Arrangement Graphs

Ebrahim Ghorbani

(joint work with B. F. Chen and K. B. Wong)

K. N. Toosi University of Technology

and

Institute for Research in Fundamental Sciences (IPM)

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Permutation

A permutation of $[n]$ is an n-permutation of $[n]$.

Arrangement graphs

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The arrangement graph $A(n, k)$ is a graph with all the k-permutations of $[n]$ as vertices where two k-permutations are adjacent if they agree in exactly $k - 1$ positions.

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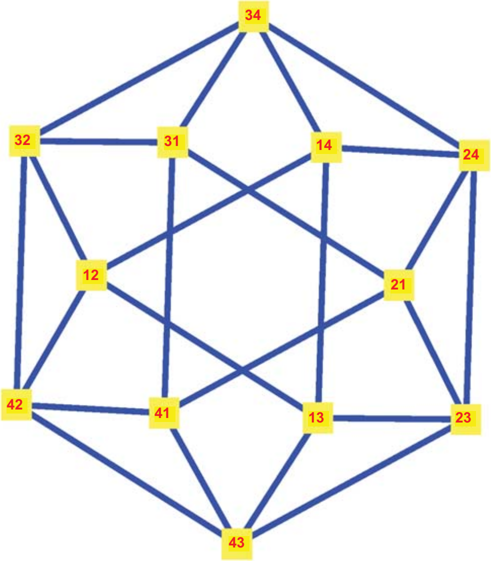
Basic properties of $A(n, k)$

- $\frac{n!}{(n-k)!}$ number of vertices;
- $k(n - k)$-regular;
- vertex transitive.

Arrangement graphs

The graph $A(4, 2)$



Arrangement graphs

Background

The family of arrangement graphs were first introduced by Day and Tripathi (1992), as an interconnection network model for parallel computation. In the interconnection network model, each processor has its own memory unit and communicates with the other processors through a topological network, i.e. a graph. For this purpose, the arrangement graphs possess many nice properties such as having small diameter, a hierarchical structure, vertex and edge symmetry, simple shortest path routing, high connectivity, etc.

Arrangement graphs

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- $A(n, 2)$ is the line graph of $K_{n,n}$ minus a perfect matching.
- $A(n, n-1)$ is $\mathrm{Cay}(\mathcal{S}_n, T_n)$ where $T_n = \{(1\,2), \ldots, (1\,n)\}$ (Abdollahi and Vatandoost (2009), Krakovski and Mohar (2012), Chapuy and Féray (2012)).

Question

What are the eigenvalues of (adjacency matrix) of $A(n, k)$?

Decomposition of permutations into cycles

Fact

Any permutation can be decomposed into disjoint cycles.

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Cycle structure of a permutation

To the cyclic decomposition of a permutation σ one may assign a list of integers consisting of the lengths of the cycles in the decomposition. This list is called the cycle type of σ.

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Cycle types: partitions of *n*

There is a one-to-one correspondence between the cycle type of permutations of $[n]$ and the partitions of the integer n.

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Examples of decompositions of some 5-permutations

Suppose $i, j > 5$.

$$(2, i, j, 5, 4) = (1\ 2\ i](3\ j](4\ 5),$$

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$$(1, 2, 3, i, j) = (1)(2)(3)(4\ i](5\ j].$$

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Any k-permutation is a product of disjoint cycles and paths. This decomposition is unique up to the order in which the cycles and paths are written.

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Definition

The decomposition of a k-permutation π as a product of disjoint cycles and paths is called the cyclic decomposition of π.

Cyclic decomposition of k-permutations

Graphical representation

To a k-permutation π, we assign a (directed) graph with vertices $[k] \cup \{\pi(1), \ldots, \pi(k)\}$ and with the set of arcs $\{(j, \pi(j)) \mid j \in [k]\}$. We call the resulting graph, the basic graph of π.

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Assume that there are integers of two kinds r and r' and we consider the ways to write n as a sum of integers of either kind where the order of terms in the sum does not matter.

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Example

$k = 2 \quad 11,\ 11',\ 1'1',\ 2,\ 2'$

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Example

$k = 2$ $\quad 11,\ 11',\ 1'1',\ 2,\ 2'$

$k = 3$ $\quad 111,\ 111',\ 11'1',\ 1'1'1',\ 21,\ 21',\ 2'1,\ 2'1',\ 3,\ 3'$

Cyclic decomposition of k-permutations

Cycle type of a k-permutation

Cycle structure or cycle type of a k-permutation π is the list consisting of the lengths of the cycles and the paths appeared in the cyclic decomposition of π. We write integers of the first kind for cycles and integers of the second kind for paths.

