α^*-Cohomology; from Deformation Quantization Theory to Classification of Translation-Invariant Non-Commutative Quantum Field Theories

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Deformation Quantization, Weil-Wigner Correspondence, Fedosov Approach and Kontsevich Formula

- **Definition 1;** By quantization of $2n$-dimensional phase space $\mathbb{R}^{2n}$, $n \geq 1$, with global coordinate chart $(x^1, \dots, x^n, p^1, \dots, p^n)$, we mean a Hilbert space $\mathcal{H}$, and a linear map $op: C^\infty(\mathbb{R}^{2n}) \to L(\mathcal{H})$, $f \mapsto \hat{f}$, such that;

  a) $\hat{1} = 1$.

  b) $\{\widehat{x^1}, \dots, \widehat{x^n}, \widehat{p^1}, \dots, \widehat{p^n}, 1\}$ generates the Heisenberg Lie algebra $\mathfrak{h}_n$.

  c) $\mathcal{H}$ is minimal, i.e. $\mathfrak{h}_n$ is represented irreducibly over $\mathcal{H}$.

  d) $\widehat{f^*} = \hat{f}^*$.

  e) $\widehat{fg} = (\hat{f}\hat{g} + \hat{g}\hat{f})/2$, for any $f, g \in C^\infty(\mathbb{R}^{2n})$.

  f) $i\hbar\widehat{\{f, g\}} = [\hat{f}, \hat{g}]$, for any $f, g \in C^\infty(\mathbb{R}^{2n})$. (Dirac's analogy)

Deformation Quantization, Weil-Wigner Correspondence, Fedosov Approach and Kontsevich Formula

- **Theorem 1; (Groenewold-van Hove No-Go Theorem)** Quantization is impossible.

- **Definition 2;** Consider a Heisenberg algebra generated by elements $\{\widehat{x^1}, \ldots, \widehat{x^n}, \widehat{p^1}, \ldots, \widehat{p^n}, 1\}$ represented irreducibly over a Hilbert space $\mathcal{H}$. Then the **Weyl quantization** of $2n$-dimensional phase space $\mathbb{R}^{2n}$, $n \geq 1$, with global Darboux's coordinate chart $(\mathbb{R}^{2n}; (x^1, \ldots, x^n, p^1, \ldots, p^n))$ is essentially a linear map $op: C^\infty(\mathbb{R}^{2n}) \to L(\mathcal{H})$, $f \mapsto \hat{f}$, with;

$$\hat{f} = \int \frac{\mathrm{d}^n\xi \mathrm{d}^n\zeta}{(2\pi)^{2n}} \int \mathrm{d}^n x \, \mathrm{d}^n p \, e^{i \sum_{i=1}^{n} \xi_i(\hat{x}_i - x_i) + \sum_{i=1}^{n} \zeta_i(\hat{p}_i - p_i)} f(x,p) \ ,$$

$f \in C^\infty(\mathbb{R}^{2n})$, mostly referred to as the **Weyl map**.

Deformation Quantization, Weil-Wigner Correspondence, Fedosov Approach and Kontsevich Formula

- **Theorem 2; (Egorov's Weak Quantization Theorem)** The Weyl map defines a quantization up to $\mathcal{O}(\hbar^2)$.

- **Theorem 3; (Weyl-Wigner correspondense)** (Wigner, 1932) The Weyl map is invertible.

- **Definition 3;** By definition the **Groenewold-Moyal star product**, mostly shown by $\star_{G-M}$, is defined with;

$$f \star_{G-M} g = op^{-1}(\hat{f}\hat{g}) \,,$$

for $f, g \in C^{\infty}(\mathbb{R}^{2n})$.

- To be well-defined, $\star_{G-M}$ should be considered over $\mathcal{S}(\mathbb{R}^{2n})$.

Deformation Quantization, Weil-Wigner Correspondence, Fedosov Approach and Kontsevich Formula

- For $2n$-dimensional phase space the Groenewold-Moyal product is given by;

$$f \star_{G-M} g(x) = \exp\left(i\theta^{\mu\nu} \frac{\partial}{\partial y^{\mu}}\bigg|_{y=0} \frac{\partial}{\partial z^{\nu}}\bigg|_{z=0} \right) f(x+y) g(x+z)\,,$$

$f, g \in \mathcal{S}(\mathbb{R}^m)$, for anti-symmetric $\theta^{\mu\nu}$ with $\theta^{\mu\,\mu+n} = \hbar/2, 0 \leq \mu \leq n-1$.

