Metric Dimension of Graphs

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Outline

1. Metric dimension

2. Applications

3. Some known results
   - Complexity
   - Specific graphs
   - Characterization
   - Graph operators
   - Bounds

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Metric representation

Definition (Slater 1975, Harary and Melter 1976)

For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices in a connected graph G and a vertex v of G, the metric representation of v with respect to W is the k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

where $d(x, y)$ represents the distance between the vertices x and y.

Metric dimension

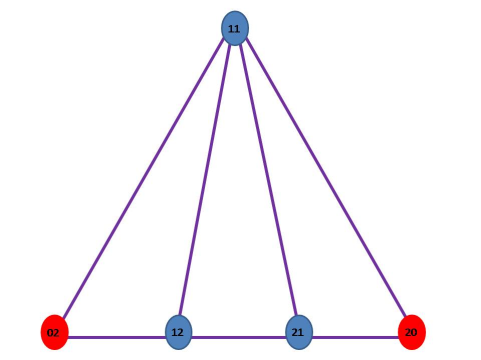
Definition (Slater 1975, Harary and Melter 1976)

A set W is called a resolving set for G if the vertices of G have distinct representations with respect to W. The members of a resolving set are called landmarks.

A resolving set containing a minimum number of vertices is called a basis for G. The number of vertices in a basis for G is its metric dimension and denoted by $\dim(G)$. If $\dim(G) = k$, then G is called a k-dimensional graph.

Warm up!

Example



A graph with metric dimension 2. Metric representations are shown on the vertices. The red vertices are landmarks.

Robot navigation (Khuller and Raghavachari 1996)

A moving point in a graph may be located by finding the distance from the point to a collection of sonar stations which have been properly positioned in the graph.

- **Problem**
  Finding a minimal sufficiently large set of labelled vertices such that robot can find its position.
- **Technique**
  Sufficiently large set of labelled vertices is a resolving set for the graph space.

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Coin Weighing (Sebo and Tannier 2004)

Given n coins, each with one of two distinct weights, determine the weight of each coin with the minimum number of weighings. Weighings a set S of coins determine how many light coins are in S and no further information.

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- **Static variant**
  The choice of sets of coins to be weighed is determined in advance.
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  In static variant the minimum number of weighings differs from $\dim(Q_n)$ by at most 1.

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Mastermind

Mastermind is a game for two player, one code setter and the code breaker. The code setter chooses a secret vector $s = [s_1, s_2, \dots, s_n] \in \{1, 2, \dots, k\}^n$. The task of code breaker is to infer the secret code by a series of questions, each a vector $t = [t_1, t_2, \dots, t_n] \in \{1, 2, \dots, k\}^n$. The code setter answer the number of positions in which the secret vector and the question agree, denoted by $a(s, t) = |\{i : \; s_i = t_i, 1 \leq i \leq n\}|$.

Mastermind (Caceres et al. 2007)

- **Problem**
  Finding the minimum number of questions required to determine the secret code, where the secret code and questions are in $\{1, 2, \ldots, k\}^n$.
- **Static variant**
  The questions are determined in advance.
- **Technique**
  In the static variant the minimum number of questions is $\dim(H_{n,k})$, where $H_{n,k} = \underbrace{K_k \Box K_k \Box \ldots \Box K_k}_{n}$ is the cartesian product of n copy of complete graph K_k.

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Scheme of proof (Caceres et al. 2007)

- Let S be the secret code, T be a question and $a(S,T)$ be the answer of T.
- S and T are vertices of $H_{n,k}$.
- $d(S,T) = n - a(S,T)$.
- The vector $(a(S,T_1), a(S,T_2), \ldots, a(S,T_m))$ uniquely determines S if and only if $(d(S,T_1), d(S,T_2), \ldots, d(S,T_m))$ uniquely determines it.
- Questions $T_1, T_2, \ldots, T_m$ are sufficient for determining each secret code if and only if $\{T_1, T_2, \ldots, T_m\}$ is a resolving set for $H_{n,k}$.

