The quantum double model as a topologically ordered phase

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Institute for Research in Fundamental Sciences (IPM)

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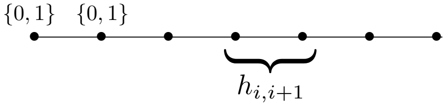
Ising model



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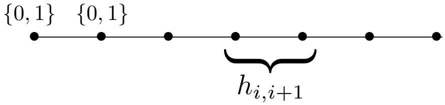

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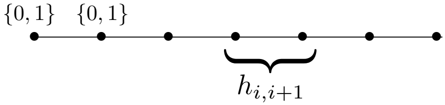

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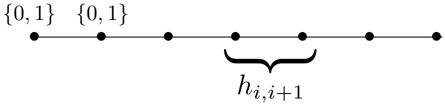

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$$x = 00\ldots0 \quad \text{and} \quad x = 11\ldots1$$

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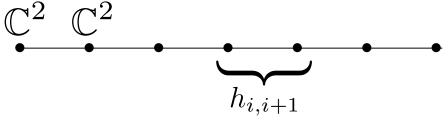

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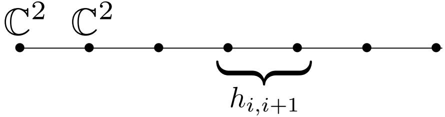

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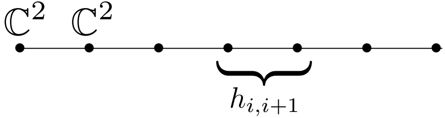

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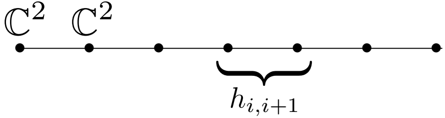

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Anyons

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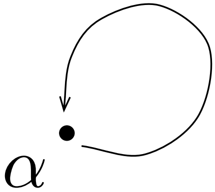

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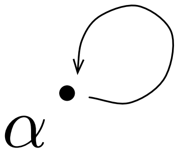

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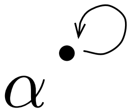

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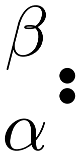

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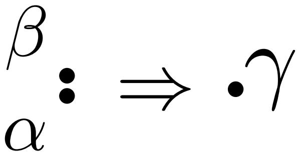

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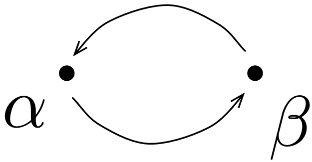

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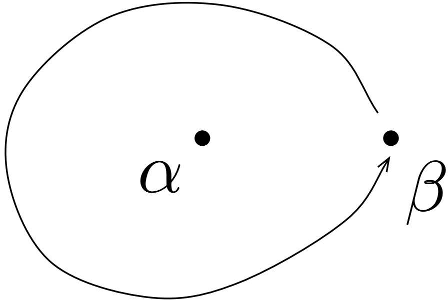

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Toric code [Kitaev '97]

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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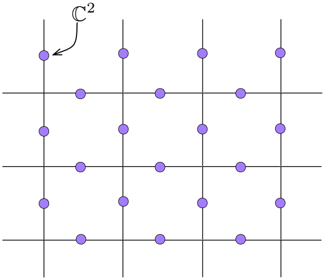

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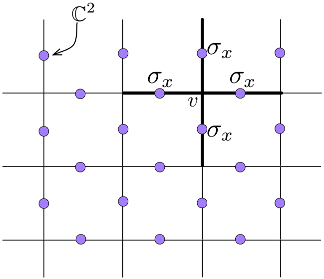

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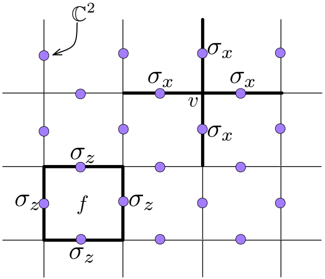

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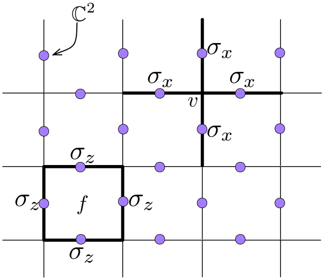

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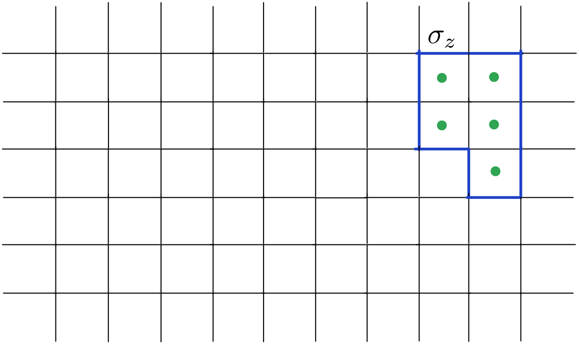
- $B_f = \sigma_z\sigma_z\sigma_z\sigma_z \quad \Rightarrow \forall v, f: \quad A_v B_f = B_f A_v$
- $H = -\sum_v A_v - \sum_f B_f$
- $|\Psi_0\rangle$ has the min energy iff $\forall v, f, \quad A_v|\Psi_0\rangle = B_f|\Psi_0\rangle = |\Psi_0\rangle$
- What is dim of ground space?