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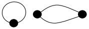
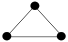
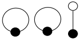
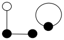
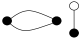
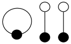
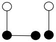
Observation

There is a one-to-one correspondence between the cycle structure of k-permutations and the partitions of k into parts of two kinds.

Cycle structure of 3-permutations

3-permutation	Decomposition	Partition of 3	Basic graph
$(1,2,3)$	$(1)(2)(3)$	111	
$(1,3,2)$	$(1)(2\ 3)$	12	
$(2,3,1)$	$(1\ 2\ 3)$	3	
$(1,2,i)$	$(1)(2)(3\ i]$	$111'$	
$(1,3,i)$	$(1)(2\ 3\ i]$	$12'$	
$(2,3,i)$	$(1\ 2\ 3\ i]$	$3'$	
$(2,1,i)$	$(1\ 2)(3\ i]$	$21'$	
$(1,i,j)$	$(1)(2\ i](3\ j]$	$11'1'$	
$(2,i,j)$	$(1\ 2\ i](3\ j]$	$1'2'$	
(i,j,ℓ)	$(1\ i](2\ j](3\ \ell]$	$1'1'1'$	

Cycle-type partition

$V(n, k) = \text{set of vertices of } A(n, k)$

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$V(n, k)$ = set of vertices of $A(n, k)$

We partition $V(n, k)$ according to the cycle type of k-permutations. So the k-permutations of each cell/part share the same cycle type. Equivalently, two k-permutations belong to the same cell if they have isomorphic basic graphs. We call this partition the cycle-type partition of $V(n, k)$.

Cycle-type partition

The cycle-type partition of 3-permutations

Type	Cell
111	$V_1 = \{(1,2,3)\}$
12	$V_2 = \{(1,3,2),(2,1,3),(3,2,1)\}$
3	$V_3 = \{(2,3,1),(3,1,2)\}$
$111'$	$V_4 = \{(1,2,i),(1,i,3),(i,2,3) \mid 4 \leqslant i \leqslant n\}$
$21'$	$V_5 = \{(2,1,i),(3,i,1),(i,3,2) \mid 4 \leqslant i \leqslant n\}$
$11'1'$	$V_6 = \{(1,i,j),(i,2,j),(i,j,3) \mid 4 \leqslant i,j \leqslant n, i \neq j\}$
$12'$	$V_7 = \{(1,3,i),(3,2,i),(1,i,2),(2,i,3),(i,2,1),(i,1,3) \mid 4 \leqslant i \leqslant n\}$
$1'1'1'$	$V_8 = \{(i,j,k) \mid 4 \leqslant i,j,k \leqslant n, i \neq j \neq k \neq i\}$
$1'2'$	$V_9 = \{(2,i,j),(3,i,j),(i,1,j),(i,3,j),(i,j,1),(i,j,2) \mid 4 \leqslant i,j \leqslant n, i \neq j\}$
$3'$	$V_{10} = \{(2,3,i),(3,1,i),(3,i,2),(2,i,1),(i,3,1),(i,1,2) \mid 4 \leqslant i \leqslant n\}$

Equitable partitions

Definition

An equitable partition of a graph G is a partition $\Pi = (V_1, \ldots, V_m)$ of the vertex set such that each vertex in V_i has the same number q_{ij} of neighbors in V_j for any i, j (and possibly $i = j$).

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Fact

Every eigenvalue of the quotient matrix Q is an eigenvalue of G.

Walk-regular graphs

Definition

A graph G is walk-regular if for every positive integer r, the number of closed walks of length r starting at a vertex v is independent of the choice of v.

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Fact

Vertex-transitive graphs are walk-regular and so are the arrangement graphs $A(n, k)$.

Equitable partitions

Theorem (Godsil and McKay, 1980)

Let G be a walk-regular graph with ν vertices. Let $\Pi = (V_1, \ldots, V_m)$ be an equitable partition of G with $|V_1| = 1$ and let Q be the quotient matrix of Π.

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(i) Every eigenvalue of G is an eigenvalue of Q.

