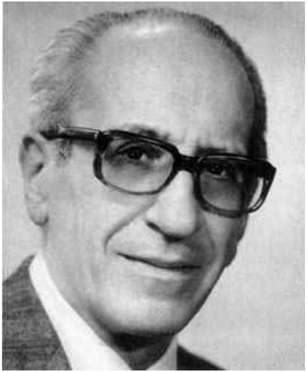
Invisibility and visibility related to Dirichlet-to-Neumann operator

Hassan Emamirad

21 Mai, 2013

Cloaking

Calderòn's inverse problem



Alberto Calderón had part of his early education in Switzerland, then attended secondary school in Mendoza, Argentina. He studied civil engineering at the University of Buenos Aires and graduated in 1947.

By A.P. Calderon

In this note we shall discuss the following problem. Let D be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with Lipschitzian boundary dD, and γ be a real bounded measurable function in D with a positive bower bound. Consider the differential operator

$$L_\gamma(W) = \nabla.(\gamma \nabla W)$$

acting on functions of $H^1(D)$ and the quadratic form $Q_\gamma(\phi)$, where the functions ϕ are restrictions to dD of functions in $H^1(\mathbb{R}^n)$, defined by

$$Q_\gamma(\phi) = \int_D \gamma (\nabla W)^2 dx \quad , W \in H^1(\mathbb{R}^n), \; W\big|_{dD} = \phi$$

$$L_\gamma W = \nabla.(\gamma \nabla W) = 0 \quad \text{in } D.$$

The problem is then to decide whether γ is uniquely determined by Q_γ and to calculate γ in terms Q_γ, if γ is indeed determined by Q_γ.

This problems originates in the following problem of electrical prospection. If D represents an in-homogeneous conducting body with electrical conductivity γ, determine γ by means of direct current steady state electrical measurements carried aut on the surface of D, that is, without penetrating D. In this physicak situation $Q_\gamma(\phi)$ represents the power necessary to maintain an electrical potential γ on dD.

Calderón's paper *On an inverse boundary problem*

In seminar on numerical analysis and its application to continuum Physics (Rio de Janeiro 1980)

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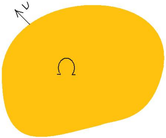

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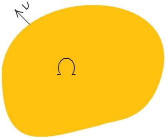

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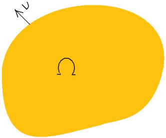

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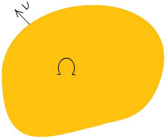

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$$f \in H^{1/2}(\partial\Omega) \mapsto \; u \in H^1(\Omega)$$

Such a function is called *γ-harmonic lifting* of f.

Dirichlet-to-Neumann operator

We define the *Dirichlet-to-Neumann operator* by

$$\Lambda_\gamma : f := u\,|_{\partial\Omega} \mapsto \frac{\partial u}{\partial \nu_\gamma} = \nu \cdot \gamma \nabla u\,|_{\partial\Omega}\ .$$

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(H1) $\gamma_{ij}(x) = \gamma_{ji}(x) \in C^\infty(\Omega)$;

(H2) There exists $0 < c_1 \leq c_2 < \infty$, such that

$$c_1 \|\xi\|^2 \leq \sum_{i,j=1}^{d} \xi_i \xi_j \gamma_{ij}(x) \leq c_2 \|\xi\|^2 \quad \xi \in \mathbb{R}^d.$$

Isotropic case

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- K. Astala and L. Päivärinta, Ann. Math. 2006.(case L^{∞}, $n = 2$).

Anisotropic case

Luc Tartar. Professor of Carnegie Mellon University.

Anisotropic case

3A. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, and let γ satisfy (1.2). For any C^1 diffeomorphism $\Phi : \Omega \to \Omega$ with

(3.1) $$\Phi(x) = x, \quad D\Phi(x) = I \quad \text{for all } x \in \partial\Omega,$$

let

$$\gamma^\Phi(\Phi(x)) = |\det(D\Phi(x))|^{-1} \cdot D\Phi(x)^t \cdot \gamma(x) \cdot D\Phi(x).$$

Then all elements of

$$\Gamma_4 = \{\gamma^\Phi : \Phi \text{ satisfies } (3.1)\}$$

give the same boundary measurements.