Loop operators



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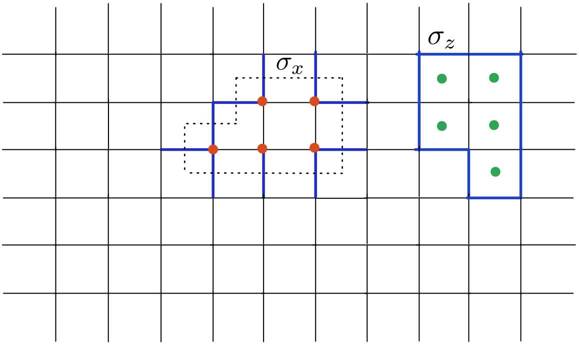

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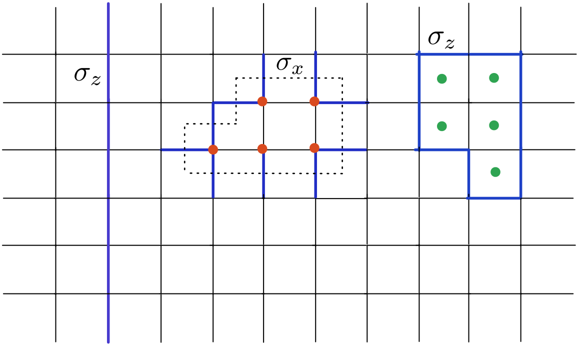

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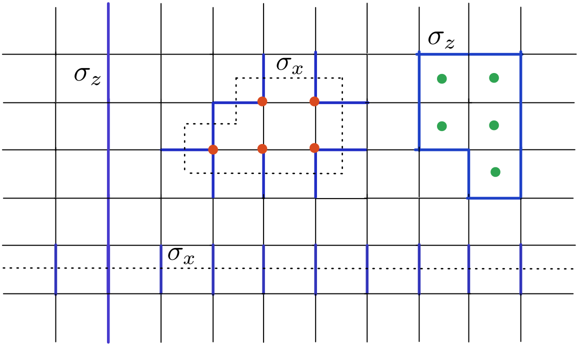

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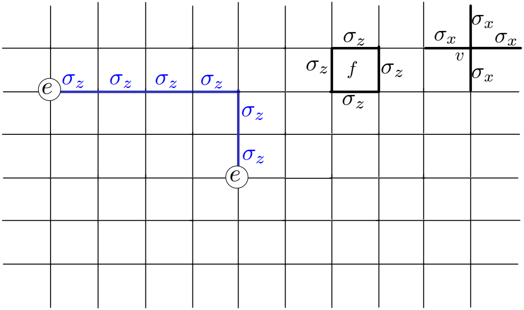
  - These 4 states are not locally distinguishable! (topological order)

String operators



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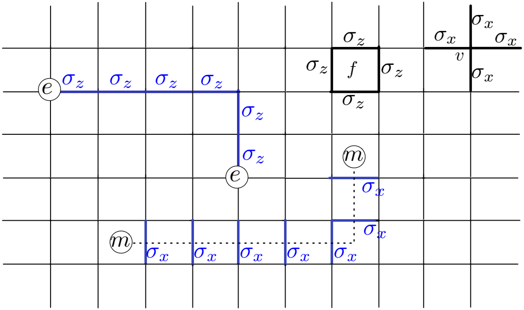

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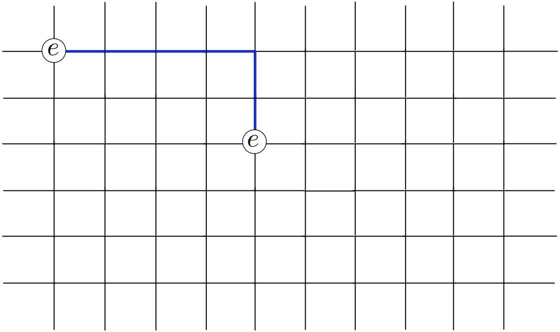
Anyons



