Variational characterization of eigenvalues of a non–symmetric eigenvalue problem governing elastoacoustic vibrations

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In this talk we consider the elastoacoustic vibration problem, which consists of determining the small amplitude vibration modes of an elastic structure coupled with an internal inviscid, homogeneous, compressible fluid.

Different formulations have been proposed to model this problem, the most obvious of which describes the structure by its relative displacement field u and the fluid by its pressure p. Thus one arrives at the following system of homogeneous time-independent partial differential equations

$$\begin{aligned}
\text{Div }\sigma(u) + \omega^2 \rho_s u &= 0 \text{ in } \Omega_s,\\
\nabla^2 p + \frac{\omega^2}{c^2} p &= 0 \text{ in } \Omega_f,\\
u &= 0 \text{ on } \Gamma_D,\\
\nabla p \cdot n_f &= 0 \text{ on } \Gamma_N,\\
\sigma(u)\, n - p\, n &= 0 \text{ on } \Gamma_I,\\
\omega^2 \rho_f u \cdot n + \nabla p \cdot n &= 0 \text{ on } \Gamma_I,
\end{aligned} \tag{1}$$

where Ω_s and Ω_f denotes the region occupied by the structure and the fluid, respectively. Γ_D and Γ_N are Dirichlet– and Neumann-parts of the outer boundary of the structure, and Γ_I the interface between the fluid and the structure. The interface boundary conditions are a consequence of an equilibrium of acceleration and force densities at the contact interface.

Although this eigenvalue problem is not self–adjoint it shares many important properties with self–adjoint models: It has a countable set of eigenvalues which are real and non–negative, and taking advantage of a Rayleigh functional (which generalizes the Rayleigh quotient for self–adjoint problems) its eigenvalues allow for the variational characterizations known from the symmetric theory [3]. Namely, they can be characterized by Rayleigh's principle, and are minmax and maxmin values of the Rayleigh functional.

Discretizing the elastoacoustic problem with finite elements where the triangulation obeys the geometric partition into the fluid and the structure domain one obtains a non–symmetric matrix eigenvalue problem which inherits the variational properties.

$$Kx := \begin{bmatrix} K_s & C \\ 0 & K_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} = \lambda \begin{bmatrix} M_s & 0 \\ -C^T & M_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} =: \lambda Mx. \tag{2}$$

The following properties can be proved:

- The eigenvalues of the discrete problem (2) are upper bounds of the corresponding eigenvalues of problem (1).
- The standard spectral approximation theory applies to prove convergence results for Galerkin type methods.
- Eigenfunctions (of problem (1) and of its adjoint problem) can be chosen to satisfy an orthogonality property.

- A Krylov–Bogoliubov type eigenvalue bound holds [4].
- For the matrix eigenvalue problem the Rayleigh functional iteration is cubically convergent as is the Rayleigh quotient iteration for linear symmetric problems [1].
- Based on the variational characterization structure preserving iterative projection methods of Jacobi–Davidson type and nonlinear Arnoldi type can be defined [1, 4].
- The automated multi-level sub-structuring method (AMLS) introduced by Bennighof for linear symmetric eigenvalue problems in structural analysis can be generalized to the non–symmetric elastoacoustic problem, and an a priori error bound can be proved using the minmax characterization [2].

References

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