We owe this remark to L. Tartar. If $L_\gamma u = 0$, then $L_{(\gamma^\Phi)} u^\Phi = 0$, with

$$u^\Phi(x) = u \circ \Phi^{-1}(x);$$

by (3.1), $u^\Phi = u$ on $\gamma^\Phi \cdot \nabla u^\Phi = \gamma \cdot \nabla u$ on $\partial\Omega$.

3B [25]. Let Ω be the unit disc in $\mathbb{R}^2$, with polar coordinates (r,θ). For any function $\alpha(r)$, let

$$\gamma^\alpha = \begin{pmatrix} \alpha \cos^2\theta + \alpha^{-1}\sin^2\theta & (\alpha-\alpha^{-1})\sin\theta\cdot\cos\theta \\ (\alpha-\alpha^{-1})\sin\theta\cdot\cos\theta & \alpha\sin^2\theta + \alpha^{-1}\cos^2\theta \end{pmatrix}.$$

Then all elements of

$$\Gamma_5 = \{\gamma^\alpha : \alpha \in L^\infty(0,1),\ \text{ess inf}\ \alpha > 0\}$$

R. Kohn and M. Vogelius, *SIAM-AMS Proceeding*, 1984

Riemannian case

Let (M, g) be an n-dimensional Riemanninan manifold with smooth boundary ∂M.

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$$\begin{cases} \Delta_g u = 0 \quad \text{in} \;\; M, \\ u = f \quad \text{on} \;\; \partial M. \end{cases}$$

where $\Delta_g u := |g|^{-1/2} \partial_j (|g|^{1/2} g^{jk} \partial_k u)$ is the Laplace-Beltrami operator, with $|g| := \det(g_{jk}),\; [g_{jk}] = [g^{jk}]^{-1}$.

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The Dirichlet-to-Neumann operator is then defined by

$$\Lambda_g : f := u\ |_{\partial M} \mapsto |g|^{1/2}\nu_j g^{jk}\frac{\partial u}{\partial x_k}\ |_{\partial M}\ .$$

Push-forward

Let

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$$Q_\gamma(f) = \int_M \gamma^{jk}(x) \frac{\partial u}{\partial x^j} \frac{\partial u}{\partial x^k} dx,$$

since by the divergence theorem

$$Q_\gamma(f) = \int_{\partial M} \Lambda_\gamma(f) f d\sigma,$$

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where

$$(F_*\gamma)^{jk}(y) = \frac{1}{\det[\frac{\partial F^j}{\partial x^k}(x)]} \sum_{p,q=1}^{n} \frac{\partial F^j}{\partial x^p}(x) \frac{\partial F^k}{\partial x^q}(x) \gamma^{pq}(x) \Big|_{x=F^{-1}(y)}$$

Definition

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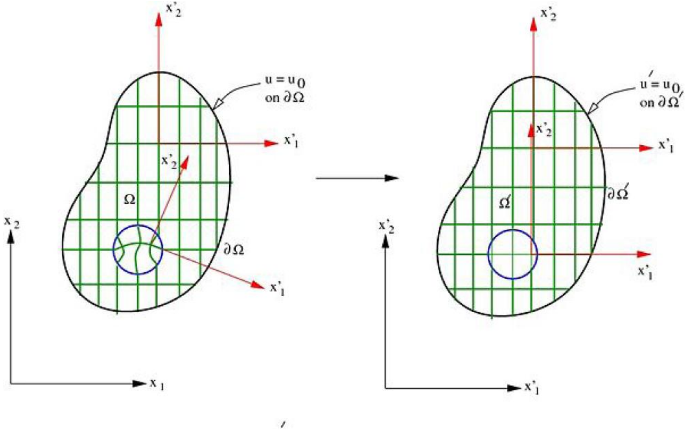

