Weak Reconstruction of Edge-Deleted Cartesian Products

Wilfried Imrich* and Marcin Wardyński
Montanuniversität Leoben, Austria

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Reconstruction

In 1960 Ulam asked

Is a graph G uniquely determined by

$$\{G \setminus x \,|\, x \in V(G)\}$$

The $G \setminus x$ are the vertex-deleted subgraphs and
$\{G \setminus x \,|\, x \in V(G)\}$ is the deck.

Reconstruction Conjecture or Ulam's Conjecture

Any two graphs on at least three vertices with the same deck are isomorphic.

Open for finite graphs.

Weak reconstruction

When reconstructing a class of graphs, the problem partitions into

recognition and weak reconstruction.

Recognition consists of showing that membership in the class is determined by the deck,
weak reconstruction consists of showing that nonisomorphic members of the class have different decks.

In 1973 Dörfler* showed that Ulam's conjecture holds for finite non-trivial Cartesian products.

In 1996 Žerovnik and I showed† that the weak reconstruction problem can already be solved from a single vertex-deleted subgraph for nontrivial, connected finite or infinite Cartesian products.

in 1999 Hagauer and Žerovnik‡ published an algorithm for weak reconstruction. They claimed complexity

$$O(mn \cdot (\Delta^2 + m \log n)),$$

where m is the size, n the order and Δ the maximum degree of the graph.

*Colloq. Math. Soc. János Bolyai, Vol. 10, Keszthely, Hungary (1973).

†Discrete Mathematics 150 (1996).

‡J. Combin. Inform. System Sci. 24 (1999).

In 2013 Kupka announced an algorithm with complexity $O(\Delta^2(\Delta^2 + m)$.

Unfortunately, it is based on erroneous argument of Hagauer and Žerovnik which was corrected by Marcin Wardyński. The corrected complexity is $O(mn + \Delta^2(\Delta^4 + m))$.

But, Marcin Wardyński not only found the erroneous argument, he also improved the complexity to $O(m(\Delta^2 + n))$.

Note that the factor $m \log n$ in the algorithm of Hagauer and Žerovnik comes from the complexity of finding the prime factors of Cartesian product graphs, which was $O(m \log n)$ then. In 2007* it was improved to $O(m)$, and so the factor $\log n$ could be dropped.

*Imrich and Peterin, Discrete Math. 307 (2007).

Edge-reconstruction

In 1964 Harary introduced the Edge Reconstruction Conjecture:

Any two graphs with at least four edges that have the same deck of edge-deleted subgraphs are isomorphic.

For products this was taken up by Dörfler 1974.

He showed that all nontrivial strong products and certain lexicographic products can be reconstructed from the deck of all edge-deleted subgraphs.

Weak edge-reconstruction

We show that the weak edge reconstruction problem can be solved from a single edge-deleted subgraph for nontrivial, connected finite or infinite Cartesian products.

For finite graphs G the reconstruction is possible in $O(mn^2)$ time, where n is the order and m the size of G.

The Cartesian product

The vertex set of the Cartesian product $G_1 \Box G_2$ is

$$V(G_1) \times V(G_2) = \{(x_1, x_2) \,|\, x_1 \in V(G_1), x_2 \in V(G_2)\}$$

and its edge-set is the set of all pairs $\{(x_1, x_2)(y_1, y_2)$, where $x_1 y_1 \in E(G_1), x_2 = y_2$ or $x_1 = y_1, x_2 y_2 \in E(G_2)$

The product is commutative, associative and has K_1 as a unit.

Given ℓ graphs we can thus write $G_1 \Box \cdots \Box G_\ell$ for their product and consider the vertices as vectors $(x_1, \ldots x_\ell)$, where $x_i \in V(G_i)$.

Then $x = (x_1, \ldots x_\ell)$ and $y = (y_1, \ldots y_\ell)$ are adjacent exactly if $\exists k$ such that $x_k y_k \in V(G_k)$ and $x_i = y_i$ for $i \neq k$.

Prime factorization

The x_i are the coordinates of x
two vertices are adjacent iff they differ in exactly one coordinate.

