j} \binom{m-3h}{p} \binom{h}{\frac{j-p}{3}}. \end{aligned} \tag{9}$$

The left-hand-side and the right-hand-side of the first equation in (9) are rational functions in h, m, and (9) is thus satisfied for all values of h and m. Setting $m = j - 1$ in (9)

$$n_j(h,j-1) = (6h+3-j)n_{j-1}(h,j-2) - 6hn_{j-1}(h-1,j-2) - q_j(h,j-1).$$

Fixing j, all the terms in $q_j(h, j-1)$ are zero $j \geq 3h+1$. If $j < 3h+1$

$$q_j(h,j-1) = \sum_{k=0}^{\lfloor \frac{j}{3} \rfloor} (-1)^{j-3k} \binom{3h-3k}{j-3k} \binom{h}{k}$$

is the coefficient of x^j in

$$\begin{aligned} \left(\sum_{k=0}^{h} (-1)^k \binom{h}{k} x^{3k} \right) \left(\sum_{\ell=0}^{\infty} \binom{3h-j-\ell}{\ell} x^\ell \right) &= (1-x^3)^h \frac{1}{(1-x)^{3h-j+1}} \\ &= (1+x+x^2)^h (1-x)^{j-1-2h}. \end{aligned}$$

For $2h+1 \leq j < 3h+1$ the above expression is a polynomial in x of degree $j-1$ and the coefficient of x^j in it is thus zero. This implies that $q_j(h,j-1) = 0$ for all h satisfying $j \geq 2h+1$. Thus, for every h, j satisfying $j \geq 2h+1$

$$n_j(h,j-1) = (6h+3-j)n_{j-1}(h,j-2) - 6hn_{j-1}(h-1,j-2). \tag{10}$$

Note that $n_j(0,m) = \binom{m}{j}$, and $n_j(0, j-1)$ is thus zero. Let h be the smallest genus such that there is some j with $j \geq 2h+1$ and $n_j(h, j-1) \neq 0$, and let j be the largest such j. Then, $j > 2h$ and $j-1 > 2(h-1)$, implying $n_{j+1}(h,j) = n_j(h-1,j-1) = 0$ by the minimality assumption on (h,j). Then (10) gives $n_j(h, j-1) = 0$. This contradiction proves the lemma. □

From Lemma 5.7, for every $j \geq 2h$

$$\begin{aligned} n_j(h,j) &= n_{j+1}(h,j+1) - n_{j+1}(h,j) \\ &= n_{j+2}(h,j+2) - n_{j+2}(h,j+1) \quad (\text{since } n_{j+1}(h,j) = 0) \\ &= ... = n_{j+3h}(h,j+3h) = 1. \end{aligned}$$

Since $n_0(h,m) = 1$, this gives the equality

$$n_0(h,m) = n_m(h,m) \quad \forall\ m \geq 2h.$$

Proposition 5.8. *For every $m \geq 2h \geq 0$ and every $0 \leq j \leq m$*

$$n_j(h,m) = n_{m-j}(h,m).$$

Proof. Denote the claim of the proposition for (j,h,m) by $\mathcal{P}(j,h,m)$, i.e. we claim that for $j = 0,...,m$ and $m \geq 2h \geq 0$, $\mathcal{P}(j,h,m)$ is true.

For $h = 0$ and $0 \leq j \leq m$, $\mathcal{P}(0,j,m)$ is trivial. Suppose that h is the smallest genus such that for some $0 \leq j \leq m$ satisfying $m \geq 2h$, $\mathcal{P}(h,j,m)$ is not true. Take m to be the largest possible value such that there is some j with $\mathcal{P}(j,h,m)$ true. Fix h,m as above and let j be the largest integer with $\mathcal{P}(j,h,m)$ true. Lemma 5.7 implies that $j < m$. Moreover, the assumptions on (j,h,m) implies that $\mathcal{P}(j+1,h,m+1)$ and $\mathcal{P}(j+1,h,m)$ are true. The String equation

$$n_j(h,m) = n_{j+1}(h,m+1) - n_{j+1}(h,m)$$

gives

$$\mathcal{P}(j+1,h,m) \ \ \& \ \ \mathcal{P}(j+1,h,m+1) \ \Rightarrow \ \mathcal{P}(j,h,m).$$

This completes the proof of the proposition. □

Since in the column of $M(m;g)$ indexed by j the coefficient of h^j is a non-zero constant, subtracting appropriate multiples of the columns corresponding to $i = 0,2,...,j-1$ from the column corresponding to j, for $j = 2,...,g+1$ kills the monomials of degree $2,...,j-1$, while leaving the determinant unchanged. The determinant

$$d_g(m) = \mathrm{Det}(M(m;g))$$

is thus equal to the determinant of a matrix of the form

$$M'(m;g) := \Big(a_j h^j + b_{j-1}(m)h\Big)_{\substack{h=0,\dots,g \\ j=0,2,3,\dots,g+1}},$$

where the constants a_j are determined in the proof of Lemma 5.6 and $b_j(m)$ is a polynomial of degree at most i in m (and with $b_{-1}(m) = 0$) for $j = 1,...,g$.

Considering the order of the coefficient c_j of m^j in $b_j(m)$ in the above process, one can easily observe that

$$\mathrm{ord}_2(c_j) = -\sum_{i=1}^{j} \mathrm{ord}_2(i!),$$

and c_j is thus always non-zero. As a consequence, $b_j(m)$ is a polynomial of degree j.