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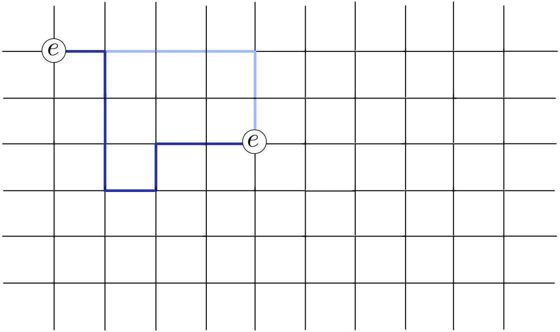

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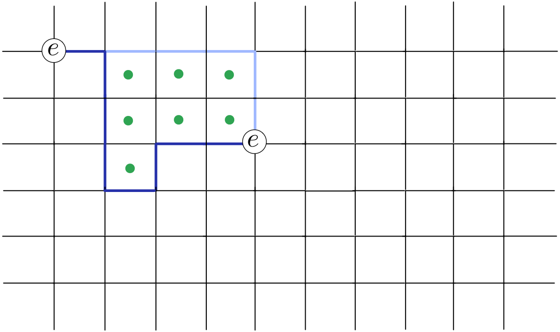

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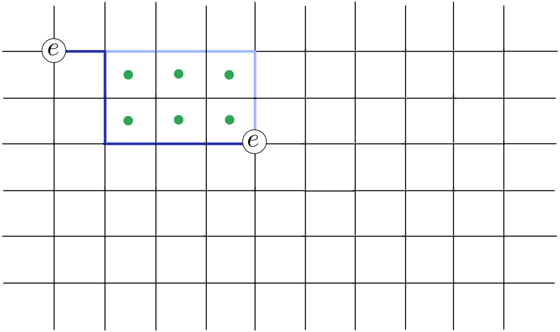

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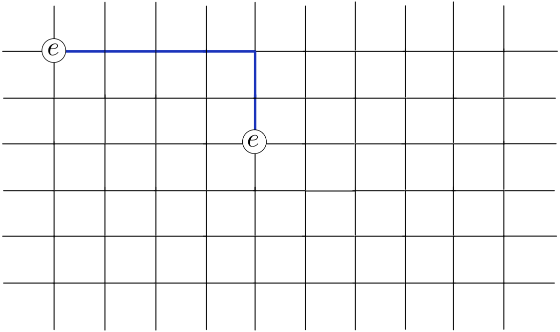

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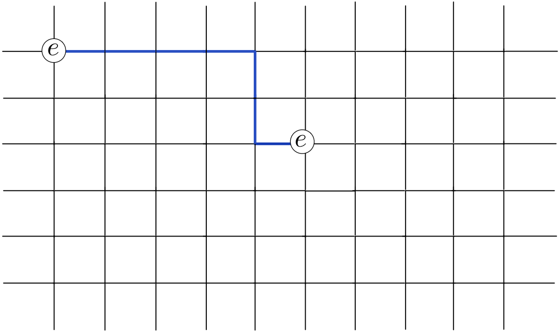

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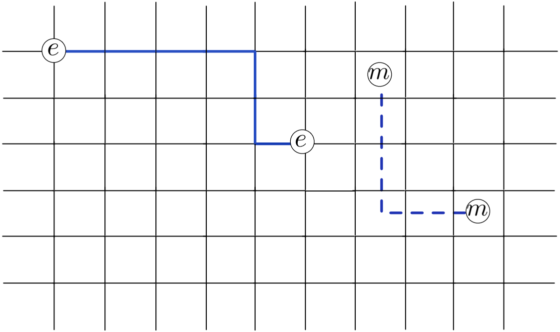

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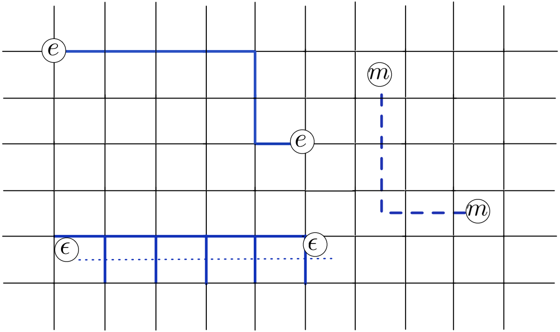

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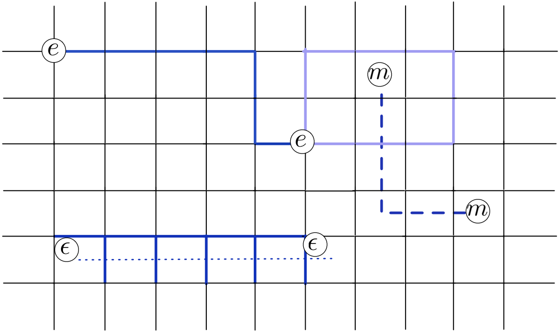

