Largest families of sets, under conditions defined by a given poset

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Let $[n] = \{1, 2, \ldots, n\}$ and let $\mathcal{F} \subset 2^{[n]}$ be a family of its subsets. Sperner proved in 1928 that if $\mathcal{F}$ contains no pair of members in inclusion that is $F_1, F_2 \in \mathcal{F}$ implies $F_1 \not\subset F_2$ then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$. Inspired by an application, Erdős generalized this theorem in the following way. Suppose that $\mathcal{F}$ contains no $k+1$ distinct members $F_1 \subset F_2 \subset \ldots \subset F_{k+1}$ then $|\mathcal{F}|$ is at most the sum of the k largest binomial coefficients of order n. This bound is, of course, tight. We will survey results of this type: determine the largest family of subsets of $[n]$ under a certain condition forbidding a given configuration described solely by inclusion among the members. This maximum is denoted by $\mathrm{La}(n, P)$ where P is the forbidden configuration. The final (hopelessly difficult) conjecture is that $\mathrm{La}(n, P)$ is asymptotically equal to the sum of the k largest binomial coefficients where the k largest levels contain no configuration P, but $k+1$ levels do.

Another related problem is the following one. A copy of the given poset P is a family $\mathcal{F}$ of subsets of $[n]$ where the embedding $f : P \to \mathcal{F}$ maps comparable elements of P into comparable subsets. Two copies $\mathcal{F}_1, \mathcal{F}_2$ of P are incomparable if no member of $\mathcal{F}_1$ is a subset of a member of $\mathcal{F}_2$ and no member of $\mathcal{F}_2$ is a subset of a member of $\mathcal{F}_1$. The maximum number of incomparable copies of P is denoted by $\mathrm{LA}(n.P)$. This quantity is asymptotically determined for all P, unlike $\mathrm{La}(n, P)$.