- **Definition 4;** Suppose that M is a Poisson manifold with Poisson structure $\Xi \in \Gamma(\Lambda^2 TM)$. A deformation quantization of M is an associative product, say $\star$, over $C^{\infty}(M)[[\hbar]]$, for formal parameter $\hbar$, such that for any $f, g \in C^{\infty}(M)$:

$$\lim_{\hbar \to 0} f \star g = fg \qquad , \qquad \lim_{\hbar \to 0} (f \star g - g \star f)/i\hbar = \Xi(\mathrm{d}f, \mathrm{d}g)\,.$$

If such star product exists over $C^{\infty}(M)[[\hbar]]$, then M is said to be quantizable.

Deformation Quantization, Weil-Wigner Correspondence, Fedosov Approach and Kontsevich Formula

- Groenewold-Moyal star product is used to quantize the symplectic planes (tori) with constant symplectic forms: ${\theta^{-1}}_{\mu\nu}\, \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}$.
- Theorem 4; (Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer, 1977) Any Poisson manifold with torsion free flat Poisson connection is quanizable.
- Theorem 5; (DeWilde, Lecomte, 1983, Kasarev, Maslov, 1984) By patching together the Darboux's charts in an appropriate setting any symplectic manifold is quantizable.
- Theorem 6; (Fedosov, 1985) Any regular Poisson manifold M (a Poisson manifold with constant RankΞ) admits a flat connection over the Weyl algebra bundle, $W(TM)$, due to the Lie algebras of Darboux's charts.

Deformation Quantization, Weil-Wigner Correspondence, Fedosov Approach and Kontsevich Formula

- **Theorem 7; (Fedosov's Theorem)** Using the Weyl structure for $\Gamma(W(TM))$ and $C^{\infty}(M)[[\hbar]]$, any regular Poisson manifold M is quantizable.

- **Theorem 8; (Kontsevich Quantization Theorem)** Any Poisson manifold M is quantizable. In fact there exists a one to one correspondence between the collection of deformation quantization and the set of Poisson structures over a manifold up to isomorphism. The quantization formula is then;

$$f \star g = fg + \sum_{n=1} \left(\frac{i\hbar}{2}\right)^{n} \sum_{\Gamma \in G_n(2)} w_{\Gamma} B_{\Gamma}(f,g) \ ,$$

$f, g \in C^{\infty}(M)$, where $G_n(2)$ is the set of oriented graphs with $2n$ segments, 2 external vertices and with no loop. w_{Γ} is the number of equivalent graphs and B_{Γ} is a partial differential operator due to graph Γ.

Deformation Quantization, Weil-Wigner Correspondence, Fedosov Approach and Kontsevich Formula

- Examples in Quantum Physics:

1. The phase space of quantum mechanics has a natural symplectic form. [Weyl-Wigner-Moyal formalism, 1949]
2. In string theory with background $B(= B_{\mu\nu} dx^{\mu} \wedge dx^{\nu})$ field the Groenewold-Moyal star product appears for B as a regular Poisson structure. [Witten, 1986]
3. The true phase space of a quantum (semi-classical) theory with an intrinsic symmetry is a Poisson manifold which can be non-regular. [Marsden-Weinstein, 1974]

Non-Commutative Field Theories, the Structure of Space-Time and Translation-Invariant $\star$ Products

- **Non-commutative** structure of space-time manifold, M, is given by:

$$\star : C_c^{\infty}(M) \otimes C_c^{\infty}(M) \to C_c^{\infty}(M)\,, f \otimes g \mapsto f \star g\,,$$

for $f, g, h \in C_c^{\infty}(M)$, given by a deformation quantization.

- For all $0 \leq \mu, \nu \leq m-1$, and for any $f, g \in C_c^{\infty}(\mathbb{R}^m)$:

$$\frac{\partial}{\partial x^{\mu}}(f \star_{G-M} g) = \left(\frac{\partial}{\partial x^{\mu}} f\right) \star_{G-M} g + f \star_{G-M} \left(\frac{\partial}{\partial x^{\mu}} g\right).$$