Subtracting h times the row corresponding to 1 from the row corresponding to h for $h = 2, ..., g$ keeps the determinant unchanged. We thus have

$$d_g(m) = \mathrm{Det}\begin{pmatrix} a_2 + b_1(m) & a_3 + b_2(m) & ... & a_{g+1} + b_g(m) \\ (2^2-2)a_2 & (2^3-2)a_3 & ... & (2^{g+1}-2)a_{g+1} \\ (3^2-3)a_2 & (3^3-3)a_3 & ... & (3^{g+1}-3)a_{g+1} \\ \vdots & \vdots & \ddots & \vdots \\ (g^2-g)a_2 & (g^3-g)a_3 & ... & (g^{g+1}-g)a_{g+1} \end{pmatrix}$$

This implies that

$$\begin{aligned} d_g(m) &= g!\left(\prod_{h=2}^{g+1} a_h\right)\mathrm{Det}\begin{pmatrix} 1 & -\widehat{b}_1(m) & -\widehat{b}_2(m) & ... & -\widehat{b}_g(m) \\ 1 & 1 & 1 & ... & 1 \\ 1 & 2 & 2^2 & ... & 2^g \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & g & g^2 & ... & g^g \end{pmatrix} \\ &= (-1)^{g+1}\left(\prod_{h=2}^{g}(a_h h!)\right)(c_g m^g + \text{lower degree terms}), \end{aligned}$$

where $\widehat{b}_j(m) = b_j(m)/a_{j+1}$. We conclude

Lemma 5.9. *The determinant $d_g(m)$ of the matrix $M(m; g)$ is a polynomial in m of degree g.*

In other words, except for at most g values of m, the matrix $M(m; g)$ is a matrix of full rank.

Theorem 5.10. *There is a non-zero constant C_g, which only depends on the genus g, such that determinant $d_g(m)$ is of the form*

$$C_g(m-2)\prod_{i=g+3}^{2g+1}(m-i).$$

Proof. By Lemma 5.9, it suffices to show that

$$d_g(2) = d_g(g+3) = d_g(g+4) = ... = d_g(2g+1) = 0.$$

The first row of $M(2; g)$ consists of the following numbers

$$(1, \binom{2}{2}, \binom{2}{3}, ..., \binom{2}{g+1}) = (1, 1, 0, ..., 0).$$

Theorem 5.1 implies that the second row of $M(2; g)$ consists of the following numbers

$$(1, n_2(1,2), n_3(1,2), ..., n_{g+1}(1,2)) = (1, \binom{2}{2} - \binom{0}{1}, 0, ..., 0).$$

Thus the first two rows of $M(2; g)$ are equal and $d_g(2) = 0$ for all $g \geq 1$.

For $g+3 \le m \le 2g+1$, let $A(m;g)$ denote the sub-matrix of $M(m;g)$ which corresponds to the rows $h = 0, ..., \lfloor \frac{m}{2} \rfloor$ and the columns $j = m - g - 1, m-g, ..., \lfloor \frac{m-1}{2} \rfloor$. Similarly, let $B(m;g)$ denote the sub-matrix of $M(m;g)$ which corresponds to the rows $h = 0, ..., \lfloor \frac{m}{2} \rfloor$ and the columns $j = \lceil \frac{m+1}{2} \rceil, \lceil \frac{m+1}{2} \rceil + 1, ..., g+1$.

Proposition 5.8 implies that the columns of $B(m;g)$ are the same as the columns of $A(m;g)$. Subtraction the jth column of $M(m;g)$ from its $(m-j)$ column for $j = m-g-1, m-g, ..., \lfloor \frac{m-1}{2} \rfloor$ produces a matrix with the same determinant, and with zeros in the block replaced for $B(m;g)$. If the determinant is non-zero the sum of the number of columns and the number of rows in $B(m;g)$ is at most $g+1$, i.e. the total number of rows in $M(m;g)$. Thus,

$$g+1 \ge \left(\left\lfloor \frac{m}{2} \right\rfloor + 1 \right) + \left(g + 2 - \left\lceil \frac{m+1}{2} \right\rceil \right)$$
$$\Leftrightarrow \quad \left\lceil \frac{m+1}{2} \right\rceil \ge \left\lfloor \frac{m}{2} \right\rfloor + 2.$$

This contradiction implies that $d_g(m) = 0$ for $m = g+3, g+4, ..., 2g+1$. Since d_g has at most g roots, it is a constant multiple of

$$(m-2) \prod_{i=g+3}^{2g+1} (m-i).$$

□

Remark 5.11. Explicit computation, using computers, shows that

$$d_g(m) = \frac{(-3)^{\binom{g+1}{2}}}{(2g+1)!! \times g!} (m-2) \prod_{i=g+3}^{2g+1} (m-i) \quad \text{for } g \le 50.$$

However, we do not have a proof of the above formula in general.

5.3. **The rank of** $\kappa_c^{3g-4+n}(\overline{\mathcal{M}}_{g,n})$ **for** $g > 1$**.** As an application of Lemma 4.3 and Theorem 5.10 we examine the kappa ring in co-degree one for $g > 1$. With $d = 3g-4+n$, we get $3g-2+n-d = 2$. We choose P to be the union of $\mathrm{P}(d;2)$ of partition of length 2 with the set of all partitions of the form $\mathbf{p}' = (a_1 > b_1 \ge b_2 > 0)$ with $b_1 + b_2 \le a_1$, and define $f(\mathbf{p}') = (a_1 \ge a_2 = b_1 + b_2 > 0)$. The order $<$ is defined by setting (d) to be the largest element in $\mathrm{P}(d,2)$, and setting $(a_1 \ge a_2 > 0)$ greater than $(b_1 \ge b_2 > 0)$ if $a_1 \ge b_1$. Let us fix $\mathbf{p} = (a_1 \ge a_2 > 0)$ and study the sub-matrices $R_f(\mathbf{p};g,n)$. In the following discussions we will prove that $R_f(\mathbf{p};g,n)$ has full row rank for $a_2 > 4$.