(ii) Let $S = \mathrm{diag}(\sqrt{|V_1|}, \sqrt{|V_2|}, \ldots, \sqrt{|V_m|})$ and $P = SQS^{-1}$. If $\{\mathbf{x}_1, \ldots, \mathbf{x}_\ell\}$ is a full set of orthonormal eigenvectors of P for the eigenvalue λ, then the multiplicity of λ as an eigenvalues of G is

$$\nu \sum_{i=1}^{\ell} (\mathbf{x}_i)_1^2,$$

where $(\mathbf{x}_i)_1$ denotes the first coordinate of $\mathbf{x}_i$.

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The cycle-type partition of $V(n, k)$ is an equitable partition of $A(n, k)$.

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Theorem (explicit description of the quotient matrix)

Suppose that $\pi \in V(n,k)$ has the cycle type $[A,B]$. Then the neighbors of π are as follows.

(i) For any $i \in A$ with multiplicity a,

- (i.1) π has $ia(n-k-|B|)$ neighbors in $[A_i, B^i]$;
- (i.2) for any $j \in B$ with multiplicity b, π has abi neighbors in $[A_i, B_j^{i+j}]$.

(ii) For any $j \in B$ with multiplicity b and for any ℓ with $1 \leqslant \ell \leqslant j$,

- (ii.1) π has b neighbors in $[A^\ell, B_j^{j-\ell}]$;
- (ii.2) for any $m \in B_j$ with multiplicity c and $m+\ell \neq j$, in $[A, B_{j,m}^{m+\ell, j-\ell}]$, π has $2bc$ neighbors if $m \neq j$ and $m-j+\ell \geqslant 1$ and has bc neighbors otherwise.
- (ii.3) in $[A, B_j^{\ell, j-\ell}]$, π has $2b(n-k-|B|)$ neighbors if $j \neq 2\ell$ and $b(n-k-|B|)$ neighbors if $j = 2\ell$.

(iii) If $B = j_1^{b_1} \dots j_h^{b_h}$, then π has $|B|(n-k-|B|) + \sum_{1 \leqslant r < t \leqslant h} b_r b_t$ neighbors in $[A,B]$ (in particular, if $B = \emptyset$, then π has no neighbor in $[A,B]$).

Cycle-type partition

Quotient matrix for $A(n,3)$

$$\begin{pmatrix}
0 & 0 & 0 & 3(n-3) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2(n-3) & 0 & n-3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3(n-3) & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & n-4 & 2 & 0 & 0 & 2(n-4) & 0 & 0 \\
0 & 1 & 0 & 1 & n-4 & 1 & 0 & n-4 & n-4 & 0 \\
0 & 0 & 1 & 0 & 1 & n-4 & 1 & 0 & 2(n-4) & 0 \\
0 & 1 & 0 & 0 & 0 & 2 & n-4 & 0 & 2(n-4) & 0 \\
0 & 0 & 0 & 2 & 2 & 0 & 0 & 2(n-5) & 2 & n-5 \\
0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 2n-9 & n-5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 6 & 3(n-6)
\end{pmatrix}$$

Cycle-type partition

Eigenvalues and eigenvectors for quotient matrix of $A(n, 3)$

eigenvalue	eigenvector
-3	$[-3(n-3),\, n-3,\, 0,\, 3,\, -1,\, \frac{-2}{n-4},\, -1,\, 0,\, \frac{1}{n-4},\, 0]$
-3	$[-(n-3),\, 0,\, 0,\, 1,\, 0,\, \frac{-1}{n-4},\, 0,\, \frac{1}{(n-4)(n-5)},\, 0,\, 0]$
-3	$[3-n,\, n-3,\, 3-n,\, 1,\, -1,\, 0,\, -1,\, 0,\, 0,\, 1]$
$n-7$	$[3(n-3),\, 3(n-3),\, 3(n-3),\, n-7,\, n-7,\, \frac{2(11-2n)}{n-4},\, n-7,\, \frac{18}{n-4},\, \frac{2(11-2n)}{n-4},\, n-7]$
$n-6$	$[-6(n-3),\, 0,\, 3(n-3),\, -2,\, 2(n-3),\, 6,\, -(n-3),\, 0,\, -3,\, n-6]$
$n-4$	$[-6(n-3),\, 0,\, 3(n-3),\, -2(n-4),\, -2(n-1),\, 2,\, n-1,\, 0,\, -1,\, n-4]$
$n-3$	$[3,\, -3,\, 3,\, 1,\, -1,\, 0,\, -1,\, 0,\, 0,\, 1]$
$2n-9$	$[3(n-3),\, 3(n-3),\, 3(n-3),\, 2n-9,\, 2n-9,\, n-9,\, 2n-9,\, -9,\, n-9,\, 2n-9]$
$2n-6$	$[-3,\, 0,\, 3/2,\, -2,\, 1,\, -1,\, -1/2,\, 0,\, 1/2,\, 1]$
$3n-9$	$[1,\, 1,\, 1,\, 1,\, 1,\, 1,\, 1,\, 1,\, 1,\, 1]$