- **Definition 1;** The star product $\star$ on $C_c^{\infty}(\mathbb{R}^m)$ is **translation-invariant** if and only if there exists a global coordinate chart, say $(\mathbb{R}^m, \{x^{\mu}\}_{\mu=0}^{m-1})$, such that for any $0 \leq \mu, \nu \leq m-1$, and any $f, g \in C_c^{\infty}(\mathbb{R}^m)$:

$$\frac{\partial}{\partial x^{\mu}}(f \star g) = \left(\frac{\partial}{\partial x^{\mu}} f\right) \star g + f \star \left(\frac{\partial}{\partial x^{\mu}} g\right).$$

Non-Commutative Field Theories, the Structure of Space-Time and Translation-Invariant $\star$ Products

- **Lemma 1;** If $\star$ is a translation-invariant product on $C_c^\infty(\mathbb{R}^m)$ then for any $f, g \in C_c^\infty(\mathbb{R}^m)$; $\mathcal{T}_a(f \star g) = \mathcal{T}_a(f) \star \mathcal{T}_a(g)$, for $\mathcal{T}_a$ any translation operator along $\left(\mathbb{R}^m, \{x^\mu\}_{\mu=0}^{m-1}\right)$, i.e.; $\mathcal{T}_a(f)(x) = f(x+a)$.

- **Theorem 1;** (Galluccio-Lizzi-Vitale) Put a global chart on $\mathbb{R}^m$, say $(\mathbb{R}^m, \{x^\mu\}_{\mu=0}^{m-1})$, then any translation-invariant star product over $C_c^\infty(\mathbb{R}^m)$ is given by;

$$(f \star g)(x) := \int \frac{\mathrm{d}^m p}{(2\pi)^m} \frac{\mathrm{d}^m q}{(2\pi)^m} \tilde{f}(q) \tilde{g}(p) e^{\alpha(p+q,q)} e^{i(p+q).x},$$

for $\alpha \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m)$ the generator of $\star$, which obeys the cyclic property:

$$\alpha(p, r+s) + \alpha(r+s, r) = \alpha(p, r) + \alpha(p-r, s),$$

for any $p, r, s \in \mathbb{R}^m$. (arXiv:0907.3640 [hep-th])

Non-Commutative Field Theories, the Structure of Space-Time and Translation-Invariant $\star$ Products

- In fact the cyclic property is equivalent to associativity of star product $\star$, i.e.; $(f \star g) \star h = f \star (g \star h), \forall f, g, h \in C_c^\infty(\mathbb{R}^m)$.

- From now on we restrict ourselves to $(\mathbb{R}^m, \{x^\mu\}_{\mu=0}^{m-1})$.

- **Lemma 2;** Suppose $S_c(\mathbb{R}^m)$ is the set of Schwartz class functions with compactly supported Fourier transforms, then $S_c(\mathbb{R}^m)$ is closed under any translation-invariant star product. The algebra then is denoted by $S_c(\mathbb{R}^m)_\star$. Translation-invariant $\star$ products can be extended to $S_{c,1}(\mathbb{R}^m) \coloneqq S_c(\mathbb{R}^m) \oplus \mathbb{C}$ as a unital algebra. The unital algebra then is denoted by $S_{c,1}(\mathbb{R}^m)_\star$.

- **Definition 2;** α is a commutative generator if it generates a commutative star product.

Non-Commutative Field Theories, the Structure of Space-Time and Translation-Invariant ⋆ Products

- **Theorem 2;** $S_{c,1}(\mathbb{R}^m)_{\star_{G-M}}$ is an algebra in the category of $U(\mathcal{P}_m)_{\chi_{G-M}}$-modules, where $\mathcal{P}_m$ is the m-Poincare Lie algebra of $\{P_\mu, M_{\mu\nu}\}_{\mu,\nu=0}^{m-1}$, and $U(\mathcal{P}_m)_{\chi_{G-M}}$ is the Drinfeld twist of its universal enveloping algebra due to the Groenewold-Moyal counital 2-cocycle $\chi_{G-M} = e^{-i\hbar^{-2}\theta^{\mu\nu}P_\mu \otimes P_\nu}$.
- **Definition 3;** A translation-invariant quantum field theory is a quantum field theory together with a translation-invariant star product, i.e. $\phi_\star^4$-theory;

$$\mathcal{L}_{\phi^4\star} = \frac{1}{2}\nabla\phi \star \nabla\phi - \frac{m^2}{2}\phi \star \phi - \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi\,.$$

- **Theorem 3;** Any Groenewold-Moyal translation-invariant quantum field theory admits the twisted Poincare symmetry due to $U(\mathcal{P})_{\chi_{G-M}}$.

