Logical laws for random graphs

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Examples

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triangle-free

$$\neg\left[\exists x_1 \exists x_2 \exists x_3 \quad (x_1 \sim x_2) \wedge (x_1 \sim x_3) \wedge (x_2 \sim x_3)\right]$$

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disconnected

$$\exists X \quad \left[\exists x \exists y \; X(x) \wedge (\neg X(y))\right] \wedge$$
$$\left[\forall x \forall y \; (X(x) \wedge [\neg X(y)]) \Rightarrow (\neg [x \sim y])\right]$$

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Variable and predicate symbols

- $x, y, x_1, x_2, \ldots$ are FO variables;
- X is a k-ary predicate variable symbol (or SO variable)

First order sentences

relational symbols $\sim, =$;

logical connectivities $\neg, \Rightarrow, \Leftrightarrow, \vee, \wedge$;

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$$\forall x \exists y \quad (x = y)$$

$$\exists x \quad \Big(\forall y \, \neg(x = y) \Rightarrow (x \sim y)\Big) \wedge$$
$$\Big(\forall \tilde{x} \, [(\forall y \, \neg(x = y) \Rightarrow (x \sim y)) \Rightarrow (x = \tilde{x})]\Big)$$

Second order sentences

relational symbols $\sim, =$;

logical connectivities $\neg, \Rightarrow, \Leftrightarrow, \vee, \wedge$;

FO variables $x, y, x_1, \ldots$;

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$$\exists X \quad \left[\forall x \exists y \forall z \; X(x,y) \wedge ([y \neq z] \Rightarrow \neg X(x,z))\right] \wedge$$

$$\left[\forall x \forall y \; (X(x,y) \Leftrightarrow X(y,x))\right]$$

Monadic second order sentences

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$$\forall X \ \left[(X \text{ is a clique}) \wedge (\forall Y \, [Y \supset X] \Rightarrow [Y \text{ is not a clique}]) \right]$$

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An existential monadic second order (EMSO) sentence is a monadic sentence such that all SO variables are in the beginning and bounded by existential quantifiers

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- If property P is defined in FO with k variables, then it is verified on n-vertex graph in $O(n^k)$ time.
- Fagin, 1973: P belongs to NP class if and only if P is defined in ESO.

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- Containing k-clique is defined in FO with k variables but not in FO with $k-1$ variables

Probabilistic approach

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Then the answer is negative.

FO zero-one law

**Theorem (Glebskii, Kogan, Liogon'kii, Talanov, 1969;
Fagin, 1976)**

Let φ *be a FO sentence.*
Let X_n *be the number of all graphs* G *on* $\{1, \ldots, n\}$ *such that* $G \models \varphi$.
Then

$$\text{either} \quad \frac{X_n}{2^{\binom{n}{2}}} \to 0, \qquad \text{or} \quad \frac{X_n}{2^{\binom{n}{2}}} \to 1.$$

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Then

$$\text{either} \quad \frac{X_n}{2^{\binom{n}{2}}} \to 0, \qquad \text{or} \quad \frac{X_n}{2^{\binom{n}{2}}} \to 1.$$

Or, in other words, $G(n, \frac{1}{2})$ obeys FO 0-1 law.

Ehrenfeucht game

- G, H — two graphs
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- k — number of rounds

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Duplicator wins if and only if

$$f : \{x_1, \ldots, x_k\} \to \{y_1, \ldots, y_k\} \text{ s.t. } f(x_i) = y_i$$

is isomorphism of $G|_{\{x_1,\ldots,x_k\}}$ and $H|_{\{y_1,\ldots,y_k\}}$.

Ehrenfeucht theorem

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Example

q.d. of

$$\exists x \quad \Big(\forall y\, \neg(x = y) \Rightarrow (x \sim y)\Big) \wedge$$

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equals 3

Ehrenfeucht theorem

Theorem (A. Ehrenfeucht, 1960)

Duplicator has a winning strategy in Ehrenfeucht game on G, H in k rounds

if and only if

for every FO sentence φ of q.d. k, either φ is true on both G, H, or φ is false on G, H

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Corollary: $G(n, \frac{1}{2})$ obeys FO 0-1 law if and only if, for every k, with asymptotical probability 1 Duplicator has a winning strategy in Ehrenfeucht game on two independent graphs $G(n, \frac{1}{2})$ and $G(m, \frac{1}{2})$ in k rounds.

k-extension property

A graph has k-**extension property** if, for every pair of disjoint sets of vertices A, B, $|A| + |B| \leq k$, there exists a vertex outside $A \sqcup B$ adjacent to every vertex of A and non-adjacent to every vertex of B.

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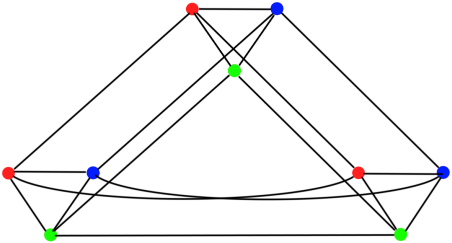

$k = 2$

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MSO logic of almost all graphs

Theorem (M. Kaufmann, S. Shelah, 1985)

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Conjecture (Le Bars, 2001): $G(n, \frac{1}{2})$ obeys 0-1 law for EMSO sentences with 2 FO variables

Le Bars conjecture is false

Theorem (S. Popova, Zhukovskii, 2019)

There exists an EMSO sentence φ with 1 monadic variable and 2 FO variables such that $\mathrm{P}(G(n, \frac{1}{2}) \models \varphi)$ does not converge.