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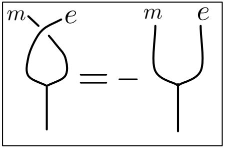

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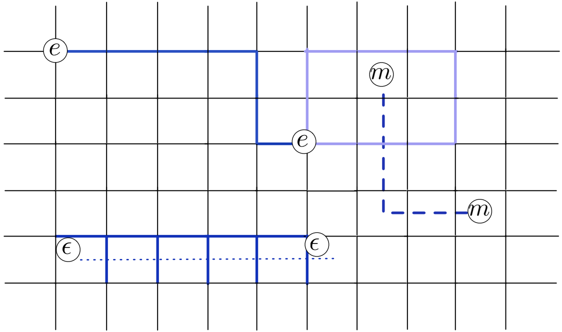

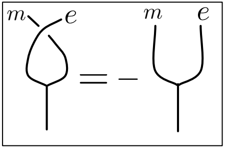

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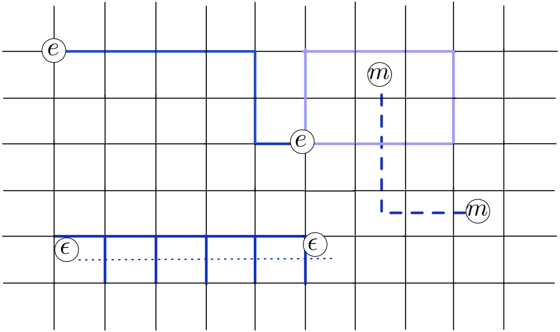

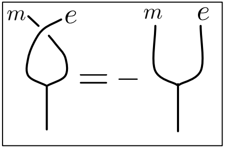

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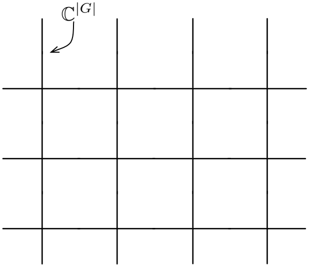
- Anyons $\{\mathbf{1}, e, m, \epsilon\}$
- These anyons correspond to 4 irreducible representations of the quantum double of $\mathbb{Z}_2$.

Generalization to an arbitrary group

- G: finite group
- $\mathbb{C}^{|G|}$: Hilbert space with orthonormal basis $\{|g\rangle : \quad g \in G\}$



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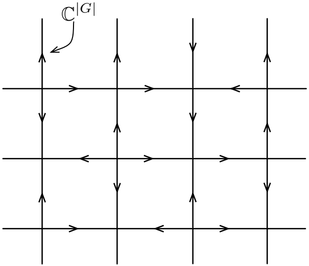

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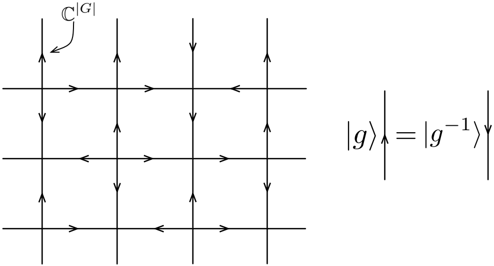

Generalization to an arbitrary group

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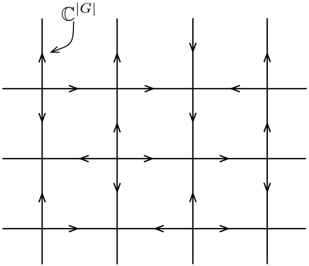
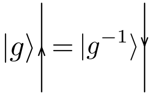
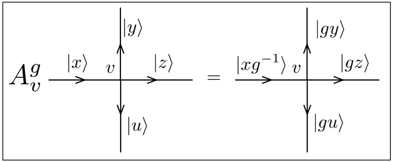
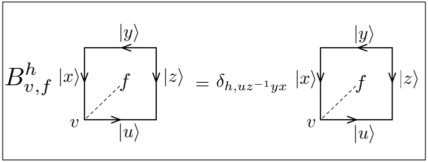
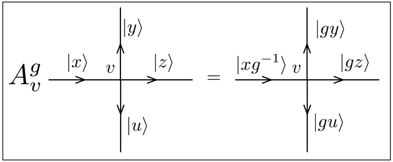


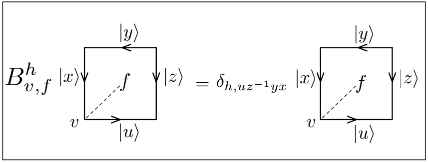






The quantum double model





- $A_v = \frac{1}{|G|}\sum_{g\in G} A_v^g, \qquad B_f = B_f^e$

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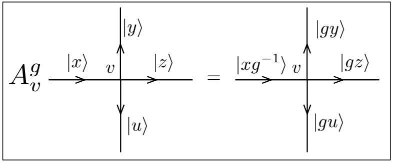