Eigenvalues of $A(n, k)$

Theorem

For $n \geqslant 4$, the eigenvalues of $A(n, 3)$ are

$$(-3)^{[n(n-2)(n-4)-1]},\ (n-7)^{[n(n-3)/2]},\ (n-6)^{[(n-2)(n-1)]},\ (n-4)^{[n(n-3)]},$$
$$(n-3)^{[(n-1)(n-2)/2]},\ (2n-9)^{[n-1]},\ (2n-6)^{[2(n-1)]},\ (3n-9)^{[1]}.$$

Eigenvalues of $A(n, k)$

Theorem

For $n \geqslant 5$, the eigenvalues of $A(n, 4)$ are as follows:

$$
\begin{array}{lll}
(-4)^{[n(n-3)(n^2-7n+8)+1]} & (n-10)^{[n(n-1)(n-5)/6]} & (n-9)^{[n(n-2)(n-4)]} \\
(n-8)^{[(n-1)(n-2)(n-3)/2]} & (n-7)^{[2n(n-2)(n-4)/3]} & (n-6)^{[n(n-1)(n-5)/2]} \\
(n-5)^{[n(n-2)(n-4)]} & (n-4)^{[(n-1)(n-2)(n-3)/6]} & (2n-14)^{[n(n-3)/2]} \\
(2n-12)^{[3(n-1)(n-2)/2]} & (2n-10)^{[3n(n-3)/2]} & (2n-8)^{[5n(n-3)/2+3]} \\
(3n-16)^{[n-1]} & (3n-12)^{[3(n-1)]} & (4n-16)^{[1]}.
\end{array}
$$

Eigenvalues of $A(n, k)$

Eigenvalues of $A(n, 5)$

$(-5)^{[n^5-15n^4+75n^3-145n^2+89n-1]}$	$(n-13)^{[n(n-1)(n-2)(n-7)/24]}$	$(n-12)^{[n(n-1)(n-3)(n-6)/2]}$
$(n-11)^{[n(n-5)(7n)^2-35n+37)/6]}$	$(n-10)^{[(n-1)(n-2)(n-3)(n-4)/6]}$	$(n-9)^{[5n(n-3)(n^2-7n+8)/4]}$
$(n-8)^{[n(n-1)(n-2)(n-7)/6]}$	$(n-7)^{[n(n-1)(7n)^2-63n+131)/6]}$	$(n-6)^{[n(n-2)(n-3)(n-5)/2]}$
$(n-5)^{[(n-1)(n-2)(n-3)(n-4)/24]}$	$(2n-19)^{[n(n-1)(n-5)/6]}$	$(2n-17)^{[4n(n-2)(n-4)/3]}$
$(2n-15)^{[(n-1)(n-2)(n-3)]}$	$(2n-14)^{[n(7n)^2-42n+50)/2]}$	$(2n-12)^{[2n(n-2)(n-4)]}$
$(2n-11)^{[5n(n-1)(n-5)/6]}$	$(2n-10)^{[(n-2)(7n^2-28n+6)/3]}$	$(3n-23)^{[n(n-3)/2]}$
$(3n-20)^{[2(n-1)(n-2)]}$	$(3n-18)^{[2n(n-3)]}$	$(3n-15)^{[(11n^2-33n+12)/2]}$
$(4n-25)^{[n-1]}$	$(4n-20)^{[4n-4]}$	$(5n-25)^{[1]}$