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There are two disjoint cliques such that

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- every vertex outside both cliques has neighbors in both.

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$x_1, \ldots, x_s$; $X_1, \ldots, X_r$ are chosen in G;
$y_1, \ldots, y_s$; $Y_1, \ldots, Y_r$ are chosen in H.

Monadic Ehrenfeucht theorem

Duplicator wins if and only if

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$G(n, \frac{1}{2})$ obeys MSO 0-1 law if and only if, for every k, with asymptotical probability 1 Duplicator has a winning strategy in MSO Ehrenfeucht game on two independent graphs $G(n, \frac{1}{2})$ and $G(m, \frac{1}{2})$ in k rounds.

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In the case of EMSO, Spoiler always plays in one graph

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for a graph H with e edges,

$$\mathrm{P}(G(n, p) = H) = p^e (1 - p)^{\binom{n}{2} - e}$$

Zero-one laws for dense random graphs

Generalization of Glebskii et al. and Fagin's 0-1 law

Let $\forall \alpha > 0 \ \min\{p, 1-p\} n^{\alpha} \to \infty$. Then $G(n, p)$ obeys FO 0-1 law.

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Generalization of Le Bars non-convergence result

Let $\forall \alpha > 0\ \min\{p, 1-p\} n^{\alpha} \to \infty$. Then $G(n, p)$ does not obey EMSO convergence law.

First order zero-one laws for sparse random graphs

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- If $1 + \frac{1}{m+1} < \alpha < 1 + \frac{1}{m}$, then EMSO 0-1 law holds.
- If $\alpha > 2$, then EMSO 0-1 law holds.
- If $\alpha = 1 + \frac{1}{m}$, then EMSO 0-1 law does not hold, but EMSO convergence law holds.

Random trees

$\mathcal{T}_n$ chosen uniformly at random from the set of all trees on $\{1, \ldots, n\}$

Theorem (G.L. McColm, 2002)

$\mathcal{T}_n$ obeys MSO 0-1 law.

The main tool

S is **pendant** in T, if there exists an edge in T such that S is a component of $T - e$

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- For every k, there exists K such that

  if, for every tree S on at most K vertices,
  T and F contain a pendant subtree isomorphic to S,

  then Duplicator wins monadic Ehrenfeucht game on G, H in k rounds.

Uniform attachment model

• $m = 1$ — random recursive tree (R.T. Smythe, H.M. Mahmoud, 1995)
• For arbitrary m, considered by B. Bollobás, O. Riordan, J. Spencer, G. Tusnády in 2000

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- $\mathcal{G}_0$ is m-clique on $\{1, \ldots, m\}$
- $\mathcal{G}_{n+1}$ is obtained from $\mathcal{G}_n$ by adding the vertex $v_n = n + m + 1$ and m edges from v_n to $\mathcal{G}_n$ chosen uniformly at random

Logic of uniform attachment: $m = 1$

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Let X_n be the number of $K_4 \setminus e$ in $\mathcal{G}_n$.
Let k be large enough, and $g(k) = \binom{k}{2}$ be the maximum possible number of $K_4 \setminus e$ in $\mathcal{G}_k$.
$\mathsf{P}(X_n \geq g(k))$ does not converge neither to 0, nor to 1.

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What about convergence?

The convergence

Theorem (Y. Malyshkin, Zhukovskii, 2019++)

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$\mathcal{G}_n$ obeys EFO convergence law

The structure: crucial properties

A connected graph on v vertices is **complex** if it contains at least $v + 1$ edges

Induced subgraph $H \sqsubset G$ is called **separated** if all its vertices having degrees at least 2 are not adjacent to any vertex outside H

The structure: crucial properties

Let K, N be large

1. With probability at least $1 - \varepsilon$, all complex subgraphs of $\mathcal{G}_n$ on at most K vertices belong to $\mathcal{G}_n|_{\{1,\dots,N\}}$
2. With asymptotical probability 1, for every **admissible** tree T on at most K vertices, $\mathcal{G}_n$ has a separated subgraph isomorphic to T such that all its vertices are outside $\{1, \dots, N\}$
3. For every **admissible** connected unicyclic graph C, the probability that $\mathcal{G}_n$ has a separated subgraph isomorphic to C such that all its vertices are outside $\{1, \dots, N\}$ converges

Preferential attachment

R. Albert, A.-L. Barabási, 1999,
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- $\mathcal{G}_0$ is m-clique on $\{1, \ldots, m\}$
- $\mathcal{G}_{n+1}$ is obtained from $\mathcal{G}_n$ by adding the vertex $v_n = n + m + 1$ and m edges independently
- the probability that i-th edge connects v_n with u is proportional to $\deg_{\mathcal{G}_n}(u)$ and equals

$$\frac{\deg_n(u)}{m(n + m - 1)}$$

Logic of preferential attachment

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