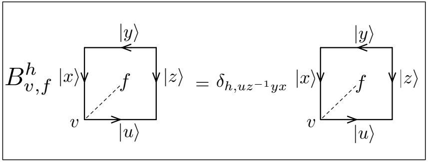

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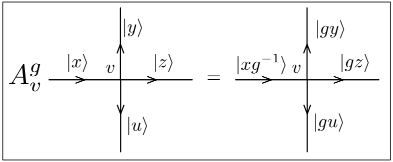

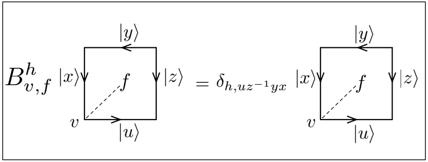

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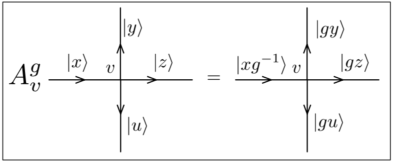

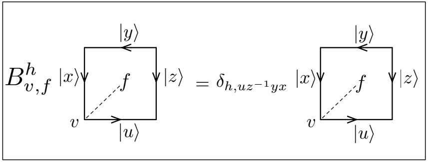

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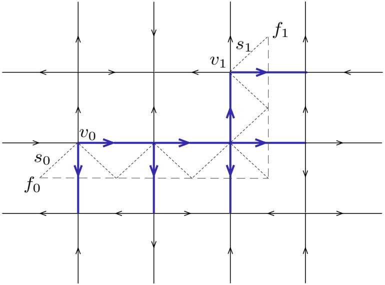
- The ground state on a planar lattice is unique

Strings Operators



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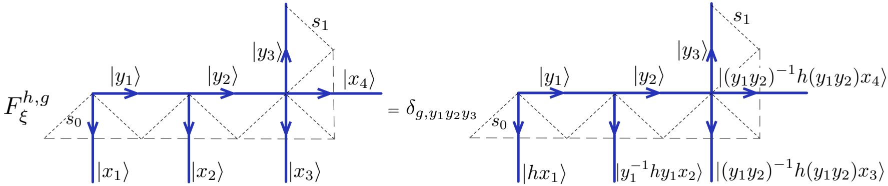

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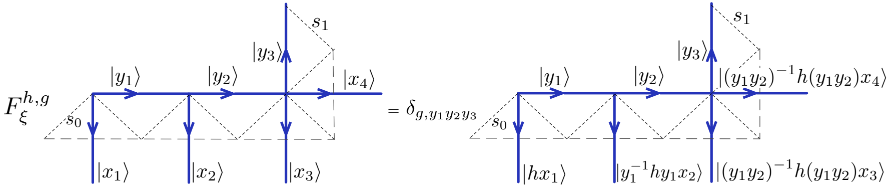

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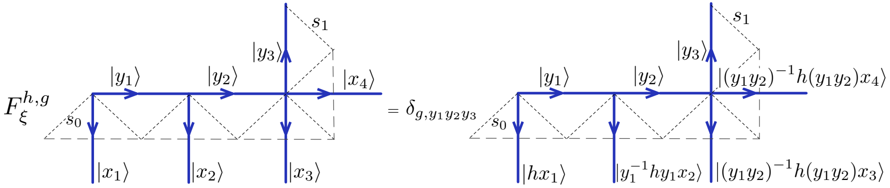

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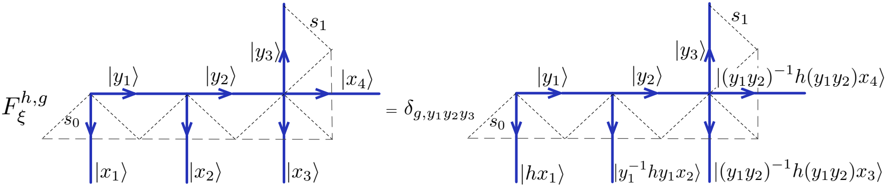

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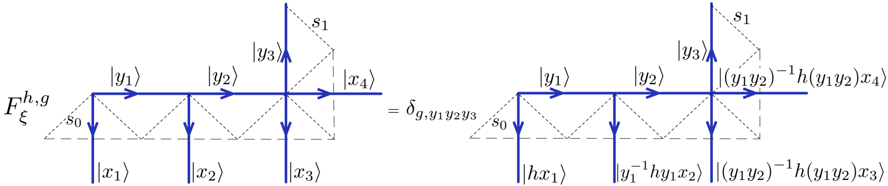

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- We obtain a system of anyons
- What are the anyon types?