α-Cohomology and Classification of Translation-Invariant $\star$ Products

- Let $C^n(\mathbb{R}^m) \subseteq C^\infty(\mathbb{R}^m \times \cdots \times \mathbb{R}^m)$, $n \geq 0$ copies of $\mathbb{R}^m$, be the complex vector spaces generated by smooth functions f with properties of:

  a) $C^0(\mathbb{R}^m) = \{0\}$.

  b) For $n = 1, f(0) = 0,$

  c) For $n = 2, f(p, 0) = f(p, p) = 0,$

  d) For $n \geq 3, f(p_1, \dots, p_{n-1}, 0) = f(p_1, \dots, p_k, q, q, p_{k+1}, \dots, p_{n-2}) = 0, k \leq n-2,$ for any $q, p, p_1, \dots, p_{n-1} \in \mathbb{R}^m$.

- Consider the linear maps $\partial_n : C^n(\mathbb{R}^m) \to C^{n+1}(\mathbb{R}^m)$, commonly denoted by ∂, defined by;

$$\partial_n f(p_0, \dots, p_n) \coloneqq \varepsilon_n \sum_{i=0}^{n} (-)^i f(p_0, \dots, p_{i-1}, \hat{p}_i, p_{i+1}, \dots, p_n) + \varepsilon_n (-)^{n+1} f(p_0 - p_n, \dots, p_{n-1} - p_n),$$

  $f \in C^n(\mathbb{R}^m)$, with $\varepsilon_n = 1$ for odd n and $\varepsilon_n = i$ for n even.

α-Cohomology and Classification of Translation-Invariant $\star$ Products

- It is easily seen that; $\partial^2 = \partial_n \circ \partial_{n-1} = 0$ for any $n \in \mathbb{N}$.

- **Definition 1;** (Galluccio-Lizzi-Vitale) By definition **α-cohomology** is the cohomology theory of complex;

$$0 = C^0(\mathbb{R}^m) \xrightarrow{\partial_0} C^1(\mathbb{R}^m) \xrightarrow{\partial_1} \dots \xrightarrow{\partial_{n-1}} C^n(\mathbb{R}^m) \xrightarrow{\partial_n} \dots \quad .$$

Then the nth α-cohomology group is defined by; $H_\alpha^n(\mathbb{R}^m) := Ker\partial_n / Im\partial_{n-1}$. As a generic convention the notation of $\alpha_1 \sim \alpha_2$ is used for two α-cohomologous n-cocycles α_1 and α_2. The cohomology class of $\alpha \in Ker\partial_n$ is shown by $[\alpha]$.

- **Lemma 1;** $\alpha \in C^2(\mathbb{R}^m)$ generates a translation-invariant star product if and only if α is a 2-cocycle, i.e.; $\partial\alpha = 0$.

α-Cohomology and Classification of Translation-Invariant $\star$ Products

- **Corollary 1;** $H_\alpha^2(\mathbb{R}^m)$ classifies translation-invariant star products over $S_{c,1}(\mathbb{R}^m)$ modulo the coboundary terms.
- It can be easily seen that if $[\alpha] = 0$ (i.e. $\alpha = \partial\beta$ for some $\beta \in C^1(\mathbb{R}^m)$) then α is a commutative generator (i.e. α generates a commutative star product).
- **Theorem 1;** α is a commutative generator if and only if $[\alpha] = 0$. (arXiv:1210.0695 [math-ph])
- **Corollary 2;** $H_\alpha^2(\mathbb{R}^m)$ classifies translation-invariant star products over $S_{c,1}(\mathbb{R}^m)$ up to commutativity.

α-Cohomology and Classification of Translation-Invariant $\star$ Products

- **Definition 2;** An algebraic homomorphism from $S_{c,1}(\mathbb{R}^m)_{\star_1}$ to $S_{c,1}(\mathbb{R}^m)_{\star_2}$, say T, is translation-invariant if for any $f \in S_{c,1}(\mathbb{R}^m)$, $T(\partial_\mu f) = \partial_\mu T(f)$. If T is invertible we may write $\star_1 \sim \star_2$.

- **Theorem 2;** $\star_1 \sim \star_2$ if and only if the generators of $\star_1$ and $\star_2$ are α-cohomologous (i.e. $\alpha_1 \sim \alpha_2$).

- **Corollary 3;** $H^2_\alpha(\mathbb{R}^m)$ classifies all translation-invariant algebraic structures over $S_{c,1}(\mathbb{R}^m)$ up to isomorphism.