Eigenvalues of $A(n, k)$

Eigenvalues of $A(n, 6)$

$(-6)^{[n^6-21n^5+160n^4-545n^3+814n^2-415n+1]}$	$(n-16)^{[n(n-1)(n-2)(n-3)(n-9)/120]}$	$(n-15)^{[n(n-1)(n-2)(n-4)(n-8)/6]}$
$(n-14)^{[n(n-1)(n-7)(7n^2-49n+78)/8]}$	$(n-13)^{[n(n-3)(n-6)(n^2-6n+6)]}$	$(n-12)^{[(n-1)(n-2)(n-5)(n^2-7n+2)/4]}$
$(n-11)^{[n(n-4)(7n^3-77n^2+217n-162)/4]}$	$(n-10)^{[n(n-1)(n-3)(n-4)(n-7)/4]}$	$(n-9)^{[n(n-1)(n-2)(n^2-12n+34)]}$
$(n-8)^{[n(n-1)(n-3)(7n^2-77n+202)/8]}$	$(n-7)^{[n(n-2)(n-3)(n-4)(n-6)/6]}$	$(n-6)^{[(n-1)(n-2)(n-3)(n-4)(n-5)/120]}$
$(2n-24)^{[n(n-1)(n-2)(n-7)/24]}$	$(2n-22)^{[5n(n-1)(n-3)(n-6)/8]}$	$(2n-20)^{[n(n-5)(2n-3)(2n-7)/2]}$
$(2n-18)^{[(7n^3-63n^2+136n-40)(n-1)/4]}$	$(2n-17)^{[2n(n-2)(n-3)(n-5)]}$	$(2n-16)^{[15n(n-1)(n-3)(n-6)/8]}$
$(2n-15)^{[4n(n-1)(n-4)(n-5)/3]}$	$(2n-14)^{[n(n-2)(13n^2-104n+171)/8]}$	$(2n-13)^{[2n(n-1)(n-3)(n-6)]}$
$(2n-12)^{[(7n^4-70n^3+217n^2-210n+20)/4]}$	$(3n-30)^{[n(n-1)(n-5)/6]}$	$(3n-27)^{[5n(n-4)(n-2)/3]}$
$(3n-23)^{[3n(n-2)(n-2)]}$	$(3n-24)^{[5(n-4)(n-1)^2/2]}$	$(3n-21)^{[10n(n-2)(n-4)/3]}$
$(3n-20)^{[3n(n-1)(n-5)/2]}$	$(3n-18)^{[(n-4)(47n^2-94n+15)/6]}$	$(4n-34)^{[n(n-3)/2]}$
$(4n-30)^{[5(n-1)(n-2)/2]}$	$(4n-28)^{[5n(n-3)/2]}$	$(4n-24)^{[(19n^2-57n+20)/2]}$
$(5n-36)^{[n-1]}$	$(5n-30)^{[5n-5]}$	$(6n-36)^{[1]}$

Eigenvalues of $A(n,7)$

$(-7)^{[n^7-28n^6+301n^5-1575n^4+4179n^3-5243n^2+2372n-1]}$
$(n-18)^{[n(n-1)(n-2)(n-3)(n-5)(n-10)/24]}$
$(n-16)^{[n(n-1)(n-8)(83n^3-996n^2+3691n-4182)/72]}$
$(n-14)^{[(n-1)(n-2)(n-6)(13n^3-156n^2+401n-10)/20]}$
$(n-12)^{[n(n-1)(n-4)(13n^3-208n^2+1003n-1348)/20]}$
$(n-10)^{[n(n-1)(n-2)(83n^3-1494n^2+8593n-15822)/72]}$
$(n-8)^{[n(n-2)(n-3)(n-4)(n-5)(n-7)/24]}$
$(2n-29)^{[n(n-1)(n-2)(n-3)(n-9)/120]}$
$(2n-25)^{[n(n-1)(n-7)(8n^2-56n+89)/6]}$
$(2n-22)^{[n(n-1)(n-2)(11n^2-132n+367)/10]}$
$(2n-20)^{[5n(n-2)(n-4)(n^2-9n+15)/3]}$
$(2n-18)^{[n(n-1)(n-3)(n-4)(n-7)]}$
$(2n-16)^{[5n(n-2)(n-4)(n^2-9n+11)/3]}$
$(2n-14)^{[(n-3)(11n^4-132n^3+469n^2-438n+20)/10]}$
$(3n-34)^{[3n(n-1)(n-3)(n-6)/4]}$
$(3n-30)^{[n(n-1)(n-2)(n-7)/4]}$
$(3n-28)^{[5(n-1)(n-2)(n-3)(n-4)/6]}$
$(3n-26)^{[7n(n-1)(n-3)(n-6)n/4]}$
$(3n-24)^{[5n(n-2)(n-3)(n-5)/2]}$
$(3n-23)^{[35n(n-1)(n-3)(n-6)/8]}$
$(4n-43)^{[n(n-1)(n-5)/6]}$
$(4n-36)^{[n(n-1)(n-5)]}$
$(4n-34)^{[14n(n-2)(n-4)/3]}$
$(4n-31)^{[7n(n-1)(n-5)/3]}$
$(5n-47)^{[n(n-3)/2]}$
$(5n-40)^{[3n(n-3)]}$
$(6n-49)^{[n-1]}$
$(7n-49)^{[1]}$