Drinfeld double of a group

- A and B operators on a site define an algebra

$$\begin{aligned} A^g A^{g'} &= A^{gg'}, \\ (A^g)^\dagger &= A^{(g^{-1})}, \\ B^h B^{h'} &= \delta_{h,h'} B^h, \\ (B^h)^\dagger &= B^h, \\ A^g B^h &= B^{(ghg^{-1})} A^g. \end{aligned}$$

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- This space gives a representation of $\mathcal{D}(G)$
- So decomposition into irreducible representations gives the anyon types
- Anyon types are in 1-to-1 correspondence with *irreducible representations* of $\mathcal{D}(G)$

Drinfeld double of a group

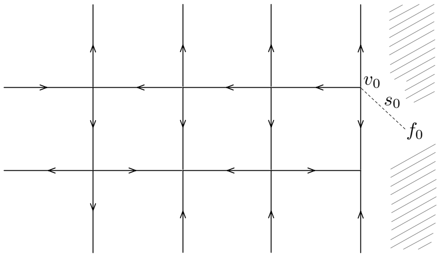
$$\mathcal{D}(G) = \mathbb{C}[G]^* \rtimes \mathbb{C}[G]$$

- is an algebra
- Rep$\mathcal{D}(G)$ is semi-simple
- co-algebra structure $\Delta : \mathcal{D}(G) \to \mathcal{D}(G) \otimes \mathcal{D}(G)$

$$\Delta(A^g) = A^g \otimes A^g, \qquad \Delta(B^h) = \sum_{h_1 h_2 = h} B^{h_1} \otimes B^{h_2}.$$

- Rep$\mathcal{D}(G)$ is a tensor category
- $\mathcal{D}(G)$ is quasi-triangular: $\quad R = \sum_{g \in G} B^g \otimes A^g$
- $R : X \otimes Y \to X \otimes Y$ gives the braiding
- Rep$\mathcal{D}(G)$ is a braided tensor category

Boundary I



- Fix a subgroup $K \subseteq G$

$$A_{s_0}^K = \frac{1}{|K|} \sum_{k \in K} A_{s_0}^k, \qquad B_{s_0}^K = \sum_{k \in K} B_{s_0}^k$$

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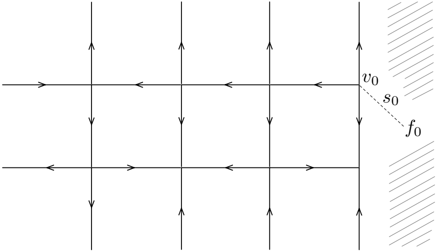

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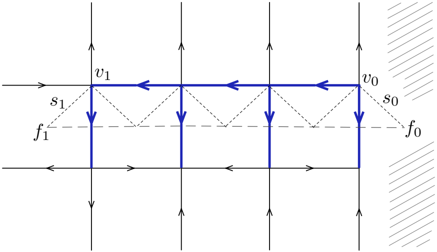
- $$H_{G,K} = -\sum_{v:\text{ internal}} A_v - \sum_{f:\text{ internal}} B_f - \sum_{s:\text{ boundary}} \left(A_s^K + B_s^K\right)$$

Condensations I



$$\mathcal{C}_\xi = \{T \in \mathcal{F}_\xi : [T, A^K_{s_0}] = [T, B^K_{s_0}] = 0\}$$

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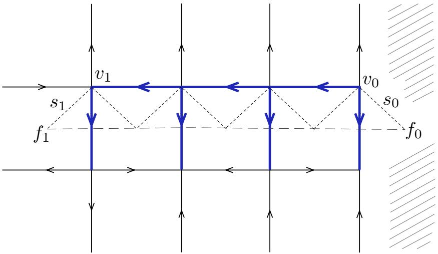

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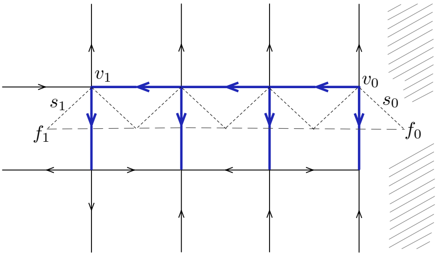

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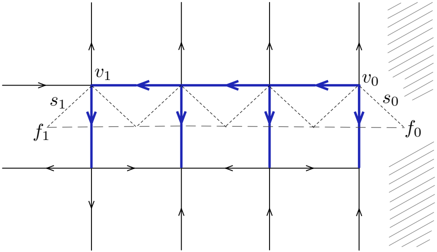