α-Cohomology and Classification of Translation-Invariant $\star$ Products

- Example in Quantum Physics:

  The Groenewold-Moyal product, $\star_{G-M}$, and the Wick-Voros product, $\star_{W-V}$, are respectively generated by 2-cocycles $\boldsymbol{\alpha_{G-M}(p,q) = iq^{\mu}\theta_{A\,\mu\nu}p^{\nu}}$ and $\boldsymbol{\alpha_{W-V}(p,q) = \alpha_{G-M}(p,q) + q^{\mu}\theta_{S\,\mu\nu}(p-q)^{\nu}}$, $p, q \in \mathbb{R}^m$, for θ_A an anti-symmetric and θ_S a symmetric real fixed matrix.

  It is clear that α_{G-M} and α_{W-V} differ in a commutative 2-cocycle. Thus, α_{G-M} and α_{W-V} are α-cohomologous and belong to the same class of $H^2_\alpha(\mathbb{R}^m)$ commonly shown by $[\boldsymbol{\alpha_{G-M}}]$. Moreover $\star_{G-M} \sim \star_{W-V}$.

The Hodge Theorem in α-Cohomology, the Harmonic Forms and the Harmonic $\star$ Products

- **Lemma 1;** To any generator α one can correspond the other generator with;

$$\alpha'(p,q) := \frac{1}{2}\big(\,\alpha(p,q) + \alpha(-p,-q)\,\big)\,,$$

for $p, q \in \mathbb{R}^m$ (i.e. $\partial\alpha' = 0$).

- **Lemma 2;** For any 2-cocycle α, $\alpha \sim \alpha'$.

- **Lemma 3;** Any generator α can be written as a sum of two following generators;

$$\alpha_-(p,q) := \frac{1}{2}\big(\,\alpha(p,q) - \alpha(-p,q-p)\,\big)\,,$$
$$\alpha_+(p,q) := \frac{1}{2}\big(\,\alpha(p,q) + \alpha(-p,q-p)\,\big)\,,$$

for $p, q \in \mathbb{R}^m$ (i.e. $\partial\alpha_- = \partial\alpha_+ = 0$).

- **Lemma 4;** For any 2-cocycle α, $\alpha \sim \alpha_-$.

The Hodge Theorem in α-Cohomology, the Harmonic Forms and the Harmonic $\star$ Products

- **Corollary 1;** For any 2-cocycle α, $\alpha \sim (\alpha')_- = (\alpha_-)' := \alpha'_-$.

- It is easy to see that for any generator α;

$$\begin{cases} \alpha'_-(p,q) = -\alpha'_-(p, p-q) \\ \alpha'_-(p,q) = \alpha'_-(-p,-q) \\ \alpha'_-(p,q) = -\alpha'_-(q,p) \end{cases} , \qquad (*)$$

for any $p, q \in \mathbb{R}^m$.

- **Theorem 1; (The Hodge Theorem in α-Cohomology)** For any cohomology class of $H^2_\alpha(\mathbb{R}^m)$ there exists a unique element which satisfies the conditions $(*)$. (arXiv:1210.0695 [math-ph])

The Hodge Theorem in α-Cohomology, the Harmonic Forms and the Harmonic $\star$ Products

- **Definition 1;** According to the Hodge theorem, any generator satisfying conditions $(*)$ is called a **harmonic form**. Also a translation-invariant star product generated by a harmonic form is called a **harmonic star product**.

- For any $[\alpha] \in H^2_\alpha(\mathbb{R}^m)$ the harmonic form is shown by α_H. Also $\star_H$ is used for the relevant harmonic star product.

- **Lemma 5;** Harmonic star products disappears under integration, i.e. for any $f, g \in S_c(\mathbb{R}^m)$;

$$\int_{\mathbb{R}^m} f \star_H g = \int_{\mathbb{R}^m} fg \ .$$

The Hodge Theorem in α-Cohomology, the Harmonic Forms and the Harmonic $\star$ Products

- Example in Quantum Physics:

  It easy to see that α_{G-M} obeys the properties of harmonic forms. Thus it is the harmonic form of α-cohomology class $[\alpha_{G-M}]$. Moreover $\star_{G-M}$ disappears under integration.

- **Lemma 6;** For any arbitrary element of $[\alpha] \in H^2_\alpha(\mathbb{R}^m)$, say α;

$$\alpha_H(p,q) = \frac{1}{2}(\,\alpha(p+q,q) - \alpha(p+q,p)\,)\,,$$

  $p,q \in \mathbb{R}^m$.