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Network Discovery (Beerliova et al. 2001)

A real world problem is the study of networks whose structure has not been imposed by a central authority but arisen from local and distributed processes. It is very difficult and costly to obtain a map of all nodes and the links between them. A commonly used technique is to obtain local view of the network from various locations and combine them to obtain a good approximation for the real network.

- **Problem**
  Determining edges and none-edges of a network.
- **Technique**
  Combining local maps of the network from landmarks.
- **Local map at a vertex v**
  The induced subgraph on the set of all edges on shortest paths between v and any other vertex.

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Theorem (Khuller and Raghavachari 1996)

The problem of finding the metric dimension of a graph is *NP*-complete. But there is a polynomial time algorithm for finding the metric dimension of a tree. Also, there is a $2\log n$-approximation algorithm for the metric dimension of each graph.

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Metric dimension of specific graphs

- $\dim(P_n) = 1$, each end vertices of P_n resolves it.
- $\dim(K_n) = n - 1$, because $\dim(K_n) + 1^{\dim(K_n)} \geq n$.
- $\dim(C_n) = 2$, each pair of adjacent vertex is a resolving set for it, and each vertex can not resolve its neighbours.
- The metric dimension of Petersen graph, P, is 3. Because $2 + 2^2 < 10 = n(P)$.

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Metric dimension of Johnson and Kneser graphs

The Kneser graph $K(n,k), n \geq 2k$, has the collection of all k-subsets of the set $[n] = \{1,\ldots,n\}$ as vertices and edges connecting disjoint subsets. The vertices of Johnson graph $J(n,k)$ is the same as Kneser graph, but two k-subsets are adjacent when their intersection has size $k-1$.

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Metric dimension of Johnson and Kneser graphs

Theorem (Valencia et al. 2005)

For every two vertices U, V in $J(n,k)$,

$$d(U,V) = k - |U \cap V|$$

For every two vertices U, V in $K(2k+b, k)$, where $|U \cap V| = s$,

$$d(U,V) = \min\left\{2\left\lceil\frac{k-s}{b}\right\rceil, 2\left\lceil\frac{s}{b}\right\rceil + 1\right\}$$

Corollary

Any resolving set for the Kneser graph $K(n,k)$ is a resolving set for $J(n,k)$. Thus, $\dim(J(n,k)) \leq \dim(K(n,k))$.

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An algebraic approach

Let $\mathcal{S} = \{S_1, \ldots, S_t\}$, where each S_i is a k-subset of $[n]$. Then the incidence matrix of $\mathcal{S}$ is the $t \times n$ matrix whose rows are the incidence vectors of $S_1, \ldots, S_t$.

Theorem (Bailey et al. 2013)

If $\mathcal{S}$ is a family of k-subsets of $[n]$ whose incidence matrix has rank n, then $\mathcal{S}$ is a resolving set for $J(n, k)$.

Corollary

For every integer n, k, the metric dimension of the Johnson graph $J(n, k)$ is at most n.

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A t-design with parameters (n, k, λ) is a pair $(X, \mathcal{B})$, where X is a set of n points, and $\mathcal{B}$ is a family of k-subsets of X, called blocks, such that any t elements of distinct points are contained in exactly λ blocks.

A symmetric design is a 2-design on n points which the number of blocks is n.

Corollary

The blocks of a symmetric design $\mathcal{D}$ with parameters (n, k, λ) form a resolving set for $J(n, k)$.