We also call x_i the projection $p_i(x)$ of x to $V(G_i)$.

A graph $G \neq K_1$ is prime or indecomposable if $G \cong A \Box B$ implies that either $A \cong K_1$ or $B \cong K_1$.

Every connected graph has a unique prime factorization with respect to the Cartesian product, up to isomorphisms and the order of the factors*.

*G. Sabidussi 1960, V.G. Vizing 1963.

Infinitely many factors

This easily extends to infinitely many factors

Let G_ι, $\iota \in I$, be a finite or infinite set of graphs and

X the set of all functions $x : I \to \bigcup_{\iota \in I} V(G_\iota)$ where $x : \iota \mapsto x_\iota \in V(G_\iota)$.

Then the Cartesian product

$$G = \prod_{\iota \in I} G_\iota$$

has X as its set of vertices, and $xy \in E(G)$

if $\exists \kappa \in I$ such that $x_\kappa y_\kappa \in E(G_\kappa)$ and $x_\iota = y_\iota$ for all $\iota \in I \setminus \{\kappa\}$.

The weak Cartesian product

The product of finitely many graphs is connected if and only if every factor is.

However, a product of infinitely many nontrivial graphs must be disconnected because it contains vertices differing in infinitely many coordinates.

No two such vertices can be connected by a path of finite length.

We call the connected components of $G = \prod_{\iota \in I} G_\iota$ containing $a \in V(\prod_{\iota \in I} G_\iota)$ the weak Cartesian product

$$G = \prod_{\iota \in I}^{a} G_\iota$$

Prime factorization with respect to the weak Cartesian product

To every connected, infinite graph G there exists a set of prime graphs $\{G_\iota \,|\, \iota \in I\}$, which are unique up to isomorphisms, such that

$$G = \prod_{\iota \in I}^{a} G_\iota$$

for an appropriate $a \in V(\prod_{\iota \in I} G_\iota)$.*

The weak Cartesian product may markedly differ from finite ones.

For example, finite graphs are vertex transitive iff all factors are, but $\prod_{\iota \in I}^{a} G_\iota$ can be vertex transitive even when all factors are asymmetric.

*Miller 1970, Imrich 1971.

Product coloring

Let $a = (a_1, \ldots, a_k)$ in $G = G_1 \,\square \cdots \square\, G_k$. Then the set of vertices

$$\{(a_1, \ldots a_{i-1}, x, a_{i+1}, \ldots, a_k) \,|\, x \in V(G_i)\}$$

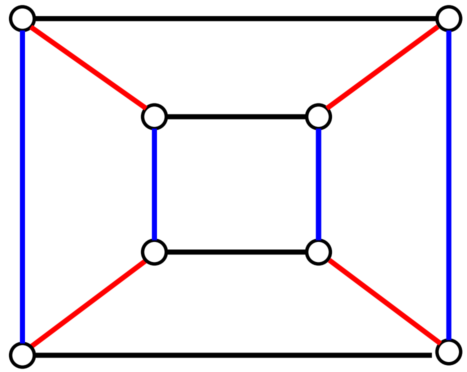
induces a subgraph of G that is isomorphic to G_i.

We call it the G_i-layer G_i^a through a.

We color the edges of $G = G_1 \,\square \cdots \square\, G_k$ with k colors c_1 to c_k, such that the edges of the G_i-layers have color c_i.

This is the product coloring of $G = G_1 \,\square \cdots \square\, G_k$. Clearly it depends on the particular representation of G.

For example, consider



It also shows that

automorphisms map sets of layers with respect to a prime factor into sets of layers with respect to prime factor.

Sets with respect to isomorphic factors can be interchanged.

Triangles are monochromatic in any product coloring.

For, let abc be a triangle. Let ab have color i and bc color j . Then a differs from c in coordinate i and j, but a, c are adjacent and can differ in only one coordinate. Hence $i = j$.

Lemma (Unique Square Lemma) *Let e, f be two incident edges of $G_1 \square G_2$ with different product colors.*
Then there exists exactly one square in $G_1 \square G_2$ containing e and f.
This square has no diagonals.