- **Lemma 7;** For any $r \in \mathbb{R}$ and any $p \in \mathbb{R}^m$; $\alpha_H(rp,p) = 0$.

- **Corollary 2;** The real line never admits non-commutative translation-invariant star product.

α^*-Cohomology and Classification of Translation-Invariant Non-Commutative Quantum Field Theories

- **Definition 1;** Star product $\star$ is complex if and only if for any $f, g \in S_{c,1}(\mathbb{R}^m)$;

$$(f \star g)^* = g^* \star f^* \, .$$

- **Lemma 1;** Generator α generates a complex translation-invariant star product if and only if for any $p, q \in \mathbb{R}^m$;

$$\alpha^*(p,q) = \alpha(-p, q-p) \, .$$

- Examples in Quantum Physics:

1. To have a real valued Lagrangian density (action) or more precisely to have a Hermitian Hamiltonian only the complex star products are allowed in quantum physics.
2. Groenewold-Moyal and Wick-Voros star products both are complex.

α^*-Cohomology and Classification of Translation-Invariant Non-Commutative Quantum Field Theories

- **Definition 2;** By definition **α^*-cohomology** is the cohomology of complex

$$0 = C_*^0(\mathbb{R}^m) \xrightarrow{\partial_0} C_*^1(\mathbb{R}^m) \xrightarrow{\partial_1} \dots \xrightarrow{\partial_{n-1}} C_*^n(\mathbb{R}^m) \xrightarrow{\partial_n} \dots \quad ,$$

where $C_*^n(\mathbb{R}^m)$ is the cochain of elements $f \in C^n(\mathbb{R}^m)$, with property

$$f^*(p_1, \dots, p_n) = f(-p_1, p_n - p_1, p_{n-1} - p_1, \dots, p_2 - p_1), \qquad p_1, \dots, p_n \in \mathbb{R}^m.$$

In fact the nth α^*–cohomology group is; $H_{\alpha^*}^n(\mathbb{R}^m) := Ker\partial_n / Im\partial_{n-1}$ (i.e. complex n-cocycles modulo complex n-coboundaries).

- **Lemma 2;** Inclusions $i_n: C_*^n(\mathbb{R}^m) \hookrightarrow C^n(\mathbb{R}^m)$, lead to a family of injections;

$$i_{n_*}: H_{\alpha^*}^n(\mathbb{R}^m) \hookrightarrow H_{\alpha}^n(\mathbb{R}^m) .$$

- **Corollary 1;** $H_{\alpha^*}^2(\mathbb{R}^m)$ classifies the complex translation-invariant star products up to commutativity.

α^*-Cohomology and Classification of Translation-Invariant Non-Commutative Quantum Field Theories

- Lemma 3; The Hodge theorem of α-cohomology can be refine to $H^2_{\alpha^*}(\mathbb{R}^m)$.
- Corollary 2; If $[\alpha]$ belongs to $H^2_{\alpha^*}(\mathbb{R}^m) \subseteq H^2_{\alpha}(\mathbb{R}^m)$, then $[\alpha^*]$ is the dual of $[\alpha]$ in the sense of;

$$[\alpha] + [\alpha^*] = 0 .$$

This is called the pure imaginary condition for α-cohomology classes.

- Theorem 1; $H^2_{\alpha^*}(\mathbb{R}^m)$ is the collection of all pure imaginary classes of $H^2_{\alpha}(\mathbb{R}^m)$.
- Corollary 3; $\dim H^2_{\alpha}(\mathbb{R}^m) = 2 \dim H^2_{\alpha^*}(\mathbb{R}^m)$.
- Corollary 4; $H^2_{\alpha}(\mathbb{R}^m) = H^2_{\alpha^*}(\mathbb{R}^m) \otimes_{\mathbb{R}} \mathbb{C}$.
- Example in Quantum Physics:

α_{G-M} is a pure imaginary complex 2-cocycle thus it must be harmonic.