A simple example

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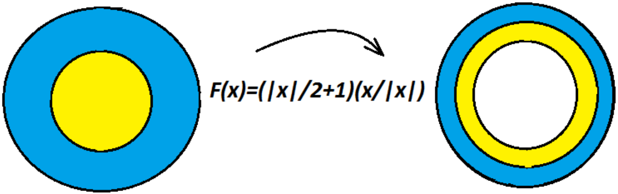

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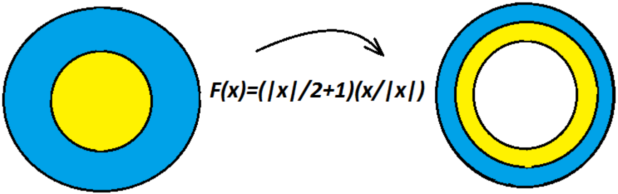
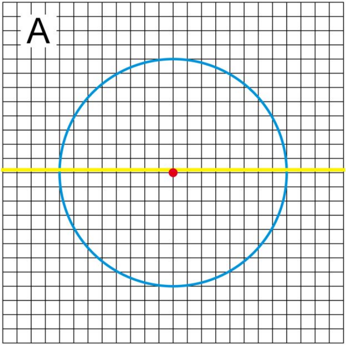
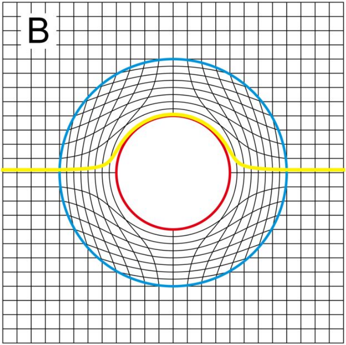
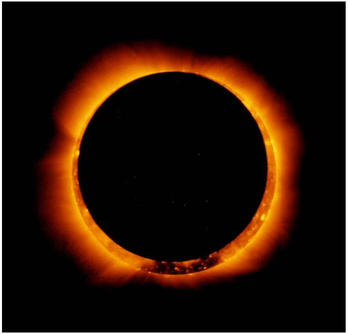


- Animation

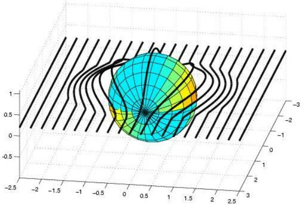
Non-conformal mapping







Eclipse.



Invisible sphere.

Riemannian point of view

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- $M_1 = B(0,2)$ the Riemannian manifold with the Euclidean metric $g_{jk} = \delta_{jk}$
- Hence, $\gamma = 1$ which corresponds to the homogeneous conductivity.
- Define a singular transformation

$$F : M_1 \setminus \{0\} \mapsto B_1, \quad F(x) = \begin{cases} \left(\frac{|x|}{2} + 1\right) \frac{x}{|x|}, & 0 < |x| < 2, \\ x & |x| \geq 2 \end{cases}$$

$$(F_*1)^{jk}(y) = \frac{1}{\det[DF(x)]} \sum_{p,q=1}^{n} \frac{\partial F^j}{\partial x^p}(x) \frac{\partial F^k}{\partial x^q}(x) \delta^{pq}(x)\Big|_{x=F^{-1}(y)}$$

- Let
$$DF(x) = \left(\frac{1}{2} + \frac{1}{|x|}\right) I - \frac{\hat{x}\hat{x}^t}{|x|}, \quad x \neq 0$$
be the Jacobian matrix at x, where I is the identity matrix and $\hat{x} = x/|x|$.