Let us first assume that $a_1 > a_2 \ge 3g-2 \ge 4$, while $a_2 \le n-2$. For a corresponding partition $\mathbf{q} = \{(g_1, m_1), (g_2, m_2)\} \in \mathrm{Q}(\mathbf{p};g,n)$, g_2 is an arbitrary genus between 0 and g, $g_1 = g - g_2$, and $m_i = a_i + 3 - 3g_i$, i.e. the matrix $R_f(\mathbf{p}; g, n)$ has precisely $g+1$ rows. The columns correspond to

$$\mathbf{p}_0 = (a_1 \geq a_2), \quad \mathbf{p}_j = (a_1 \geq (a_2 - j) \geq j) \qquad j = 1, ..., \left\lfloor \frac{a_2}{2} \right\rfloor \geq \left\lfloor \frac{3g}{2} \right\rfloor - 1.$$

The $(g+1) \times (\lfloor a_2/2 \rfloor + 1)$ matrix $R_f(\mathbf{p}; g, n)$ consists of the entries m_{hj}, with $h = 0, ..., g$ and $j = 0, ..., \lfloor a_2/2 \rfloor$, which are given by

$$m_{hj} = \begin{cases} \left(\int_{\overline{\mathcal{M}}_{g-h,a+3h}} \psi_1^{a_1+1}\right)\left(\int_{\overline{\mathcal{M}}_{h,a_2-3h+4}} \psi_1^{a_2+1}\right) & \text{if } j = 0 \\ \left(\int_{\overline{\mathcal{M}}_{g-h,a+3h}} \psi_1^{a_1+1}\right)\left(\int_{\overline{\mathcal{M}}_{h,a_2-3h+5}} \psi_1^{a_2-j+1}\psi_2^{j+1}\right) & \text{if } j \neq 0, \end{cases}$$

where $a = a_1 - 3g + 4$. Thus, the rank of $R_f(\mathbf{p}; g, n)$ is equal to the rank of the matrix $N(a_2 + 2; g)$, where

$$N(m; g) := \Big(n_j(h, m)\Big)_{\substack{h=0,\dots,g \\ j=0,2,3,\dots,\lfloor m/2 \rfloor}}.$$

Since $m = a_2+2 \geq 3g \geq 6$, the matrix $M(g; m)$ is a sub-matrix of $N(g; m)$. This sub-matrix is full-rank by Theorem 5.10, and the rank $r_f(\mathbf{p}; g, n)$ of $R_f(\mathbf{p}; g, n)$ is equal to $g+1$. Thus,

$$\sum_{\substack{\mathbf{p}=(a_1>a_2)\in P(d) \\ 3g-2\leq a_2\leq n-2}} r_f(\mathbf{p}; g, n) \geq \left(\sum_{\substack{\mathbf{p}=(a_1>a_2)\in P(d) \\ 3g-2\leq a_2\leq n-2}} (g+1)\right)$$

When $a_1 = a_2 \geq 3g - 2$, the possible combinatorial cycles correspond to the values $0 \leq g_2 \leq \lfloor g/2 \rfloor$, and with a similar argument we have

$$r_f(\mathbf{p}; g, n) = \left\lfloor \frac{g+2}{2} \right\rfloor.$$

For arbitrary values of $a_2 > 4$, for

$$\mathbf{q} = \big\{(g_1, m_1), (g_2, m_2)\big\} \in Q(\mathbf{p}; g, n)$$

we have

$$\max\left\{0, \left\lceil \frac{a_2 + 2 - n}{3} \right\rceil \right\} \leq g_2 \leq \min\left\{ \left\lfloor \frac{a_2+2}{3} \right\rfloor, g \right\}.$$

Let $\mathrm{row}_f(\mathbf{p}; g, n)$ denote the number of rows in $R_f(\mathbf{p}; g, n)$. The matrix $R_f(\mathbf{p}; g, n)$ consists of the multiples of a subset of the rows in the matrix $N(a_2 + 2; h)$ where $h = \min\{g, \lfloor (a_2 + 2)/3 \rfloor\}$. Its rank is thus equal to $\mathrm{row}_f(\mathbf{p}; g, n)$.

Finally, we gather the rows and the columns corresponding to the partitions of the form $(a_1 \geq a_2 > 0)$ with $a_2 \in \{0, 1, 2, 3, 4\}$ in one block (see Remark 4.4). The corresponding partitions of d consist of the following list:

$$A = \big\{(d), (d-1, 1), (d-2, 2), (d-3, 3), (d-4, 4)\big\}.$$

For every $\mathbf{q} \in \mathrm{Q}(d; g, n)$ and every partition $\mathbf{p}$ in

$$B = \Big\{(d),(d-1,1),(d-2,2),(d-2,1,1),(d-3,3),(d-3,2,1),(d-3,1,1,1),\\ (d-4,4),(d-4,3,1),(d-4,2,2),(d-4,1,1,1,1),(d-4,2,1,1)\Big\}$$

if $\langle \mathbf{p}, \mathbf{q} \rangle \neq 0$ then $\mathbf{p}(\mathbf{q}) \in A$. There are 11 such $\mathbf{q} \in \mathrm{Q}(d; g, n)$ which, together with the first 11 partitions of d in the above set of 12 elements, determine an 11×11 sub-matrix of $R_f(d; g, n)$.

Using the explicit formulas in Table 1 for some of the ψ integrals, one may compute the determinant of the above 11×11 matrix. Surprisingly, the determinant is independent of d and equals

$$-\frac{2592(g-1)^2(4928g^4 - 275516g^3 - 437138g^2 + 62924g - 334941)}{42109375}.$$

Thus, the aforementioned 11×11 matrix is always of rank 11.

Gathering the above information one arrives at the following theorem.