α^*-Cohomology and Classification of Translation-Invariant Non-Commutative Quantum Field Theories

- **Theorem 2; (The Quantum Equivalence Theorem)** Consider two complex translation-invariant star products $\star_1$ and $\star_2$. Then $\star_1 \sim \star_2$ if and only if there exists a fixed $\beta \in C^\infty(\mathbb{R}^m)$, with $\beta(0) = 0$, such that for any $n \geq 1$, the equality

$$\tilde{G}_{\star_1\,\text{conn.}}(p_1, \dots, p_n) = e^{\sum_{i=1}^{n} \beta(p_i)} \tilde{G}_{\star_2\,\text{conn.}}(p_1, \dots, p_n)$$

holds for any given renormalizable quantum field theory, where $G_{\text{conn.}}$ is any connected n-point function, $G_{\star\,\text{conn.}}$ is its non-commutative version for the star product $\star$ and $\tilde{G}_{\star\,\text{conn.}}(p_1, \dots, p_n)$ is its Fourier transform for the modes $\{p_i\}_{i=1}^{n}$. Therefore α^*-cohomology produces the most general classification of translation-invariant quantum field theories via the view points of quantum physics. (arXiv:1210.0695 [math-ph])

α^*-Cohomology and Classification of Translation-Invariant Non-Commutative Quantum Field Theories

- **Corollary 5;** All the quantum behaviors of Wick-Voros non-commutative quantum field theories coincide thoroughly with those of Groenewold-Moyal ones.
- **Corollary 6;** The Grosse-Wulkenhaar approach and the method of $1/p^2$ for renormalizing the Groenewold-Moyal non-commutative ϕ^4-theory also work well for Wick-Voros non-commutative ϕ^4-theory.
- **Theorem 3; (The 2nd Version of Quantum Equivalence Theorem)** Consider two complex translation-invariant star products $\star_1$ and $\star_2$. Then $\star_1 \sim \star_2$ if and only if for any renormalizable quantum field theory $\mathcal{L}$, its two translation-invariant (non-commutative) version $\mathcal{L}_{\star_1}$ and $\mathcal{L}_{\star_2}$ have the same scattering matrix.

α^*-Cohomology and Classification of Translation-Invariant Non-Commutative Quantum Field Theories

- **Theorem 4;** Suppose that $\star$ is generated by complex 2-cocycle α. Then the structures of all quantum behaviors of a translation-invariant quantum field theory with star product $\star$, such as (non-) renormalizability, unitarity, causality, locality, the structure of UV/IR mixing, the forms of singularities of n-point functions, ..., are thoroughly explained by the harmonic form of $[\alpha] \in H^2_{\alpha^*}(\mathbb{R}^m)$.

- **Corollary 7;** $H^2_{\alpha^*}(\mathbb{R}^m)$ classifies the structures of all quantum behaviors of translation-invariant quantum field theories.

- **Corollary 8;** Any commutative translation-invariant ϕ^4-theory (gauge theory) is local, causal, unitary and renormalizable.

Harmonic Forms, Groenewold-Moyal ⋆ Products and Classification of Quantum Behaviors

- Recall that any Groenewold-Moyal 2-cocycle is a complex harmonic form.

- **Theorem 1; (The Theorem of Harmonic Forms)** Any complex harmonic form is a Groenewold-Moyal 2-cocycle. (arXiv:1210.1004 [math-ph])

- **Corollary 1;** $\dim H^2_{\alpha^*}(\mathbb{R}^m) = m(m-1)/2$. More precisely;

$$H^2_{\alpha^*}(\mathbb{R}^m) = \{\theta \in \mathbb{M}_{m\times m}(\mathbb{R}) | \theta \text{ is anti} - \text{symmetric}\}.$$

- **Corollary 2;** $\dim H^2_{\alpha}(\mathbb{R}^m) = m(m-1)$. More precisely;

$$H^2_{\alpha}(\mathbb{R}^m) = \{\theta \in \mathbb{M}_{m\times m}(\mathbb{C}) | \theta \text{ is anti} - \text{symmetric}\}.$$

In fact any harmonic form α_H, is given by an anti-symmetric matrix θ_A, i.e. $\alpha_H(p,q) = p^{\mu}\theta_{A\,\mu\nu}q^{\nu}, p, q \in \mathbb{R}^m$.

Harmonic Forms, Groenewold-Moyal ⋆ Products and Classification of Quantum Behaviors

- **Corollary 3;** Any complex generator α can be uniquely written as;

$$\alpha = \alpha_{G-M} + \partial\beta \ ,$$

for $\beta \in C^1_*(\mathbb{R}^m)$ and for $\alpha \sim \alpha_{G-M}$.