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Theorem (Bailey et al. 2013)

Suppose there exists a Steiner System $(k-1)-(n,k,1)$, where $n \geq 4k-2$. Then its block form a resolving set for $K(n,k)$. Thus,

$$\dim(K(n,k)) \leq \frac{1}{k}\binom{n}{k-1}.$$

A combinatorial approach

A partial geometry with parameter (s, t, α), $pg(s, t, \alpha)$, is a pair $(\mathcal{P}, \mathcal{L})$, consisting of a set of points $\mathcal{P}$ and a set of lines $\mathcal{L}$ satisfying the following conditions:

- any line is incident with $s + 1$ points, and the intersection of any two lines is at most a single point;
- any point is incident with $t + 1$ lines, and any two points are in at most one line;
- if a point p and a line L are not incident, then exactly α points of L are collinear with p and exactly α lines incident with p are concurrent with L.

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Theorem (Bailey et al. 2013)

Let Γ be a partial geometry $pg(s,t,\alpha)$ with point set $\mathcal{P}$ and line set $\mathcal{L}$ and $t > s$. Then $\mathcal{L}$ is a resolving set for the Kneser graph $K(v, s+1)$.

For $s = q-1$, $t = q$ and $\alpha = q$ we have $v = q^2$.

Corollary

If $q \geq 3$ is a prime power, then $\dim(K(q^2, q)) \leq q^2 + q$.

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The metric dimension of ...	using ...	is bounded by ...
$J(n,k)$	partitioning $[n]$	$\lfloor k(n+1)/(k+1)\rfloor$
$K(2k+1,k)=O_{k+1}$		$2k$
$K(n,k)$		$\lceil \frac{n}{2k-1}\rceil(\binom{2k-1}{k}-1)$
$K(n,k)$, diameter 3		$2\binom{n-k}{k}$
$J(n,k)$	k-set system whose incidence matrix has rank n	n
	(n,k,λ) symmetric design	
$J(q^2+q+1,q+1)$	projective plane of order q	q^2+q+1
$J(4m-1,2m-1)$, $K(4m-1,2m-1)=O_{2m}$	Hadamard design	$4m-1$
$K(n,3)$	Steiner triple STS(n)	$n(n-1)/6$
$K(n,k)$	Steiner triple STS$(k-1,k,n)$	$\binom{n}{k-1}/k$
$K(v,s+1)$, $v=(s+1)(st+\alpha)/\alpha$	partial geometry pg(s,t,α)	$(t+1)(st+\alpha)/\alpha$
$K(q^2,q)$	affine plane of order q	$q(q+1)$
$K(q^3,q)$	generalized quadrangle	$q^2(q+2)$
$K((q+1)(q^3+1),q+1)$		$(q^2+1)(q^3+1)$
$K((q^2+1)(q^5+1),q^2+1)$		$(q^3+1)(q^5+1)$
$K(n,4)$	toroidal grid $C_b\Box C_b$	$2ab=2n$
$K(n,5)$		
$K(n,6)$		

Characterization

Theorem (Chartrand et al. 2000)

let G be a connected graph of order $n \geq 1$. Then

- $\dim(G) = 1$ if and only if $G = P_n$. Let $\{w\}$ be a basis, for each v, $r(v|W) = d(v, w)$, thus there exists u with $d(u, w) = n - 1$, so $G = P_n$.
- $\dim(G) = n - 1$ if and only if $G = K_n$,otherwise there are $u, v, w \in V(G)$ such that $u \sim v$ and $u \nsim w$ that is $V(G) \setminus \{v, w\}$ is a resolving set.
- for $n \geq 4$, $\dim(G) = n - 2$ if and only if $G = K_{r,s}$, $r, s \geq 1$, $G = K_r \vee \overline{K_s}$, $r \geq 1, s \geq 2$, or $G = K_r \vee (K_1 \cup K_s)$, $r, s \geq 1$.

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Characterization

- Graphs with metric dimension two are characterized. (Sudhakara and Kumar 2009)
- Graphs with metric dimension $n(G) - diam(G)$ are characterized. (Hernando et al. 2010)
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$$\lim_{n \to \infty} dim(Q_n) . \frac{\log n}{n} = 2.$$

Theorem (Caceres et al. 2007)

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$dim(Q_n)$	2	3	4	4	5	6	6	≤ 7	≤ 10

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Some bounds for metric dimension

Certainly, if G is a nontrivial connected k-dimensional graph of order n then $1 \leq k \leq n-1$.