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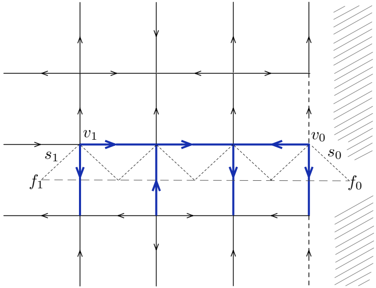
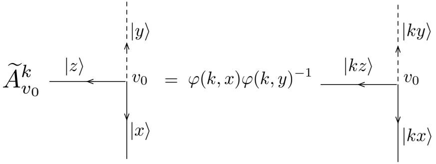
- $\forall T \in \mathcal{C}_\xi : T|\Psi_0\rangle$ is a single-site excited state
- An anyon at s_1 disappears after moving to the boundary
- A full characterization of $\mathcal{C}_\xi$ and condensed anyons are known

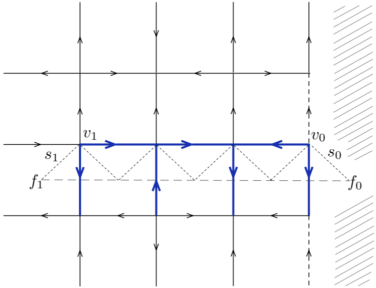
Boundary & Condensation II

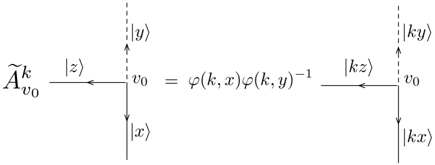




- $K \subseteq G$, $\quad \varphi$ a 2-cocycle of K:
  $\varphi : K \times K \to \mathbb{C} \quad$ s.t. $\quad \varphi(kl, m)\varphi(k, l) = \varphi(k, lm)\varphi(l, m)$

Boundary & Condensation II





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- $\tilde{A}_s^K = \frac{1}{|K|} \sum_{k \in K} \tilde{A}_s^k$
-
$$\tilde{H}_{G,K} = -\sum_{v:\text{ internal}} A_v - \sum_{f:\text{ internal}} B_f - \sum_{s:\text{ boundary}} \left(\tilde{A}_s^K + B_s^K\right)$$

Condensations vs Algebras

Operators A^g & B^h	Drinfeld doubld $\mathcal{D}(G)$
Anyon types	Rep$\mathcal{D}(G)$
Fusion of anyons	Tensor product of representations
Braiding of anyons	Quasi-triangularity of $\mathcal{D}(G)$
Boundary, Condensation	Algebra $\mathcal{C}_\xi = \{T \in \mathcal{F}_\xi : [T, \tilde{A}_s^K] = [T, B_s^K] = 0\}$

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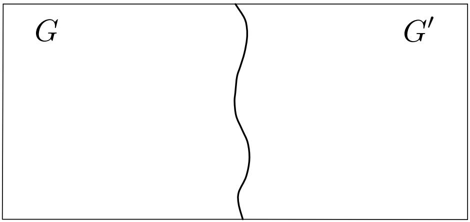
- As a representation of $\mathcal{D}(G)$, $\mathcal{C}_\xi$ is an *indecomposable separable commutative algebra*.

Theorem [Davydov '10] All *maximal indecomposable separable commutative* algebras of Rep$\mathcal{D}(G)$ are in 1-to-1 correspondence with pairs (K, φ) where $K \subseteq G$ is a subgroup and φ is a 2-cocycle of K.

Domain walls vs boundaries



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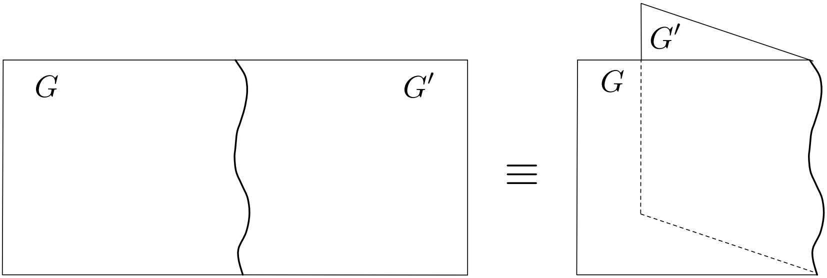