- **Corollary 4;** According to quantum equivalence theorem for any general translation-invariant non-commutative quantum field theory there is a particular Gronwold-Moyal non-commutative quantum field theory with exactly the same quantum effects and physical out-comings such as n-point functions and the scattering matrix.

- **Corollary 5;** Studying the Groenewold-Moyal non-commutative quantum field theories covers the whole domain of translation-invariant non-commutative quantum field theories.

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- The domain of translation-invariant star products can consistently be extended to the space of polynomials over coordinate functions.

- **Lemma 1;** Suppose that $\star$ is a translation-invariant star product generated by 2-cocycle α. Then the non-commutative structure of space-time, i.e. $[x^\mu, x^\nu]_\star := x^\mu \star x^\nu - x^\nu \star x^\mu$, $\mu, \nu = 0, \dots, m-1$, is thoroughly determined by the α-cohomology class of generator α, $[\alpha] \in H^2_\alpha(\mathbb{R}^m)$. Moreover the matrix of commutators is the anti-symmetric matrix of α_H.

- **Corollary 6;** Consider two translation-invariant star products $\star_1$ and $\star_2$. Then $\star_1 \sim \star_2$ if and only if $\star_1$ and $\star_2$ lead to the same non-commutative structure of space-time.

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- **Corollary 7; (Kontsevich's Theorem for Translation-Invariant Star Products)** The non-commutative structure of space-time is sufficient for studding the structure of quantum behavior for any translation-invariant quantum field theory.

  In fact from quantum physics point of view the only fundamental data is the non-commutative structure of space-time, but not the star product.

- **Corollary 8;** There is no translation-invariant non-commutative star product on $S_{c,1}(\mathbb{R}^m)$ which is commutative at the level of coordinate functions. Thus, due to path integral formalism where the integration is taken over $S_{c,1}(\mathbb{R}^m)$, commutative space-time never admits non-commutative translation-invariant quantum field theories.

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- **Corollary 9;** Due to Grosse-Wulkenhaar approach and the method of $1/p^2$ any translation-invariant non-commutative version of ϕ^4 -theory is renormalizable.
- **Corollary 10;** Any proposal for renormalizing the Groenewold-Moyal (non-commutative) gauge theories extends thoroughly to the collection of all translation-invariant non-commutative gauge theories.
- **Corollary 11;** Any translation-invariant star product is reflected by the Weyl-Wigner correspondence via a modified version of Weyl map, i.e.;

$$\hat{f} = \int \frac{\mathrm{d}^m p}{(2\pi)^m} \int \mathrm{d}^m x \, e^{i \sum_{i=1}^{m} p_i (\hat{x}_i - x_i)} f(x) \, e^{\beta(p)} \,,$$

$f \in C^\infty(\mathbb{R}^m)$, for 1-cochain β and $[\hat{x}_\mu, \hat{x}_\nu] = \theta_{A\,\mu\nu}$, $0 \le \mu, \nu \le m-1$, leads to generator; $\alpha(p,q) = p^\mu \theta_{A\,\mu\nu} q^\nu + \partial\beta(p,q)$.

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- **Corollary 12;** For any translation-invariant star product $\star$, $S_{c,1}(\mathbb{R}^m)_\star$ is an algebra in the category of $U(\mathcal{P}_m)_\chi$-modules, where $\mathcal{P}_m$ is the m-Poincare Lie algebra of $\{P_\mu, M_{\mu\nu}\}_{\mu,\nu=0}^{m-1}$, and $U(\mathcal{P}_m)_\chi$ is the Drinfeld twist of its universal enveloping algebra due to counital 2-cocycle;

$$\chi = \chi_{G-M}(1\otimes e^{\beta(\vec{P})})(e^{\beta(\vec{P})}\otimes 1)(e^{-\beta(\vec{P}\otimes 1+1\otimes\vec{P})}).$$

- **Theorem 2;** Any translation-invariant quantum field theory admits the twisted Poincare symmetry due to $U(\mathcal{P})_\chi$.
- **Lemma 2;** For any translation-invariant star product $\star$, χ and χ_{G-M} are Hopf algebra cohomologous in cohomology space $H^2(U(\mathcal{P}_m))$ $(\chi = \chi_{G-M}\partial e^{\beta(\vec{P})})$. Moreover, $H^2\big(U(\mathcal{T}_m)\big) = H^2_\alpha(\mathbb{R}^m)$ for $\mathcal{T}_m$ the translation algebra of $\{P_\mu\}_{\mu=0}^{m-1}$.

The End