It implies that opposite edges of any square have the same color.

An easy consequence of the Unique Square Lemma is the following.

If there is an edge from a vertex of G_i^a to one of G_i^b, then the edges between G_i^a and G_i^b induce an isomorphism between G_i^a and G_i^b.

Convexity

A subgraph $W \subseteq G$ is convex in G if every shortest G-path between vertices of W lies entirely in W.

Proposition *A subgraph W of $G = G_1 \Box \ldots \Box G_k$ is convex if and only if $W = U_1 \Box \ldots \Box U_k$, where each U_i is convex in G_i.*

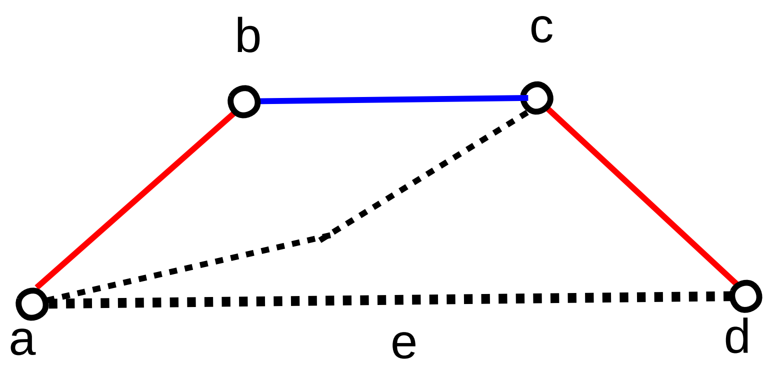
Every layer G_i^a is convex in G.

Edge deleted nontrivial Cartesian products are prime

Let G be graph and $e \in E(G)$.

Then the edge-deleted graph $G \setminus e$ is defined on the same set of vertices as G and $E(G \setminus e) = E(G) \setminus \{e\}$.

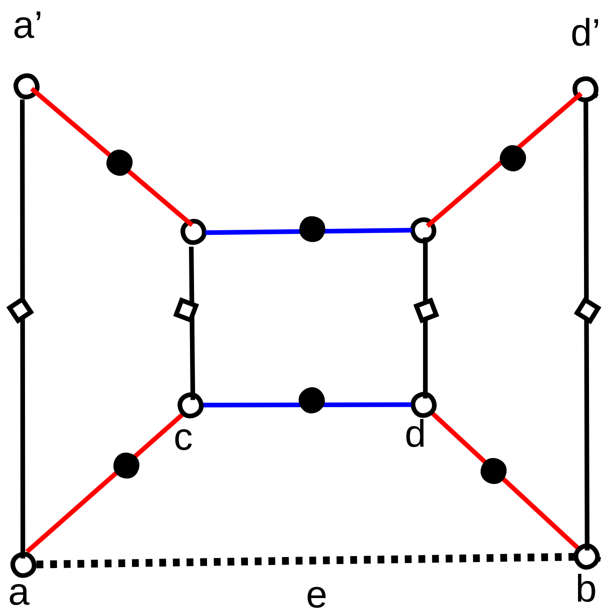
Lemma *Let G be a nontrivial Cartesian product and $e \in E(G)$. Let $e = ad$ and $abcda$ be a product square in G. If $G \setminus e$ is a Cartesian product, then the path $abcd$ must be monochromatic in any product coloring of $G \setminus e$.*



Lemma *Let G be a nontrivial Cartesian-product and $e \in E(G)$. Then $G \setminus e$ is prime.*

Proof (Outline) Let $G = A \square B$ and let c_A, c_B be the product colors of G. Suppose that e is contained in an A-layer and set $e = ad$, where $abcda$ is a product square of $A \square B$.

We assume that $G \setminus e$ is not prime, say $G \setminus e = X \square Y$, with product colors c_X, c_Y, and lead this to a contradiction.