Theorem (Chartrand et al. 2000)

For positive integers d and n with $d < n$, define $f(n,d)$ as the least positive integer k such that $k + d^k \geq n$. Then for a connected graph G of order $n \geq 2$ and diameter d,

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- Let $u = v_0, v_1, \ldots, v_d = v$ be a path of length d in G.
- For each $i, 1 \leq i \leq d$, $d(u, v_i) = i$.
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- Let B be a basis of G of size k.
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Theorem (Chartrand et al. 2000)

If G is a connected graph, then $\lceil \log_3(\Delta(G)+1) \rceil \leq \dim(G)$.

Definition

A set of vertices S is a determining set of a graph G if every automorphism of G is uniquely determined by its action on S. The determining number of G, $det(G)$, is the minimum cardinality of a determining set of G.

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A set of vertices S is a dominating set of a graph G if every vertex not in S has a neighbour in S. The domination number of G, $\gamma(G)$, is the cardinality of a minimum dominating set of G.

Theorem (Bagheri, Jannesari and O., 2013)

If G is a graph of order n, then $\dim(G) \leq n - \gamma(G)$. Moreover, $\dim(G) = n - \gamma(G)$ if and only if G is a complete graph or a complete bipartite graph $K_{s,t}$, $s, t \geq 2$.

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Two vertices $u, v \in V(G)$ are called false twin vertices if $N(u) = N(v)$.

- In every connected graph there exists a minimum dominating set with no pair of false twin vertices.
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Example

The following example shows that this bound can give a better upper bound for $\dim(G)$ compared to the upper bound $n - diam(G)$.

Example

Let G be a connected graph of order $3k+1$, $k \geq 6$, obtained from the wheel W_k by replacing each spoke by a path of length three (i.e. every spoke subdivided twice). It is easy to see that $\gamma(G) = k+1$ and $diam(G) \leq 6$. Hence, we have $\dim(G) \leq n - \gamma(G) = 2k$ and $\dim(G) \leq 3k+1 - diam(G)$.

A list of new bounds for metric dimension

Corollary

For every connected graph G of order n and girth g,

- if $g \geq 5$, then $\dim(G) \leq n - \delta(G)$.
- if $g \geq 6$, then $\dim(G) \leq n - 2\delta(G) + 2$.
- $\dim(G) \leq n - \left\lceil \frac{n}{1+\Delta(G)} \right\rceil$.
- if G has degree sequence $(d_1, d_2, \ldots, d_n)$ with $d_i \geq d_{i+1}$, then $\dim(G) \leq n - \min\{k \mid k + (d_1 + d_2 + \cdots + d_k) \geq n\}$.
- if $\delta(G) \geq 2$ and $g \geq 7$, then $\dim(G) \leq n - \Delta(G)$.
- if $\mu_n \geq \mu_{n-1} \geq \cdots \geq \mu_1$ be the eigenvalues of Laplacian matrix of G, then $\dim(G) \leq n - \frac{n}{\mu_n(G)}$.

Randomly k-dimensional graphs

Definition

A connected graph G is called randomly k-dimensional graph if each k-set of vertices of G is a basis of G.

Theorem (Chartrand et al. 2000)

A graph G is randomly 2-dimensional if and only if G is an odd cycle.

Chartrand et al. provided the following question.
Question. Are there only randomly k-dimensional graphs other than complete graph and odd cycles?

Randomly k-dimensional graphs

Theorem (Jannesari and O. 2012)

Let G be a graph with $dim(G) = k > 1$. Then, G is a randomly k-dimensional graph if and only if G is a complete graph K_{k+1} or an odd cycle.

Thank you