Theorem 5.12. *For $g > 1$ and $d = 3g - 4 + n$, the rank of $\kappa_c^d(\overline{\mathcal{M}}_{g,n})$ is equal to*

$$\left\lceil \frac{(n+1)(g+1)}{2} \right\rceil - 1.$$

Proof. For $d = 3g - 4 + n$, the lower bound on the rank is given by the sum of the rank of the above 11×11 matrix, plus the following sum

$$\sum_{\substack{\mathbf{p}=(a_1 \geq a_2 > 0)\\ 5 \leq a_2 \leq (3g-4+n)/2}} r_f(\mathbf{p}; g, n).$$

The above computations may then be used to compute this lower bound explicitly. Let $\mathrm{row}(d; g, n)$ denote the number of rows in $R(d; g, n)$. For $g > 1$ we thus have

$$\begin{aligned} \mathrm{rank}(R(d;g,n)) &\geq 11 + \sum_{\mathbf{p}=(a_1 \geq a_2 > 4)} r_f(\mathbf{p}; g, n) \\ &= \mathrm{row}(d; g, n) = \left\lceil \frac{(n+1)(g+1)}{2} \right\rceil - 1. \end{aligned} \tag{11}$$

Since $\mathrm{rank}(R(d; g, n))$ can not be larger than $\mathrm{row}(d; g, n)$, the inequality in (11) is in fact an equality. □

The following point of view was brought to the attention of the authors by one of the anonymous referees. By the work of Al-Aidroos [1] it is known that $R^{3g+n-4}(\overline{\mathcal{M}}_{g,n})$ is dual to $R^1(\overline{\mathcal{M}}_{g,n})$ and that the latter is generated by $\kappa_1, \psi_1, \ldots, \psi_n$ and the boundary divisors in $A^1(\overline{\mathcal{M}}_{g,n})$ [7]. In particular, a

TABLE 1. Small ψ integrals

$\mathbf{p} = (a_1, ..., a_k)$	The integral $g! \times 24^g \times \int_{\overline{\mathcal{M}}_{g,d(\mathbf{p})+3-3g}} \prod_{i=1}^{k} \psi_i^{a_i}$
(d)	1
$(d, 2)$	$\binom{d+2-g}{2} + \frac{g(2g+3)}{5}$
$(d, 3)$	$\binom{d+3-g}{3} + \binom{d+3-g}{1}\frac{g(2g+3)}{5} - \frac{g(8g^2+60g+37)}{105}$
$(d, 4)$	$\binom{d+4-g}{4} + \binom{d+4-g}{2}\frac{g(2g+3)}{5} - \binom{d+4-g}{1}\frac{g(8g^2+60g+37)}{105} + \frac{g(g+1)(2g+3)(2g+5)}{70}$
$(d, 5)$	$\binom{d+5-g}{5} + \binom{d+5-g}{3}\frac{g(2g+3)}{5} - \binom{d+5-g}{2}\frac{g(8g^2+60g+37)}{105} + \binom{d+5-g}{1}\frac{g(g+1)(2g+3)(2g+5)}{70} - \frac{g(2g+3)(8g^3+84g^2+55g+84)}{1155}$
$(d, 2, 2)$	$6\binom{d+4-g}{4} + \binom{d+4-g}{2}\frac{2g(2g+3)}{5} + \frac{g(4g^3-4g^2-41g-9)}{25}$
$(d, 2, 2, 2)$	$90\binom{d+6-g}{6} + \binom{d+6-g}{4}\frac{18g(2g+3)}{5} + \binom{d+6-g}{2}\frac{3g(4g^3-4g^2-41g-9)}{25} + \frac{g(8g^5-60g^4-70g^3+1275g^2+1067g+30)}{125}$
$(d, 2, 2, 2, 2)$	$2520\binom{d+8-g}{8} + \binom{d+8-g}{6}\frac{360g(2g+3)}{5} + \binom{d+8-g}{4}\frac{36g(4g^3-4g^2-41g-9)}{25} + \binom{d+8-g}{2}\frac{4g(8g^5-60g^4-70g^3+1275g^2+1067g+30)}{125} + \frac{g(16g^7-288g^6+1192g^5+7440g^4-57671g^3-120522g^2-34677g-20490)}{625}$
(d,3,2)	$10\binom{d+5-g}{5} + \binom{d+5-g}{3}\frac{4g(2g+3)}{5} - \binom{d+5-g}{2}\frac{g(8g^2+60g+37)}{105} + \binom{d+5-g}{1}\frac{g(4g^3-4g^2-41g-9)}{25} - \frac{g(2g+3)(8g^3+12g^2-467g-78)}{525}$
$(d, 4, 2)$	$15\binom{d+6-g}{6} + \binom{d+6-g}{4}\frac{7g(2g+3)}{5} - \binom{d+6-g}{3}\frac{3g(8g^2+60g+37)}{105} + \binom{d+6-g}{2}\frac{g(76g^3+44g^2-419g-51)}{350} - \binom{d+6-g}{1}\frac{g(2g+3)(8g^3+12g^2-467g-78)}{525} + \frac{g(g+2)(2g+1)(2g+3)(2g^2-11g-61)}{350}$
$(d, 3, 3)$	$20\binom{d+6-g}{6} + \binom{d+6-g}{4}\frac{8g(2g+3)}{5} - \binom{d+6-g}{3}\frac{2g(8g^2+60g+37)}{105} + \binom{d+6-g}{2}\frac{2g(4g^3-4g^2-41g-9)}{25} - \binom{d+6-g}{1}\frac{2g(2g+3)(8g^3+12g^2-467g-78)}{525} + \frac{g(64g^5+384g^4-13376g^3-76224g^2-71315g-15933)}{11025}$
$(d, 3, 2, 2)$	$210\binom{d+7-g}{7} + 10g(2g+3)\binom{d+7-g}{5} - \frac{6g(8g^2+60g+37)}{35}\binom{d+7-g}{4} + \frac{7g(4g^3-4g^2-41g-9)}{25}\binom{d+7-g}{3} - \frac{2g(2g+3)(8g^3+12g^2-467g-78)}{525}\binom{d+7-g}{2} + \frac{g(8g^5-60g^4-70g^3+1275g^2+1067g+30)}{125}\binom{d+7-g}{1} - \frac{g(32g^6-176g^5-3300g^4+24440g^3+96943g^2+53031g+12780)}{2625}$

kappa class $\kappa \in \kappa^{3g+n-4}(\overline{\mathcal{M}}_{g,n})$ is trivial if and only if it is combinatorially trivial and

$$\int_{[\overline{\mathcal{M}}_{g,n}]} \kappa\kappa_1 = \int_{[\overline{\mathcal{M}}_{g,n}]} \kappa\psi_i = 0, \quad i = 1, \ldots, n.$$