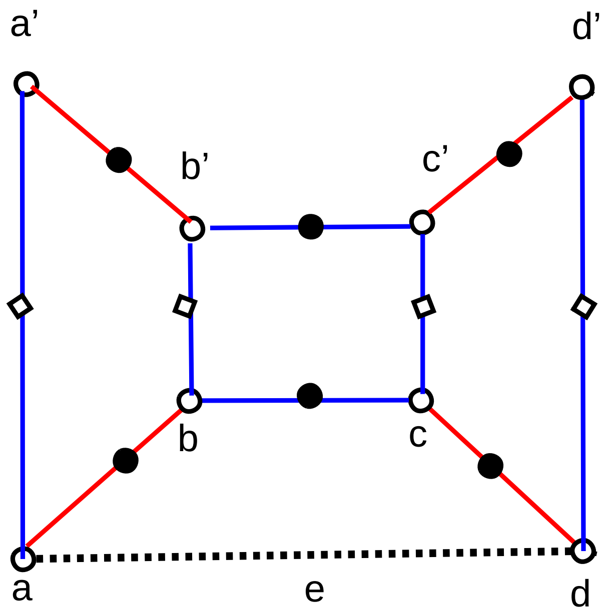


By the previous Lemma all edges of the path $abcd$ in $G \setminus e$ have the same color in any product coloring of $G \setminus e$. We choose the notation such that they are in an X-layer, so their color is c_X.

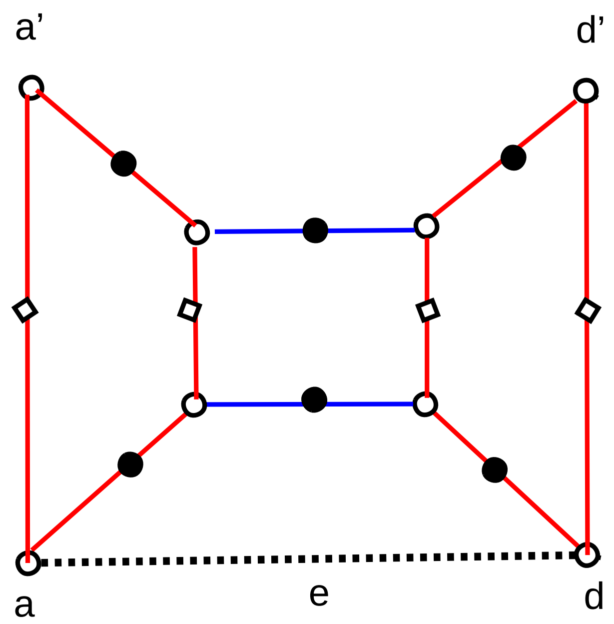
There must be at least one edge incident to b with color c_Y. Let this edge be aa'.

Now we have to consider two different cases regarding the position of aa' in G, namely whether aa' belongs to an A-layer or to a B-layer.

Consider the case where aa' is in a B-layer.



And now the case where aa' is in an A-layer.



