Eigenvalues of Graphs

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Matrices associated with graphs

The Adjacency Matrix A:

$$A(i,j) = \begin{cases} 1 & (i,j) \text{ is an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

The matrix A is symmetric, so its eigenvalues are real.

Example

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$P_G(\lambda) = \begin{vmatrix} \lambda & -1 & 0 & 0 \\ -1 & \lambda & -1 & 0 \\ 0 & -1 & \lambda & -1 \\ 0 & 0 & -1 & \lambda \end{vmatrix} = \lambda^4 - 3\lambda^2 - 1$$

Eigenvalues:

$$\frac{1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}$$

Matrices associated with graphs

The Laplacian Matrix L:

$$L(i,j) = \begin{cases} \text{deg}(v_i) & \text{if} \quad i = j, \\ -1 & \text{if} \quad \{i,j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix L is positive semidefinite, so its eigenvalues are real and nonnegative.

Example: G **=The wheel** W_5

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix}$$

Eigenvalues: 0, 3, 3, 5, 5

Eigenvalues: -2,0,0,0,2

Spectral characterization

Given a graph G,

Is there any graph H which share the eigenvalues with G?

In other words,

Is G determined uniquely by its eigenvalues?

(If so, G is called DS)

The main problem

Are almost all graphs DS?

The ratio of known DS graphs to all graphs is zero.

order	#graphs	percentage of DS graphs
2	2	100
3	4	100
4	11	100
5	34	94.1
6	156	93.6
7	1044	89.5
8	12346	86.1
9	274668	81.4
10	12005168	78.7
11	1018997864	78.9
12	165091172592	81.2

Brouwer, Haemers, Spence, 2009

Fact

Spectral characterization of graphs is a very hard problem.

Even for special families of graphs, the problem is challenging.

There are at least 11,084,874,829 graphs with eigenvalues $24, 5^{18}, -3^{38}$. (Kaski, Ostergard, 2004).

Theorem (Shrikhande, 1959)

The Hamming graph $H(2,n) \simeq K_n \times K_n$ is DS if and only if $n \neq 4$.

Theorem (Bang, van Dam and Koolen, 2008)

The Hamming graph $H(3,n) \simeq K_n \times K_n \times K_n$ is DS for $n \geqslant 36$.

Graph isomorphism problem

Given two graphs G_1 and G_2, determine whether they are isomorphic.

This is one of the most important problems in graph theory and complexity theory.

The complexity of this problem is unknown.

Graph isomorphism and Eigenvalues

Two isomorphic graphs have the same eigenvalues.

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Graph isomorphism and Eigenvalues

Two isomorphic graphs have the same eigenvalues.

But the converse is not true!

Conjecture (van Dam, Haemers, 2003)

Almost any two nonisomorphic graphs have different eigenvalues.

Theorem (Babai, Grigoryev, Mount, 1982)

The isomorphism test of graphs with bounded eigen-value multiplicity can be done in polynomial time.

Main Problems in Graph Spectra

-Spectral characterization

-Graphs with some bounded eigenvalues

-Graphs with bounded number of eigenvalues

-Integral graphs

-Eigenvalues and graph structure

-Eigenvalues and graph parameters

-Extremal graphs with respect to some eigenvalues

⋮

Eigenvalues in Graph Theory

Eigenvalues can be used to estimate some parameters of graphs which are hard to find, i.e. their determinations are NP-complete problems.

Examples

Chromatic number: $1 - \frac{\lambda_1}{\lambda_n} \leqslant \chi \leqslant 1 + \lambda_1$

Independence number: $\frac{n}{1+\lambda_1} \leqslant \alpha \leqslant n \frac{-\lambda_1 \lambda_n}{\delta^2 - \lambda_1 \lambda_n}$

Shannon capacity: $c \leqslant \frac{-n\lambda_n}{k-\lambda_n}$ (k-regular graphs)

Eigenvalues in Ranking

In a network, important people have many connections. Also important people have connections to many other important people. If one models this and says that up to some constant of proportionality one's importance is the sum of the importances of one's neighbors in the graph, then the vector giving the importance of each vertex becomes an eigenvector of the graph, necessarily the Perron-Frobenius eigenvector.

Eigenvalues in Quantum Physics

A network of quantum particles with fixed couplings is modeled with a graph. If the state of the particle in vertex u can be transferred to the particle in vertex v without any information loss, then we say that a perfect state transfer occurs from u to v.

Mathematical Model

Let G be a graph with adjacency matrix A. We define the matrix function $H(t)$ by

$$H(t) := exp(itA) = \sum_{n \geqslant 0} i^n A^n \frac{t^n}{n!}.$$

We say we have perfect state transfer from vertex u to a different vertex v at a time τ if

$$|H(\tau)_{u,v}| = 1.$$

The main problem is to characterize the pairs of vertices in graphs where perfect state transfer takes place.

Godsil proves that if perfect state transfer from u to v in a graph G occurs, then

(1) Any automorphism of G that fixes u must fix v, and conversely.

(2) The subgraphs $G \setminus u$ and $G \setminus v$ are cospectral.

(3) If ρ is the spectral radius of G, then ρ^2 is an integer.