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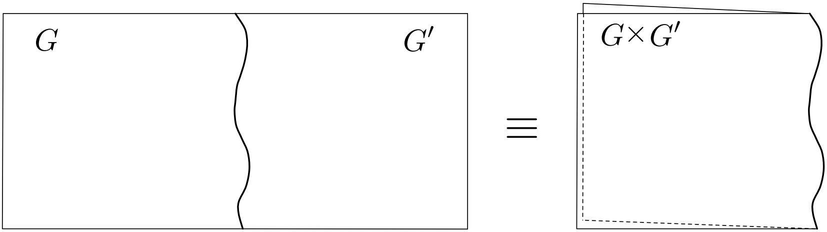
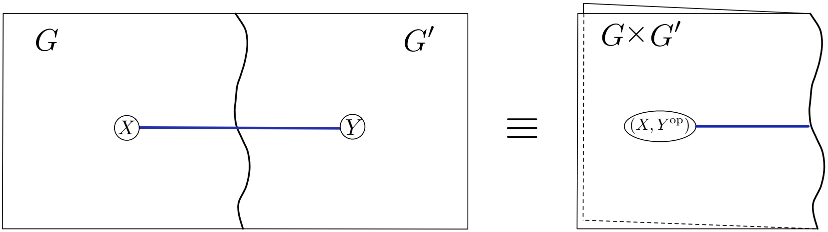


- A domain wall between the G-phase and the G'-phase is defined by $U \subseteq G \times G'$ and a 2-cocycle φ of U

Domain walls vs boundaries



- A domain wall between the G-phase and the G'-phase is defined by $U \subseteq G \times G'$ and a 2-cocycle φ of U
- Boundaries can be used to study equivalences of phases corresponding to different groups

Example: S_3

- $S_3 = \langle \sigma, \tau : \sigma^2 = \tau^3 = e, \sigma\tau = \tau^{-1}\sigma \rangle$.
- $\mathcal{D}(S_3)$ has 8 irreducible representations:

	A	B	C	D	E	F	G	H
conjugacy class	e	e	e	$\overline{\sigma}$	$\overline{\sigma}$	$\overline{\tau}$	$\overline{\tau}$	$\overline{\tau}$
irrep of the centralizer	$\mathbf{1}$	*sign*	π	$\mathbf{1}$	$[-1]$	$\mathbf{1}$	$[\omega]$	$[\omega^*]$

- Fusion rules

$\otimes$	A	B	C	D	E	F	G	H
A	A	B	C	D	E	F	G	H
B	B	A	C	E	D	F	G	H
C	C	C	$A\oplus B\oplus C$	$D\oplus E$	$D\oplus E$	$G\oplus H$	$F\oplus H$	$F\oplus G$
D	D	E	$D\oplus E$	$A\oplus C\oplus F\oplus G\oplus H$	$B\oplus C\oplus F\oplus G\oplus H$	$D\oplus E$	$D\oplus E$	$D\oplus E$
E	E	D	$D\oplus E$	$B\oplus C\oplus F\oplus G\oplus H$	$A\oplus C\oplus F\oplus G\oplus H$	$D\oplus E$	$D\oplus E$	$D\oplus E$
F	F	F	$G\oplus H$	$D\oplus E$	$D\oplus E$	$A\oplus B\oplus F$	$H\oplus C$	$G\oplus C$
G	G	G	$F\oplus H$	$D\oplus E$	$D\oplus E$	$H\oplus C$	$A\oplus B\oplus G$	$F\oplus C$
H	H	H	$F\oplus G$	$D\oplus E$	$D\oplus E$	$G\oplus C$	$F\oplus C$	$A\oplus B\oplus H$

A non-trivial symmetry of the $\mathbf{F}_q^+ \rtimes \mathbf{F}_q^\times$-phase

- For every finite field $\mathbf{F}_q$, there exists a non-trivial symmetry for $\mathbf{F}_q^+ \rtimes \mathbf{F}_q^\times$
- For $q = 2, 3$ we obtain the groups $\mathbb{Z}_2$ and S_3 respectively

A non-trivial symmetry of the $\mathbf{F}_q^+ \rtimes \mathbf{F}_q^\times$-phase

- For every finite field $\mathbf{F}_q$, there exists a non-trivial symmetry for $\mathbf{F}_q^+ \rtimes \mathbf{F}_q^\times$
- For $q = 2, 3$ we obtain the groups $\mathbb{Z}_2$ and S_3 respectively

- $U = \left\{ \big((a_1, \alpha), (a_2, \alpha^{-1}) \big) : \quad a_1, a_2 \in \mathbf{F}_q^+, \ \ \alpha \in \mathbf{F}_q^\times \right\}$
- For $g = \big((a_1, \alpha), (a_2, \alpha^{-1}) \big)$ and $h = \big((b_1, \beta), (b_2, \beta^{-1}) \big)$

$$\varphi(g, h) = \omega^{\mathrm{tr}_p(\alpha a_2 b_1)}$$

where p is a prime number and q is a power of p, $\mathrm{tr}_p : \mathbf{F}_q \to \mathbf{F}_p$ is the trace function, and ω is a p-th root of unity

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