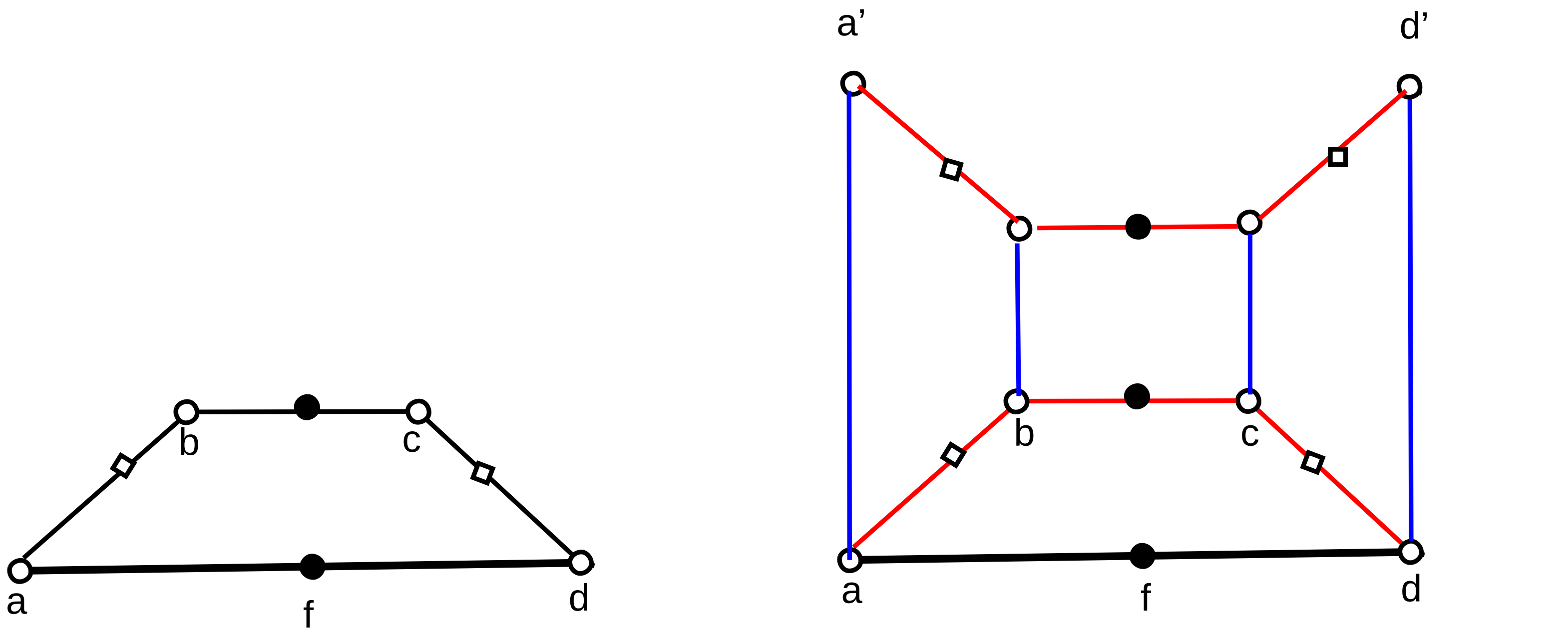
Reconstruction when G does not have K_2 as a factor

Lemma *Let G be a nontrivial Cartesian product and f an edge not in $G \setminus e$. If G does not have K_2 as a factor, then $G \setminus e \cup f$ is prime unless $f = e$.*

Proof Let $G = A \,\square\, B$ and $H = G \setminus e \cup f$ be a Cartesian product $X \,\square\, Y$.

Let $f = ad$, $G \setminus e \cup f = X \,\square\, Y$ and f be in an X-layer. f must be in a product square, say $abcda$, and where bc also has color c_X, and ab, cd have color c_Y.

