Hello

Turning Yablo's Paradoxes into Modality Theorems*

SAEED SALEHI

University of Tabriz & IPM

http://SaeedSalehi.ir/

*A Joint Work with AHMAD KARIMI.

IPM-Isfahan workshop on
Various Aspects of Modality
12 May 2016

The Liar Paradox

$\mathfrak{L}$: The Sentence $\mathfrak{L}$ is Untrue.

Or, $\mathfrak{L}$ is True IF AND ONLY IF $\mathfrak{L}$ is Untrue. So, $\mathfrak{L} \Longleftrightarrow \neg\mathfrak{L}$

$$\texttt{Propositional Logic} \vdash \neg(p \longleftrightarrow \neg p).$$

Theorem (Tarski)

If all the formulas can be coded by some terms in a language $\mathcal{L}$ ($\#: \mathcal{L}\text{-}\texttt{Formulas} \to \mathcal{L}\text{-}\texttt{Terms}, \varphi \mapsto \#\varphi$) and the diagonal lemma holds for a consistent $\mathcal{L}$-theory T (for any $\Psi(x) \in \mathcal{L}\text{-}\texttt{Formulas}$ there is some $\psi \in \mathcal{L}\text{-}\texttt{Sentences}$ such that $T \vdash \psi \leftrightarrow \Psi(\#\psi)$) then there can be no TRUTH PREDICATE *in $\mathcal{L}$ for T (an $\mathcal{L}$-formula $\mathfrak{T}(x)$ such that for any $\varphi \in \mathcal{L}\text{-}\texttt{Sentences}$, $T \vdash \varphi \leftrightarrow \mathfrak{T}(\#\varphi)$).*

Proof.

Take $\mathfrak{L}$ to be the diagonal sentence of $\neg\mathfrak{T}(x)$. Then

$T \vdash \mathfrak{L} \longleftrightarrow \neg\mathfrak{T}(\#\mathfrak{L}) \longleftrightarrow \neg\mathfrak{L}$ ※ ❑

Russell's Paradox

Is This Set a Member of Itself or not?

The Set of All Sets that are not Members of Themselves.

Theorem (Invalidity of "unrestricted" Comprehension Principle)

For some formula $\varphi(x)$ *the set* $\{x \mid \varphi(x)\}$ *does not exist.*

Proof.

Let $\varphi(x) =$"$x \notin x$". ❑

Proof.

$\varphi(x) =$"$\exists y\big[x = \mathscr{P}(y) \wedge x \notin y\big]$"
$\varphi(x) =$"$\exists y\big[x = y \times y \wedge x \notin y\big]$"
$\varphi(x) =$"$\exists y\big[x = \{y\} \wedge x \notin y\big]$" $\cdots$ ❑

$\{\hbar(y) \mid \hbar(y) \notin y\}$

Russell's Paradox—Theoremized

$$\texttt{Set Theory} \vdash \neg\exists y \forall x \big(x \in y \longleftrightarrow x \notin x \big).$$

Indeed, the proof does not make any essential use of $\in$.
Any binary relation will do:

$$\texttt{First-Order Logic} \vdash \neg\exists y \forall x \big(\Re(x,y) \longleftrightarrow \neg\Re(x,x) \big).$$

Russell's Popularization of his paradox:

Barber's Paradox

Shaves All and Only Those Who Cannot Shave Themselves.

$$\texttt{Second-Order Logic} \vdash \neg\exists Z^{(2)} \exists y \forall x \big(Z_{(x,y)} \longleftrightarrow \neg Z_{(x,x)} \big).$$

Russell's Paradox vs. the Liar's

Russell's Paradox $\equiv$

$$\neg\exists y\forall x\big(\Re(x,y)\longleftrightarrow\neg\Re(x,x)\big)\equiv$$

$$\forall y\exists x\,\neg\big(\Re(x,y)\longleftrightarrow\neg\Re(x,x)\big)\equiv$$

$$\bigwedge\nolimits_y\bigvee\nolimits_x\neg\big(\Re(x,y)\leftrightarrow\neg\Re(x,x)\big)\equiv$$

$$\bigwedge\nolimits_y\bigvee\nolimits_{x\neq y}\neg\big(\Re(x,y)\leftrightarrow\neg\Re(x,x)\big)\vee\neg\big(\Re(y,y)\leftrightarrow\neg\Re(y,y)\big)\equiv$$

$$\bigwedge\nolimits_y\Big[\neg\big(\Re(y,y)\leftrightarrow\neg\Re(y,y)\big)\vee\bigvee\nolimits_{x\neq y}\neg\big(\Re(x,y)\leftrightarrow\neg\Re(x,x)\big)\Big]\equiv$$

$$\bigwedge\nolimits_y\Big[\text{The Liar}_{\Re(y,y)}\vee\text{A Formula}\Big]$$

Russell's Paradox and Self-Reference

B. Russell, On Some Difficulties in the Theory of Transfinite Numbers and Order Types, *Proceedings of the London Mathematical Society* 4:1 (1907) 29–53.

Given a property ϕ and a function f, such that, if ϕ belongs to all the members of u $[\forall x \in u : \phi(x)]$, f'u $[f(u)]$ always exists, has the property ϕ, and is not a member of u $[f(u)\downarrow \in \{x \mid \phi(x)\} \setminus u]$; the the supposition that there is a class w of all terms having the property ϕ $[w = \{x \mid \phi(x)\}]$ and that f'w exists $[f(w)\downarrow]$ leads to the conclusion that f'w both has and has not the property ϕ $[\phi(f(w)) \& \neg\phi(f(w))]$.