Let us denote the functional on $\kappa^d(\overline{\mathcal{M}}_{g,n})$ which are defined as

$$\kappa \mapsto \int_{[\overline{\mathcal{M}}_{g,n}]} \kappa_1\kappa \;\text{ and }\; \kappa \mapsto \int_{[\overline{\mathcal{M}}_{g,n}]} \psi_1\kappa = \cdots = \int_{[\overline{\mathcal{M}}_{g,n}]} \psi_n\kappa$$

by $\widehat{\kappa}_1$ and $\widehat{\psi}$, respectively. The rank of $\kappa^{3g+n-4}(\overline{\mathcal{M}}_{g,n})$ is equal to $\lceil\frac{(g+1)(n+1)}{2}\rceil + \epsilon(g,n)$, where $\epsilon(g,n) \in \{0,1,-1\}$. For $g = 2$ the quotient map from $\kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$ to its combinatorial quotient is an isomorphism by the work of authors [3], and we find $\epsilon(2,n) = -1$.

Theorem 5.13. *For $g > 2$ and $n > 0$ the rank of $\kappa^{3g+n-4}(\overline{\mathcal{M}}_{g,n})$ is equal to*

$$\left\lceil \frac{(n+1)(g+1)}{2} \right\rceil + 1.$$

Proof. We continue to use the notation set in the proof of Theorem 5.12 and set $d = 3g + n - 4$. In order to prove the theorem we first show that the linear functional $\widehat{\kappa}_1$ is independent from the $\lceil\frac{(g+1)(n+1)}{2}\rceil - 1$ functionals corresponding to the integration over stable weighted graphs in $\overline{\mathcal{M}}_{g,n}$.

The aforementioned functional may be used to add a row to the matrix $R(d;g,n)$. Denote the new matrix by $R'(d;g,n)$. Correspondingly, we may define $R'_f(d;g,n)$. The 12 partitions in B determine 12 columns in $R'(d;g,n)$. The partitions $\mathbf{q} \in \mathrm{Q}(d;g,n)$ with $\mathbf{p}(\mathbf{q}) \in A$ and the particular linear functional constructed above determine 12 rows of $R'(d;g,n)$. These 12 rows and 12 columns determine a 12×12 sub-matrix X of $R'(d;g,n)$ and the proof of Theorem 5.12 may be copied to show that

$$\begin{aligned} \operatorname{rank}(R'(d;g,n)) &\geq \operatorname{rank}(X) + \sum_{\mathbf{p}=(a_1\geq a_2>4)} r_f(\mathbf{p};g,n) \\ &= \operatorname{rank}(X) + \left\lceil \frac{(n+1)(g+1)}{2} \right\rceil - 12. \end{aligned} \tag{12}$$

The entries of X may be computed from an extended version of Table 1, which contains the integrals of ψ classes corresponding to the following 7 partitions as well:

$$(d,6), (d,5,2), (d,4,3), (d,4,2,2), (d,3,3,2), (d,3,2,2,2) \text{ and } (d,2,2,2,2,2).$$

The determinant of X may subsequently be computed. Surprisingly, the determinant is once again independent of d and only depends on g:

$$\det(X) = \frac{331776\ g(g-2)(g-1)^3 q(g)}{922063720703125}$$

where $q(g) = 14764032g^7 + 43415424g^6 + 158902720g^5 + 928792100g^4 - 21119392g^3 - 4217474459g^2 - 4089074775g - 1301969475.$

Since $q(g)$ does not have any integer roots, the matrix X is invertible for $g > 2$. This means that for $g > 2$, $\epsilon(g, n)$ is either 0 or 1. Furthermore, if $\epsilon(g, n) = 0$ the functional $\widehat{\psi}$ is a linear combination of $\widehat{\kappa}_1$ and the linear functionals corresponding to integration over divisors.

Next, we refine the above argument by adding another row to $R'(d; g, n)$, which corresponds to $\widehat{\psi}$. Denote the resulting matrix by $R''(d; g, n)$. Let

$$B' = B \cup \{(d-5,5), (d-5,4,1), (d-5,3,2), (d-5,1,1,1,1,1)\}$$

and $A' = A \cup \{(d-5,5)\}$. The sets A' and B' correspond to a 16×16 sub-matrix Y of $R''(d; g, n)$, while the argument of Theorem 2 implies that

$$\begin{aligned}\operatorname{rank}(R''(d; g, n)) &\geq \operatorname{rank}(Y) + \sum_{\mathbf{p}=(a_1 \geq a_2 > 5)} r_f(\mathbf{p}; g, n) \\ &= \operatorname{rank}(Y) + \left\lceil \frac{(n+1)(g+1)}{2} \right\rceil - 15.\end{aligned}$$

Once again, the entries of Y may be computed from an extended version of Table 1, and the determinant of Y may subsequently be computed. This time, the determinant is no longer independent of d and we find

$$\det(Y) = \frac{g(g-1)^3(g-2)(d-3g+1)(d-3g+2)p(g)\left(d^7 + \sum_{i=0}^{6} p_i(g)d^i\right)}{65262637440000000000},$$

where $p(g) = 14764032g^7 + 43415424g^6 + 158902720g^5 + 928792100g^4 - 21119392g^3 - 4217474459g^2 - 4089074775g - 1301969475$

and $p_i(g)$ are polynomials of g with rational coefficients for $i = 0, 1, ..., 6$.

Fix the genus $g > 2$. Since $d = 3g - 4 + n$, when the number n of the marked points is sufficiently large the determinant of Y is nonzero. In other words, $\widehat{\kappa}_1, \widehat{\psi}$ and the functionals corresponding to stable weighted graphs are independent when n is sufficiently large. This proves the theorem when n is sufficiently large.