$(n-19)^{[n(n-1)(n-2)(n-3)(n-4)(n-11)/720]}$
$(n-17)^{[n(n-1)(n-2)(n-9)(23n^2-207n+439)/60]}$
$(n-15)^{[n(n-7)(11n^4-154n^3+739n^2-1400n+844)/16]}$
$(n-13)^{[7n(n-5)(n^4-16n^3+80n^2-151n+89)/6]}$
$(n-11)^{[(11n^2-165n+592)(n-1)(n-2)(n-3)n/16]}$
$(n-9)^{[n(n-1)(n-3)(n-4)(23n^2-299n+941)/60]}$
$(n-7)^{[(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)/720]}$
$(2n-27)^{[n(n-1)(n-2)(n-4)(n-8)/5]}$
$(2n-23)^{[n(n-3)(n-6)(17n^2-102n+101)/8]}$
$(2n-21)^{[5(n-1)(n-3)(n-4)(3n^2-21n+2)/8]}$
$(2n-19)^{[7(n-7)(11n^2-77n+122)n(n-1)/20]}$
$(2n-17)^{[7n(n-1)(7n^3-98n^2+427n-568)/20]}$
$(2n-15)^{[7n(n-1)(n-7)(3n^2-21n+34)/8]}$
$(3n-37)^{[n(n-1)(n-2)(n-7)/24]}$
$(3n-31)^{[n(n-5)(73n^2-365n+382)/24]}$
$(3n-29)^{[7n(n-1)(n-3)(n-6)/4]}$
$(3n-27)^{[5n(n-3)(5n^2-35n+44)/4]}$
$(3n-25)^{[7n(n-1)(n^2-9n+19)/2]}$
$(3n-22)^{[7n(n-1)(n-2)(n-7)/12]}$
$(3n-21)^{[(75n^4-750n^3+2233n^2-1958n+120)/8]}$
$(4n-39)^{[2n(n-2)(n-4)]}$
$(4n-35)^{[5(n-1)(n-2)(n-3)/2]}$
$(4n-32)^{[5n(n-2)(n-4)]}$
$(4n-28)^{[(52n^3-312n^2+425n^2-60)/3]}$
$(5n-42)^{[3(n-1)(n-2)]}$
$(5n-35)^{[(29n^2-87n+30)/2]}$
$(6n-42)^{[6(n-1)]}$

Cayley Graphs

Definition

Let G be a finite group and S be a subset of G such that $1 \notin S$ and $s \in S \Rightarrow s^{-1} \in S$. The Cayley graph $\mathrm{Cay}(G, S)$ is the graph which has the elements of G as its vertices and

$$u \sim v \Leftrightarrow v = su, \ \exists s \in S.$$

Group Characters

Definition

If $\Psi : G \to \mathrm{GL}_n(\mathbb{C})$ is an homorphism, then the function

$$\chi : G \to \mathbb{C}$$
$$\chi(g) = \mathrm{trace}\,(\Psi(g))$$

is called a character of G.

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χ (or Ψ) is called reducible if

$$\exists n_1, n_2,\ n = n_1 + n_2,\ \forall g \in G, \Psi(g) \in \mathrm{GL}_{n_1}(\mathbb{C}) \oplus \mathrm{GL}_{n_2}(\mathbb{C}).$$

Normal Cayley Graphs

A Cayley graph $\mathrm{Cay}(G, S)$ is said to be normal if S is closed under conjugation, i.e.

$$\forall s \in S \;\; \forall g \in G, \;\; gsg^{-1} \in S.$$

Normal Cayley Graphs

A Cayley graph Cay(G, S) is said to be normal if S is closed under conjugation, i.e.

$$\forall s \in S \;\; \forall g \in G, \;\; gsg^{-1} \in S.$$

Theorem (Babai (1979), Diaconis and Shahshahani (1981))

The eigenvalues of a normal Cay(G, S) are given by

$$\eta_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s),$$

where χ ranges over all the irreducible characters of G. Moreover, the multiplicity of η_χ is $\chi(1)^2$.

Normal Cayley Graphs of $\mathcal{S}_n$

$\mathcal{S}_n$ denotes the symmetric group on $[n]$.

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Facts

- If C is a conjugacy class in $\mathcal{S}_n$, then for any irreducible characters of χ of $\mathcal{S}_n$,

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- The characters of $\mathcal{S}_n$ are integral valued.