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- $$\det[DF(x)] = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{|x|}\right)^{n-1} = \frac{(|x|+2)^{n-1}}{2^n |x|^{n-1}}$$

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- $$(F_*1)(y) = \frac{2^n |x|^{n-1}}{(|x|+2)^{n-1}} \left[\left(\frac{1}{4} + \frac{1}{|x|} + \frac{1}{|x|^2} \right)(I - \hat{x}\hat{x}^t) + \frac{\hat{x}\hat{x}^t}{4} \right]$$

  where the right-hand side is evaluated at

  $$x = F^{-1}(y) = 2(|y|-1)\frac{y}{|y|}.$$

Electromagnetic cloaking

Maxwell's equations

Maxwell equations

$$\operatorname{curl} H := \nabla \times H = (\sigma - \mathrm{i}\omega\epsilon)E, \qquad \operatorname{curl} E := \nabla \times E = \mathrm{i}\omega\mu H,$$

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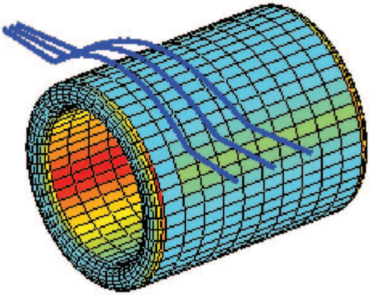
where

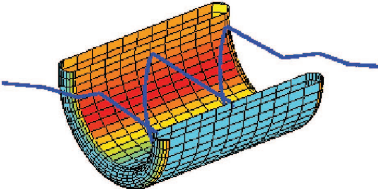
- E and H are the electric and magnetic complex vector fields;
- σ, ϵ and μ are real-valued, the electrical electrical conductivity tensor;

$$(F_*\gamma)^{jk}(y) = \frac{1}{\det[\frac{\partial F^j}{\partial x^k}(x)]} \sum_{p,q=1}^{n} \frac{\partial F^j}{\partial x^p}(x) \frac{\partial F^k}{\partial x^q}(x) \gamma^{pq}(x)\Big|_{x=F^{-1}(y)}$$

Metamaterial



Rays travelling outside of a wormhole.



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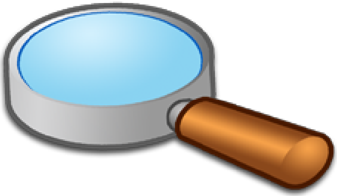

Dirichlet-to-Neumann semigroup

Dirichlet-to-Neumann semigroup acts as a magnifying glass

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$$\begin{cases} \nabla \cdot (\gamma \nabla u(t,\cdot)) = 0, & \text{for every } t \in \mathbb{R}^+, \text{ in } \Omega, \\ \partial_t u + \nu \cdot \gamma \nabla u = 0, & \text{for every } t \in \mathbb{R}^+, \text{ on } \partial\Omega, \\ u(0,\cdot) = f, & \text{on } \partial\Omega. \end{cases}$$

$$\mathrm{e}^{-t\Lambda_\gamma} f := u(t,x)\big|_{\partial\Omega}, \quad \text{for every } f \in \partial X$$

Lax representation

P. D. Lax, *Functional Analysis* Wiley Inter-science, New-York, 2002 (Chapter 36).

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Let u be the harmonic lifting of f in the n-dimensional unit ball B.

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The Lax semigroup is defined by

$$\mathrm{e}^{-t\Lambda_1} f(\omega) = u(e^{-t}\omega) \ \ \textit{for}\ \omega \in S^{n-1}. \tag{2}$$

Approximating family

P. R. Chernoff, Note on product formulas for operator semigroups. *J. Funct. Analysis.* **2** (1968), 238–242.

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Théorème (Chernoff's product formula)

Let X be a Banach space and $\{V(t)\}_{t\geq 0}$ be a family of contractions on X with $V(0) = I$. Suppose that the derivative $V'(0)f$ exists for all f in a set $\mathcal{D}$ and that the closure Λ of $V'(0)\big|_{\mathcal{D}}$ generates a (C_0) semigroup $S(t)$ of contractions. Then, for each $f \in X$,

$$\lim_{n\to\infty} V\left(\frac{t}{n}\right)^n f = S(t)f,$$

uniformly for t in compact subsets of $\mathbb{R}^+$.