Keep $g > 2$ fixed and suppose that the claim is not true over some $\overline{\mathcal{M}}_{g,n}$. Choose n to be the largest positive integer such that $R''(3g - 4 + n; g, n)$ is not full-rank. Since $\widehat{\psi}$ is independent from the other functionals over

$\kappa^{3g-3+n}(\overline{\mathcal{M}}_{g,n+1})$ there is some $\kappa \in \kappa^{3g-3+n}(\overline{\mathcal{M}}_{g,n+1})$ such that κ is combinatorially trivial,

$$\int_{[\overline{\mathcal{M}}_{g,n+1}]} \kappa(\kappa_1 - \psi_{n+1}) = 0 \quad \text{and} \quad \int_{[\overline{\mathcal{M}}_{g,n+1}]} \kappa\psi_1 \neq 0.$$

Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ denote the map which forgets the last marking. Then $\pi^*\kappa_1 = \kappa_1 - \psi_{n+1}$ and we find

$$\int_{[\overline{\mathcal{M}}_{g,n}]} \pi_*(\kappa)\kappa_1 = \int_{[\overline{\mathcal{M}}_{g,n+1}]} \kappa\pi^*(\kappa_1) = 0 \quad \text{and}$$
$$\int_D \pi_*(\kappa) = \int_{\pi^{-1}(D)} \kappa = 0 \quad \forall \text{ divisor } D \text{ in } \overline{\mathcal{M}}_{g,n}.$$

Since the rank of $\kappa^{3g-4+n}(\overline{\mathcal{M}}_{g,n})$ is $\lceil (g+1)(n+1)/2 \rceil$, every kappa class in co-degree one over $\overline{\mathcal{M}}_{g,n}$ which is combinatorially trivial and is in the kernel of $\widehat{\kappa}_1$ is trivial (by the first part of the proof). The above equalities thus imply that $\pi_*(\kappa) = 0$. Note that $\psi_1 = \pi^*(\psi_1) + \delta_{1,n+1}$ where $\delta_{1,n+1}$ denotes the divisor which may be identified with $\overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{0,3}$, and the markings 1 and $n+1$ are both placed on the genus zero component. We thus find

$$0 \neq \int_{[\overline{\mathcal{M}}_{g,n+1}]} \kappa\psi_1 = \int_{\delta_{1,n+1}} \kappa + \int_{[\overline{\mathcal{M}}_{g,n+1}]} \kappa\pi^*(\psi_1) = \int_{[\overline{\mathcal{M}}_{g,n}]} \pi_*(\kappa)\psi_1 = 0$$

This contradiction completes the proof. □

Theorem 5.12 and its proof imply that

$$R^{3g+n-4}(\overline{\mathcal{M}}_{g,n})^{S_n} = \kappa^{3g+n-4}(\overline{\mathcal{M}}_{g,n})$$

and that this vector space is spanned by $\kappa_{\mathbf{p}}$ with $\mathbf{p} \in P \cup B'$.

6. The asymptotic behavior of the ranks

The behavior of the rank of $\kappa_c^d(\overline{\mathcal{M}}_{g,n})$ for arbitrary values of g, n and d seems to be more complicated. Table 2 illustrates the computations for genus $0, 1, 2$ in co-degrees $2, 3, 4, 5$ and 6 when the number n of the marked points is less than or equal to 10. The second author has a computer program for computing the relevant kappa integrals, as well as the rank of the matrix $R(d; g, n)$. Computations beyond these tables require large memory and are relatively time consuming even over very fast computers.

We apply the strategy of the previous section, and study the asymptotic behavior of the rank of $\kappa_c^d(\overline{\mathcal{M}}_{g,n})$ instead, when the genus g and the co-degree $e = 3g - 3 + n - d$ are fixed. Since n grows large, we may use Theorem 1. The number of elements in $P(d, e+1)$, as d grows large, is asymptotic to $\binom{d+e}{e}/(e+1)!$. The number of rows in the matrix $R(d; g, n)$, i.e. the number of elements in $\mathrm{Q}(d; g, n)$, is thus asymptotic to either of

$$|\mathrm{P}(d, e+1)|\binom{g+e}{e} \quad \text{and} \quad \frac{\binom{n+e}{e}\binom{g+e}{e}}{(e+1)!}.$$

Table 2. The tables illustrate the rank of $\kappa^*\left(\overline{\mathcal{M}}_{g,n}\right)$ in co-degrees $2,3,4,5$ and 6 respectively.

Co-degree=2										
Genus \ Points	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	1	1	2	3	4	5
1	0	1	1	2	3	5	7	10	13	17
2	2	3	5	7	11	15	21	28	36	45

Co-degree=3										
Genus \ Points	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	1	1	2	3	5
1	0	0	1	1	2	3	5	7	11	15
2	1	2	3	5	7	11	15	22	30	42

Co-degree=4										
Genus \ Points	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	1	1	2	3
1	0	0	0	1	1	2	3	5	7	11
2	1	1	2	3	5	7	11	15	22	30

Co-degree=5										
Genus \ Points	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	1	1	2
1	0	0	0	0	1	1	2	3	5	7
2	0	1	1	2	3	5	7	11	15	22

Co-degree=6										
Genus \ Points	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	1	1
1	0	0	0	0	0	1	1	2	3	5
2	0	0	1	1	2	3	5	7	11	15

Set $h = g+2$. As n grows large, the asymptotic growth of $\mathrm{P}(d, e+1)$ is the same as the growth of the number of partitions $\mathbf{p} = (a_0 \geq a_1 \geq ... \geq a_e)$ satisfying

$$a_e > 2eh, \quad a_i \leq a_{i-1} - 2h \quad \text{for } i = 1, ..., e-1.$$

Denote the set of all such partitions by $\mathrm{P}_h(d; e+1)$.