Corollary

The eigenvalues of a normal $\mathrm{Cay}(\mathcal{S}_n, S)$ are integers.

Normal Cayley Graphs of $\mathcal{S}_n$

In general, if S is not closed under conjugation, then the eigenvalues of $\text{Cay}(\mathcal{S}_n, S)$ may not be integers.

Normal Cayley Graphs of $\mathcal{S}_n$

In general, if S is not closed under conjugation, then the eigenvalues of $\mathrm{Cay}(\mathcal{S}_n, S)$ may not be integers.

Problem

Find conditions on S, so that the eigenvalues of $\mathrm{Cay}(\mathcal{S}_n, S)$ are integers.

Integral Cayley Graphs of $\mathcal{S}_n$

Let $T \subseteq [n]$ and

$$\mathcal{S}_n(T) = \{\sigma \in \mathcal{S}_n \mid \sigma \text{ only moves elements of } T\}.$$

$\mathcal{S}_n(T)$ is a subgroup of $\mathcal{S}_n$ isomorphic to $\mathcal{S}_{|T|}$.

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$\mathcal{S}_n(T)$ is a subgroup of $\mathcal{S}_n$ isomorphic to $\mathcal{S}_{|T|}$.

Definition

A subset S of $\mathcal{S}_n$ is said to be nicely separated if there exists a normal subset S_0 and a partition $T_1, \ldots, T_\ell$ of $[n]$ such that

$$S = S_0 \setminus \left(\bigcup_{i=1}^{\ell} \mathcal{S}_n(T_i) \right).$$

Integral Cayley Graphs of $\mathcal{S}_n$

Lemma

If S is nicely separated, then $\sigma \in S$ implies that $\sigma^{-1} \in S$.

Integral Cayley Graphs of $\mathcal{S}_n$

Lemma

If S is nicely separated, then $\sigma \in S$ implies that $\sigma^{-1} \in S$.

Theorem

If S is nicely separated, then $\mathrm{Cay}(\mathcal{S}_n, S)$ is integral.

Integral Cayley Graphs of $\mathcal{S}_n$

Let $r \geqslant 2$ and $\mathrm{Cy}(r)$ be the set of all r cycles in $\mathcal{S}_n$ which do not fix 1, i.e.

$$\mathrm{Cy}(r) = \{\alpha \in \mathcal{S}_n \mid \alpha(1) \neq 1 \text{ and } \alpha \text{ is an } r\text{-cycle}\}.$$

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For instance, $\mathrm{Cy}(2) = \{(1\ 2), (1\ 3), \ldots, (1\ n)\}$.

Integral Cayley Graphs of $\mathcal{S}_n$

- It was conjectured by Abdollahi and Vatandoost (2009) that the eigenvalues of $\text{Cay}(\mathcal{S}_n, \text{Cy}(2))$ are integers, and contain all integers in the range from $-(n-1)$ to $n-1$.

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- The second part of the conjecture was proved by Krakovski and Mohar (2012).
- Chapuy and Feray (2012) pointed out that the conjecture could be proved by using Jucys–Murphy elements.

Integral Cayley Graphs of $\mathcal{S}_n$

Theorem

For all $2 \leqslant r \leqslant n$, the eigenvalues of $\mathrm{Cay}(\mathcal{S}_n, \mathrm{Cy}(r))$ are integers.

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Theorem

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Proof

$\text{Cy}(r)$ is nicely separated with $T_1 = \{1\}$ and $T_2 = \{2, 3, \dots, n\}$.

Integrality of Eigenvalues of $A(n, k)$

We may assume that $n > k \geqslant 2$ as $A(k, k)$ is the empty graph with k vertices and $A(n, 1)$ is the complete graph with n vertices.

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Theorem

Let

$$S = \{(i, j) \mid i \in [k], j \in [n] \setminus [k]\}.$$

The eigenvalues of $\mathrm{Cay}(\mathcal{S}_n, S)$ are integral.

Integrality of Eigenvalues of $A(n, k)$

Let $T = [n] \setminus [k]$ and

$$\mathcal{S}_n = \bigcup_{i=1}^{\ell} \mathcal{S}_n(T)\alpha_i$$

be the disjoint union of all the right cosets of $\mathcal{S}_n(T)$. Note that $\ell = n!/(n-k)!$.

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Observation

There is a one-to-one correspondence between the cosets of $\mathcal{S}_n(T)$ and the k-permutations of $[n]$.