Euler Explicit Scheme

H. Emamirad and M. Sharifitabar, On explicit representation and approximations of Dirichlet-to-Neumann semigroup. *Semigroup Forum* **86** (2013), 192–201.

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(EES)
$$\begin{cases} div(\gamma \nabla u^m) = 0 & \text{in } \Omega, \\ \frac{1}{\Delta t}\left(u^{m+1} - u^m\right) + \gamma \frac{\partial u^m}{\partial n} = 0 & \text{on } \partial\Omega, \\ u(x,y,0) = h(x,y) & \text{on } \partial\Omega. \end{cases}$$

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$$V(t)f(x) = \begin{cases} (1-\alpha)u(x) + \alpha u(x - \alpha^{-1}t\gamma(x)\nu(x)), & 0 \le t \le \alpha T, \\ V(\alpha T)f(x), & t > \alpha T, \end{cases}$$

Euler Explicit Scheme

H. Emamirad and M. Sharifitabar, On explicit representation and approximations of Dirichlet-to-Neumann semigroup. *Semigroup Forum* **86** (2013), 192–201.

(EES)
$$\begin{cases} div(\gamma\nabla u^m) = 0 & \text{in } \Omega, \\ \frac{1}{\Delta t}\left(u^{m+1} - u^m\right) + \gamma\frac{\partial u^{\boldsymbol{m}}}{\partial n} = 0 & \text{on } \partial\Omega, \\ u(x,y,0) = h(x,y) & \text{on } \partial\Omega. \end{cases}$$

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$$u^{m+1} = V(\Delta t)u^m.$$

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Since any x with $|x| = 1$ belongs to $\partial\Omega$, we have

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\frac{\partial u^{m+1}}{\partial \nu_\gamma} \approx \frac{u^{m+1}(x) - u^{m+1}(x - \Delta x \gamma(x) x)}{\Delta x}. \qquad (3)
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By replacing (3) in **(EIS)**, we get

$$
\left(1 + \frac{\Delta t}{\Delta x}\right) u^{m+1}(x) - \frac{\Delta t}{\Delta x} u^{m+1}(x - \Delta x \gamma(x) x) = u^m(x). \qquad (4)
$$

Euler Implicit Scheme

$$W(t)f(x) = \begin{cases} (1+\alpha)u(x) - \alpha u(x - \alpha^{-1} t\gamma(x)\nu(x)), & 0 \leq t \leq \alpha T, \\ W(\alpha T)f(x), & t > \alpha T, \end{cases} \tag{5}$$

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$V(t)$ satisfies the assumptions of the Chernoff's theorem.

The variational formulation of this problem can be obtained by multiplying both sides of the dynamic boundary condition by a test function v and by using the divergence theorem, we get

$$\int_{\Omega} \gamma \nabla u^{m+1} \nabla v dx - \int_{\partial\Omega} \gamma \frac{\partial u^{m+1}}{\partial n} v d\sigma = 0,$$

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$$\int_\Omega \Delta t \gamma \nabla u^{m+1} \nabla v dx + \int_{\partial\Omega} u^{m+1} v - \int_{\partial\Omega} u^m v d\sigma = 0, \tag{7}$$

which is of the form

$$a(u^{m+1}, v) = \ell(v),$$

where

$$a(u^{m+1}, v) = \int_\Omega \Delta t \gamma \nabla u^{m+1} \nabla v dx + \int_{\partial\Omega} u^{m+1} v d\sigma$$

is the bilinear form with the unknown of the problem u^{m+1} and

$$\ell(v) = \int_{\partial\Omega} u^m v d\sigma.$$

Numerical illustration.

F. Hecht and O. Pironneau, A finite element software for PDE : FreeFem++, avaible online, http://www.freefem.org/ff++.

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Here we have taken the boundary function

$$f(x, y) = x^4 + y^2 \sin(2\pi y).$$