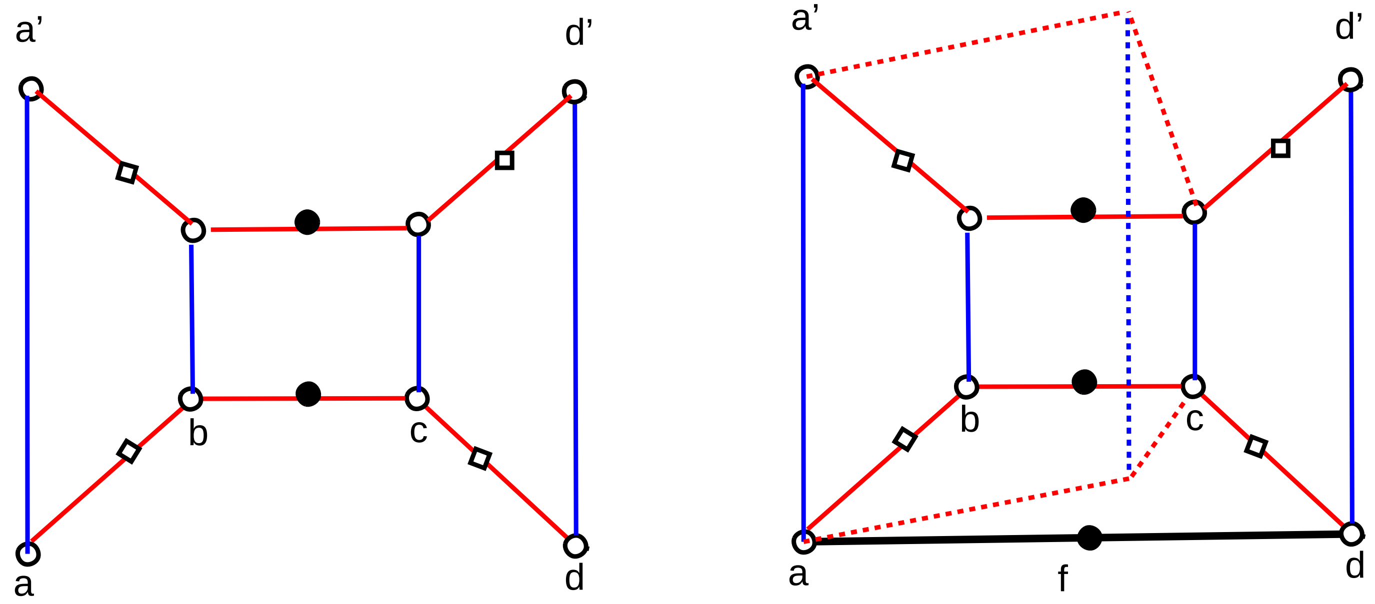
Now we ask whether the edges ab and bc have different colors in the original graph G.



If that were the case, then ab and bc must span a product square $abcg$ in G, but then the edges ab and bc would span two different squares in $G \setminus e \cup f$, which is not possible.

Similarly we argue that there can be no product-square $bcdg$ in G, hence the path $abcd$ is monochromatic in G. Without loss of generality we can assume that $abcd$ is colored c_A.

Clearly there must be a path $a'b'c'd'$ and edges aa', bb', cc', dd' in G, where $abcd$ has color c_A and the other edges have color c_B. Let R be this subgraph of G.



If the edge $a'd'$ is in G, then it is in the A-layer that contains $a'b'c'd'$ by convexity. Clearly this means that ad is also in G, because the edges aa', bb', cc', dd' induce a (partial) isomorphism between the A-layer through a' and that through a. If $e = ad$, then $f = e$ and H and G are isomorphic. If $ad \neq e$, then f joins two vertices of $E(G \setminus e)$, contrary to assumption. So we can assume that $a'd' \notin E(G)$.

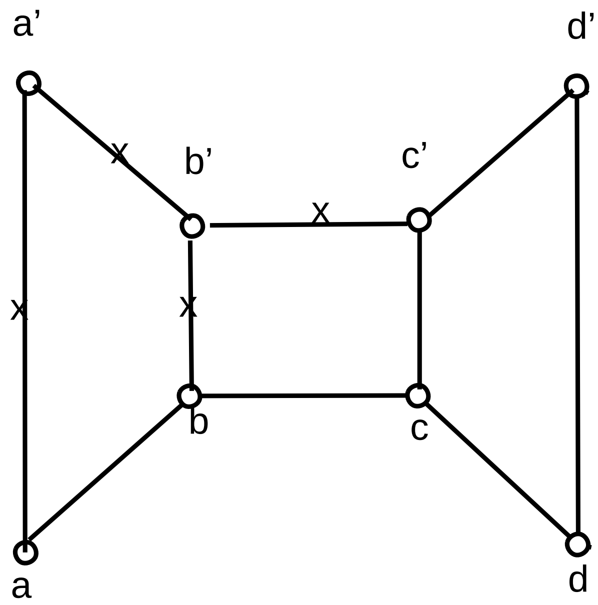
Assume now that R is in H. Because $a'b'$ and $c'd'$ have color c_Y, but not $b'c'$, there are product squares $a'b'c'x'a'$ and $b'c'd'y'$ in H. Neither $x' \neq d'$ nor $y' \neq a'$ can hold, because $a'd' \notin E(G)$. Furthermore $x \neq y$, otherwise c_X would be equal to c_Y.