Integrality of Eigenvalues of $A(n, k)$

Lemma

In $\mathrm{Cay}(\mathcal{S}_n, S)$ we have:

- For all $1 \leqslant i \leqslant \ell$, $\mathcal{S}_n(T)\alpha_i$ is an independent set.

Integrality of Eigenvalues of $A(n, k)$

Lemma

In $\text{Cay}(\mathcal{S}_n, S)$ we have:

- For all $1 \leqslant i \leqslant \ell$, $\mathcal{S}_n(T)\alpha_i$ is an independent set.
- For $1 \leqslant i, j \leqslant \ell$, $i \neq j$ if there exists an edge between $\mathcal{S}_n(T)\alpha_i$ and $\mathcal{S}_n(T)\alpha_j$, then there is a matching between $\mathcal{S}_n(T)\alpha_i$ and $\mathcal{S}_n(T)\alpha_j$.

Integrality of Eigenvalues of $A(n, k)$

- By Lemma, $(\mathcal{S}_n(T)\alpha_1, \ldots, \mathcal{S}_n(T)\alpha_\ell)$ is an equitable partition of Cay$(\mathcal{S}_n, S)$.

Integrality of Eigenvalues of $A(n, k)$

- By Lemma, $(\mathcal{S}_n(T)\alpha_1, \ldots, \mathcal{S}_n(T)\alpha_\ell)$ is an equitable partition of Cay$(\mathcal{S}_n, S)$.
- Let $Q = [q_{ij}]$ be the quotient matrix. If we represent the right coset $\mathcal{S}_n(T)\alpha_i$ by the vector $(\alpha_i(1), \alpha_i(2), \ldots, \alpha_i(k))$, then

$$\begin{aligned} q_{ij} = 1 \Leftrightarrow\ & (\alpha_i(1), \alpha_i(2), \ldots, \alpha_i(k)) \text{ and} \\ & (\alpha_j(1), \alpha_j(2), \ldots, \alpha_j(k)) \text{ differ in exactly 1 position.} \end{aligned}$$

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- $Q =$ adjacency matrix of $A(n, k)$.

Integrality of Eigenvalues of $A(n, k)$

- By Lemma, $(\mathcal{S}_n(T)\alpha_1, \ldots, \mathcal{S}_n(T)\alpha_\ell)$ is an equitable partition of $\mathrm{Cay}(\mathcal{S}_n, S)$.
- Let $Q = [q_{ij}]$ be the quotient matrix. If we represent the right coset $\mathcal{S}_n(T)\alpha_i$ by the vector $(\alpha_i(1), \alpha_i(2), \ldots, \alpha_i(k))$, then

$$q_{ij} = 1 \Leftrightarrow (\alpha_i(1), \alpha_i(2), \ldots, \alpha_i(k)) \text{ and } (\alpha_j(1), \alpha_j(2), \ldots, \alpha_j(k)) \text{ differ in exactly 1 position.}$$

- $Q =$ adjacency matrix of $A(n, k)$.
- Any eigenvalue of $A(n, k)$ is an eigenvalues of $\mathrm{Cay}(\mathcal{S}_n, S)$.

Integrality of Eigenvalues of $A(n, k)$

Corollary

The eigenvalues of Arrangement Graph $A(n, k)$ are integers.

Generalized Arrangement Graph $A(n, k, r)$

One can generalize the arrangement graphs $A(n, k)$ in a natural way.

Generalized Arrangement Graph $A(n, k, r)$

One can generalize the arrangement graphs $A(n, k)$ in a natural way.

The arrangement graph $A(n, k, r)$ is a graph with all the k-permutations of $[n]$ as vertices where two k-permutations are adjacent if they differ in exactly r positions.

Open Problems

Problem

What are the eigenvalues of the arrangement graph $A(n, k, r)$?

Open Problems

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What are the eigenvalues of the arrangement graph $A(n, k, r)$?

Conjecture

The eigenvalues of the arrangement graphs $A(n, k, r)$ consists entirely of integers.

Open Problems

We can prove that $-k$ is the smallest eigenvalue of $A(n, k)$ with multiplicity $\mathcal{O}(n^k)$ for fixed k.

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We can prove that $-k$ is the smallest eigenvalue of $A(n,k)$ with multiplicity $\mathcal{O}(n^k)$ for fixed k.

Conjecture

For any integer k, there is an integer n_0 such that for all $n \geqslant n_0$, $-k$ is the only negative eigenvalue of $A(n,k,1)$.

Happy 75th Birthday

Dr. Khosrovshahi!