For $\mathbf{p} = (a_0 \geq a_1 \geq \ldots \geq a_e > 0) \in \mathrm{P}_h(d; e+1)$ let $P(\mathbf{p}) \subset \mathrm{P}(d, 2e+1)$ denote the set of partitions

$$\mathbf{p}' = (a_0 > a_1 - b_1 > ..., a_e - b_e > b_e > b_{e-1} > \ldots > b_1), \quad (i-1)h < b_i \leq ih.$$

For every $\mathbf{p}' \in P(\mathbf{p})$ define $f(\mathbf{p}') = \mathbf{p}$. This gives a function

$$f : P = \bigcup_{\mathbf{p} \in \mathrm{P}_h(d;e+1)} P(\mathbf{p}) \longrightarrow \mathrm{P}_h(d; e+1).$$

Equip $\mathrm{P}_h(d; e+1)$ with the lexicographic order, setting $\mathbf{p} = (a_0 > ... > a_e > 0)$ less than $\mathbf{p} = (a'_0 > ... > a'_e > 0)$ if there is some $i \geq 0$ such that $a_j = a'_j$ for $j = 0, ..., i-1$ and $a_i < a'_i$ (while P is partially ordered with $\lhd$).

Although f is not a fine assignment and the second condition in Definition 4.2 may fail, it differs from a fine assignment in a controllable way, as will be discussed below. Let

$$\mathbf{p}' = (a_0, a_1 - b_1, ..., a_e - b_e, b_e, ..., b_1)$$

and suppose that $\mathbf{p}'' \in \mathrm{P}_h(d; e+1)$ refines $\mathbf{p}'$ while $\mathbf{p}'' < \mathbf{p} = (a_0, ..., a_e)$. Then

$$\mathbf{p}'' = (a_0 \geq a_1 \geq ... \geq a_{i-1} \geq a_i - b_i \geq c_{i+1} \geq ... \geq c_e > 0)$$

for some $i > 0$ and some positive integers $c_{i+1}, ..., c_e$. Associated with every $\mathbf{p}, \mathbf{p}'$ and $\mathbf{p}''$ as above, and every $\mathbf{q} \in \mathrm{Q}(\mathbf{p}''; g, n)$ we put $\frac{1}{\Lambda(\mathbf{q})}\langle \mathbf{p}', \mathbf{q}\rangle$ in a matrix $E_i(d; g, n)$ as the entry corresponding to the row indexed by $\mathbf{q}$ and the column indexed by $\mathbf{p}'$. For $i = 1, ..., e$ the rows of the matrix $E_i(d; g, n)$ are labelled by

$$Q = \bigcup_{\mathbf{p} \in \mathrm{P}_h(d;e)} \mathrm{Q}(\mathbf{p}; g, n),$$

while its columns are labelled by P.

The sub-matrix $S(d; g, n)$ of $R(d; g, n)$ determined by the columns corresponding to $P \subset \mathrm{P}(d)$ and the rows corresponding to $Q \subset \mathrm{Q}(d; g, n)$ is thus a sum

$$S(d; g, n) = T(d; g, n) + \sum_{i=1}^{e} E_i(d; g, n),$$

where $T(d;g,n)$ is an upper triangular matrix with respect to the total order $<$. Lemma 4.3 implies that

$$\mathrm{rank}(T(d;g,n)) \geq \sum_{\mathbf{p}\in \mathrm{P}(d;e)} r_f(\mathbf{p};g,n)$$

Proposition 6.1. *There is a subset $A \subset \mathrm{P}_h(d;e+1)$ of size $\frac{(eh)^2}{2}|\mathrm{P}(d;e+1)|$ such that for every $\mathbf{p} \in \mathrm{P}_h(d;e+1) \setminus A$*

$$r_f(\mathbf{p};g,n) = \binom{g+e}{e}.$$

Proof. Consider the matrix $S(\mathbf{p};g,n)$ (containing $R_f(\mathbf{p};g,n)$ as a sub-matrix) whose rows are in correspondence with all assignments $(g_0, g_1, ..., g_e)$ to the $(e+1)$ components of the combinatorial cycle, without any restriction on their sum, i.e. we consider all tuples $\mathbf{q} = ((m_0, g_0), ..., (m_e, g_e))$ such that $a_i = 3g_i - 3 + m_i$ and $0 \leq g_i \leq g$. The matrix $R_f(\mathbf{p};g,n)$ is a sub-matrix of $S(\mathbf{p};g,n)$ while they both have the same number of columns. As a result, if the row rank of $S(\mathbf{p};g,n)$ is full, so is the row rank of $R(\mathbf{p};g,n)$.

Let $M_k(m;g)$ denote the sub-matrix of $N(m;g)$ which consists of the columns corresponding to the values $j = k, k+1, ..., k+g$. The matrix $S(\mathbf{p};g,n)$ has at least the same rank as the matrix $N_1 \otimes N_2 \otimes ... \otimes N_e$, where $N_1 = M(m;g)$ and for $i > 1$ N_i is the matrix $M_{ih-h+1}(a_i+2;g)$. The determinant $d_g^k(m) = \mathrm{Det}(M_k(m;g))$ is a polynomial with

$$\deg\left(d_g^k(m)\right) \leq \frac{(g+1)(2k+g)}{2}$$

by Lemma 5.6. Moreover, Proposition 5.8 implies that

$$d_g^k(k+g) = (-1)^{\binom{g}{2}}\mathrm{Det}\begin{pmatrix} n_0(0,k+g) & n_1(0,k+g) & \dots & n_g(0,k+g) \\ n_0(1,k+g) & n_1(1,k+g) & \dots & n_g(1,k+g) \\ \vdots & \vdots & \ddots & \vdots \\ n_0(0,k+g) & n_1(0,k+g) & \dots & n_g(0,k+g) \end{pmatrix}.$$