By convexity x' and y' are in $A^{a'}$. But then, by the isomorphism of layers, we have vertices x, y in A^a and squares $abcxa$ and $bcdy'$. At least one of those squares does not contain e, and is thus in H. It contains two edges that are also in $abcd$, in contradiction to the Unique Square Lemma.

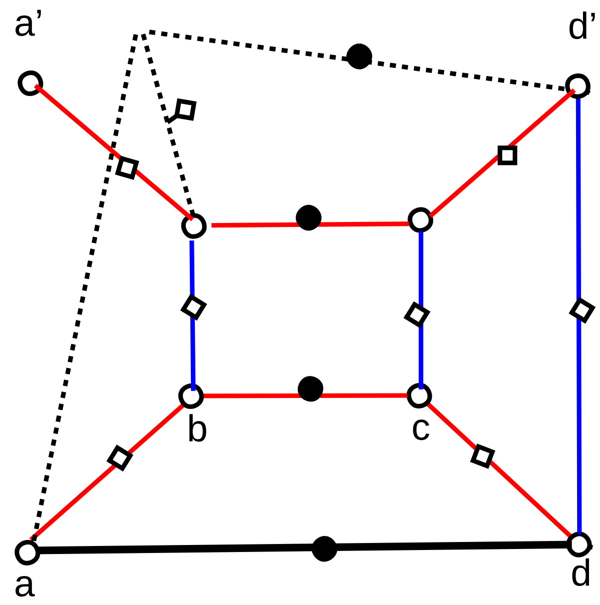
Hence, R is not in H, which means that one of its edges is e. Notice that this implies $f \neq e$. Because $abcd$ is in H we have the following possibilities for e: $e = aa', dd'$, $e = bb', cc'$, $e = a'b', cc'$, or $e = b'c'$. By the symmetry of R it suffices to treat $e = aa', bb', a'b'$, and $e = b'c'$.

We will show that H is prime in all these cases.

Consider the case $e = aa'$



Posssible positions of e in R



The case $e = aa'$

Clearly $b'c'$ and $c'd'$ have different colors in H and there is a poduct square $b'c'd'y'b'$.

Because $a'd'$ is not in $E(G)$ the vertex $y' \neq a'$.

By the isomorphism of layers there is a square $bcdy$ without diagonals in A^a, contrary to the uniqueness of the product square $abcda$.

The other cases are treated similarly.

Reconstruction when G has a factor K_2

In the above proof the fact that G has no factor K_2 is only used when $e = b'c'$.

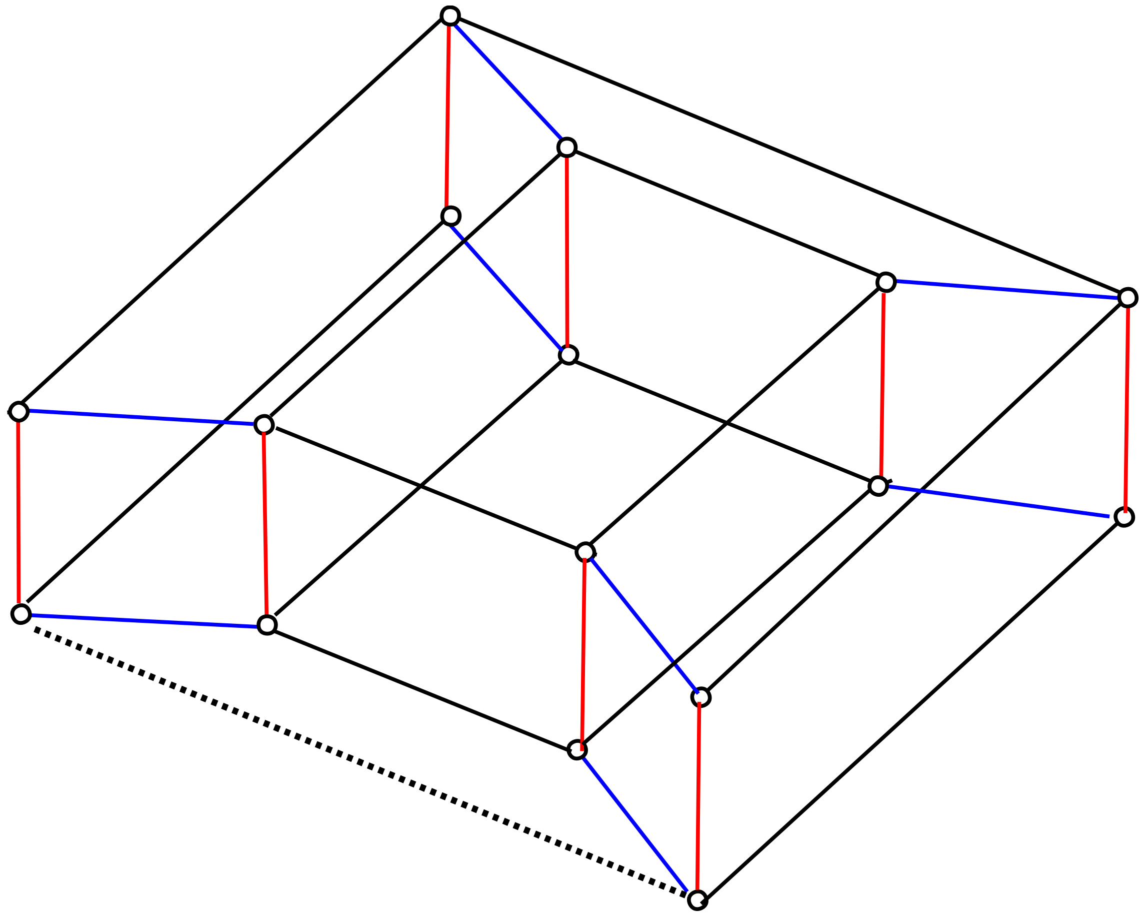
Then one gets through if $B \neq K_2$.

If $B = K_2$, but not Y, then one can interchange the roles of G and $G \setminus e \cup f$, resp. the roles of e and f.

In all these cases $G \setminus e \cup f$ is prime unless $f = e$.

Thus the reconstruction is unique in a very strong sense:
One can identify exactly two vertices a, b in $G \setminus e$ such that $G \setminus e \cup f$ is a product.

When G has at least two factors K_2, then several choices of f are possible, but the reconstruction is still unique up to isomorphisms.



Theorem (Main Theorem) *Let G be a connected, nontrivial Cartesian product and e an edge of G.*

Suppose insertion of an edge f into $G \setminus e$ yields a Cartesian product $H = G \setminus e \cup f$. Then H is isomorphic to G.

If G has at most one factor K_2, then $f = e$. If G has more than one factor K_2, then one can characterize all possibilities for the insertion of f.

Weak Cartesian product

This result also holds for infinite graphs, we never needed the finiteness of the factors.

Reconstruction complexity

Given a graph $G \setminus e$ one can try all possible extensions by an edge f and check whether they yield a Cartesian product.

If the order of G is n, there are $O(n^2)$ possibilities for f.

Because prime factorization is linear time and space in the size m of G^*, the reduction is possible in $O(mn^2)$ time and space.

Within the same time and space complexities one can also determine all possible reconstructions.

*Peterin and I, 2007.

Compare this with the fact that the complexity for weak (vertex) reconstruction is $O(m(\Delta^2 + n))$.

So one should be able to improve the above complexity of $O(mn^2)$ for edge-reconstruction.

What else is there to do for weak reconstruction ?

One can show that deletion of several edges with a common endpoint from a Cartesian product yields a prime graph.

If one deletes all incident edges, then one has the case of a vertex deleted subgraph, which has been treated, also from the algorithmic side.

The case when at least two edges incident with a vertex are deleted, but not all, is open.

And what is open for recognition?

The recognition for infinite connected Cartesian products is open.

The edge-recognition for finite and infinite Cartesian products is also open.

Probably one does not need the entire deck.

THANK YOU FOR YOUR ATTENTION