Since the degree of $n_i(h,m)$ is i, the right-hand-side of the above equality is a Van-der-Monde matrix, and the determinant is non-zero. The polynomial $d_g^k(m)$ is thus non-trivial and has at most $\frac{(g+1)(2k+g)}{2}$ roots. Let A_g^k denote the set of integer roots of $d_g^k(m)$. Thus, N_i is invertible unless a_i+2 belongs A_g^{ih-h+1}, and

$$|A_g^{ih-h+1}| = \frac{(g+1)(2ih-2h+2+g)}{2} \leq (i-\frac{1}{2})h^2.$$

The set of partitions $(a_0 > a_1 > ... > a_e) \in \mathrm{P}_h(d;e+1)$ such that $a_i \in A_g^{ih-h+1}$ is at most of size

$$|\mathrm{P}_h(d;e)||A_g^{ih-h+1}| \leq |\mathrm{P}(d;e)|h^2(i-\frac{1}{2}).$$

Consequently, for $\mathbf{p}$ outside a set of size

$$\sum_{i=2}^{e}(i-\frac{1}{2})|\mathrm{P}(d;e)|h^2 = \frac{(eh)^2}{2}|\mathrm{P}(d;e)|$$

every N_i is a full-rank matrix and the rank of $R_f(\mathbf{p};g,n)$ is equal to the number of its rows, i.e. $\binom{g+e}{e}$. □

Since $|\mathrm{P}(d;e)|$ is asymptotic to $\binom{n+e-1}{e-1}/e!$, Proposition 6.1 implies that the rank of $T(d;g,n)$ is asymptotic to

$$|\mathrm{P}_h(d;e+1)|\binom{g+e}{e} \simeq \frac{\binom{n+e}{e}\binom{g+e}{e}}{(e+1)!}.$$

In order to complete a computation of the asymptotic behavior of $r(d;g,n)$ it suffices to study the difference between the rank of $T(d;g,n)$ and the rank of $S(d;g,n)$.

Proposition 6.2. *With the above notation*

$$\lim_{n\to\infty}\frac{\mathrm{rank}(E_i(d;g,n))}{\binom{n+e}{e}} = 0.$$

Proof. Define a function $f_i : Q \subset \mathrm{Q}(d;g,n) \to \mathrm{Q}(d;g,n-1)$ as follows. Let $\mathbf{q} = \{(g_i,m_i)\}_{i=0}^{e} \in Q$ be a multi-set with $3g_i + m_i > 3g_j + m_j$ if $i < j$. Define

$$f_i : Q \subset \mathrm{Q}(d;g,n) \to \mathrm{Q}(d;g,n-1)$$
$$f_i(\mathbf{q}) := \Big((g_0+g_i, m_0+m_i-3), (g_1,m_1), ..., \widehat{(g_i,m_i)}, \dots, (g_e,m_e)\Big),$$

where the hat over (g_i,m_i) means that it is omitted from the sequence. Let $Q_i = f_i(Q) \subset \mathrm{Q}(d;g,n-1)$. Suppose that

$$\mathbf{q} = \Big\{(g_i,m_i)\Big\}_{i=0}^{e+1}, \mathbf{q}' = \Big\{(g_i',m_i')\Big\}_{i=0}^{e+1}, \quad \text{and} \quad f_i(\mathbf{q}) = f_i(\mathbf{q}').$$

Furthermore, let $3g_j + m_j \geq 3g_k + m_k$ and $3g_j' + m_j' \geq 3g_k' + m_k'$ if $j < k$. For every $\mathbf{p} = (a_0, a_1 - b_1, ..., a_e - b_e, b_e, ..., b_1)$ such that $\langle \mathbf{p}, \mathbf{q}\rangle$ is non-zero in $E_i(d;g,n)$,

$$\mathbf{p}(\mathbf{q}) = (a_0 \geq a_1 \geq ... \geq a_{i-1} \geq a_i - b_i \geq c_{i+1} \geq ... \geq c_e > 0),$$

for some integers $c_{i+1}, ..., c_e$. Let

$$\widehat{\mathbf{p}} = (a_{i+1}-b_{i+1}, ..., a_e - b_e, b_e, ..., b_i), \quad \widehat{\mathbf{q}} = ((g_{i+1},m_{i+1}), \dots (g_e,m_e)).$$

Then,

$$\langle \mathbf{p},\mathbf{q}\rangle = \frac{1}{24^{g_0+g_i}\times g_0! \times g_i!}\left(\prod_{j=1}^{i-1}\int_{\overline{\mathcal{M}}_{g_j,m_j+2}}\psi_1^{b_j+1}\psi_2^{a_j-b_j+1}\right)\langle\widehat{\mathbf{p}},\widehat{\mathbf{q}}\rangle$$
$$= \langle \mathbf{p}',\mathbf{q}\rangle.$$

Thus, the rows in $E_i(d;g,n)$ which correspond to $\mathbf{q}$ and $\mathbf{q}'$ are identical. Consequently, the rank of $E_i(d;g,n)$ is bounded above by $|Q_i|$, which is in turn less than or equal to the cardinality of $\mathrm{Q}(d;g,n-1)$. But the latter cardinality is asymptotic to

$$\frac{\binom{g+e-1}{e-1}\binom{n+e}{e-1}}{e!}.$$

The proposition follows immediately. □

Theorem 6.3. *The rank of the kappa ring $\kappa_c^*(\overline{\mathcal{M}}_{g,n})$ in co-degree e, as the number n of the marked points becomes large, is asymptotic to*

$$\frac{\binom{n+e}{e}.\binom{g+e}{e}}{(e+1)!}$$

Proof. Proposition 6.1 implies that asymptotically, the rank is greater than or equal to

$$|\mathrm{P}_h(d;e+1)|\binom{g+e}{e} - \sum_{i=1}^{e}\mathrm{rank}(E_i(d;g,n)).$$

By Proposition 6.2, the matrices $E_i(d;g,n)$ do not change the asymptotic, and $r(3g-3+n-e;g,n)$ is asymptotically greater than or equal to

$$\frac{\binom{n+e}{e}.\binom{g+e}{e}}{(e+1)!}.$$

Since the number of rows in $R(d;g,n)$ follows the same asymptotic behavior the proof is complete. □

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