This generalization is important, because it covers all the contradictions [paradoxes] that have hitherto emerged in the subject.

Russell and Self-Reference

$$u \subseteq \{x \mid \phi(x)\} \Longrightarrow f(u)\!\downarrow\in \{x \mid \phi(x)\} \setminus u$$
$$w = \{x \mid \phi(x)\} \,\&\, f(w)\!\downarrow \Longrightarrow \phi(f(w)) \,\&\, \neg\phi(f(w))$$

Definition (Productive)

A set A is *productive*, if there exists a (partial) computable function $f : \mathbb{N} \to \mathbb{N}$ such that for every n, if $\mathcal{W}_n$ (the n-th RE set) is a subset of A, then $f(n)\!\downarrow\in A \setminus \mathcal{W}_n$. $\mathcal{W}_n \subseteq A \Longrightarrow f(n)\!\downarrow\in A \setminus \mathcal{W}_n$

`Creative`: an RE set whose complement is *productive*.

E. L. Post, Recursively Enumerable Sets of Positive Integers and their Decision Problems, *Bulletin of the American Mathematical Society* 50:5 (1944) 284–316.

"… every symbolic logic is incomplete […]. The conclusion is unescapable that even for such a fixed, well defined body of mathematical propositions, *mathematical thinking is, and must remain, essentially creative.*"

Paradoxes and Self-Reference / Circularity

A General Belief:

all the paradoxes involve self-reference / circularity

(in a way or another).

YABLO's Paradox $Y_1, Y_2, Y_3, \cdots$

For all n, Y_n is True if and only if All Y_k's for $k > n$ are Untrue.

$$Y_1: \quad Y_2, Y_3, Y_4, \cdots \text{ are all untrue.}$$
$$Y_2: \quad Y_3, Y_4, Y_5, \cdots \text{ are all untrue.}$$
$$Y_3: \quad Y_4, Y_5, Y_6, \cdots \text{ are all untrue.}$$
$$\vdots$$

- If some Y_m is true, then $Y_{m+1}, Y_{m+2}, Y_{m+3}, \cdots$ are all untrue. Whence Y_{m+1} is untrue but also true (by $\bigwedge_{i \geqslant m+2} Y_i$).
- If all Y_k's are untrue, then $Y_0, Y_1, Y_2, \cdots$ are true!

Paradox(es) without Self-Reference?

- S. YABLO, Paradox without Self-Reference, *Analysis* (1993).

- On Paradox without Self-Reference, *Analysis* (1995).
- Is Yablo's Paradox Liar-Like?, *Analysis* (1995).
- Is Yablo's Paradox Non-Circular?, *Analysis* (2001).
- Paradox without satisfaction, *Analysis* (2003).
- There are Non-Circular Paradoxes (but Yablo's is not one of them), *The Monist* (2006).
- The Elimination of Self-Reference: Generalized Yablo-Series and the Theory of Truth, *Journal of Philosophical Logic* (2007).
- The Yablo Paradox and Circularity, *Análisis Filosófico* (2012).
- Equiparadoxicality of Yablo's Paradox and the Liar, *Journal of Logic Language and Information* (2013).

Yablo's Paradoxes

YABLO's Paradoxes $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \cdots$

(always)	$\mathcal{Y}_n \iff \forall i > n\, (\mathcal{Y}_i \texttt{ is untrue})$
(sometimes)	$\mathcal{Y}_n \iff \exists i > n\, (\mathcal{Y}_i \texttt{ is untrue})$
(almost always)	$\mathcal{Y}_n \iff \exists i > n\, \forall j \geqslant i\, (\mathcal{Y}_j \texttt{ is untrue})$
(infinitely often)	$\mathcal{Y}_n \iff \forall i > n\, \exists j \geqslant i\, (\mathcal{Y}_j \texttt{ is untrue})$

(almost always):

- If some Y_m is true, then for some $k > m$, all $Y_k, Y_{k+1}, Y_{k+2}, \cdots$ are untrue. Whence Y_{k+1} is simultaneously true and untrue!
- If all Y_k's are untrue, then $Y_0, Y_1, Y_2, \cdots$ are true!

Theoremizing Yablo's Paradox

J. Ketland, Yablo's Paradox and ω-Inconsistency, *Synthese* 145:3 (2005) 295–302.

$$\{\forall x\exists y(x<y), \forall x,y,z(x<y<z\rightarrow x<z)\} \vdash \neg\forall x\big(\varphi(x)\leftrightarrow\forall y[x<y\rightarrow\neg\varphi(y)]\big).$$

More generally,

Theorem (First-Order Logic)

$$\forall x\exists y\big(x\Re y\wedge\forall z[y\Re z\rightarrow x\Re z]\big)\vdash\neg\forall x\big(\varphi(x)\leftrightarrow\forall y[x\Re y\rightarrow\neg\varphi(y)]\big)$$

Proof.

If $\forall x\big(\varphi(x)\leftrightarrow\forall y[x\Re y\rightarrow\neg\varphi(y)]\big)$ then for any $a\Re b$ with $\forall z(b\Re z\rightarrow a\Re z)$, we have $\varphi(a)\Rightarrow\neg\varphi(b)\&\neg\varphi(c)$ for any c with $b\Re c$ (and so $a\Re c$) a contradiction with the arbitrariness of c. So, $\neg\varphi(a)$ for every a, hence $\varphi(a)$ for any a, contradiction! ❑

Theoremizing Yablo's Paradox

Theorem (Second-Order Logic)

$$\forall x \exists y \big(x \Re y \wedge \forall z [y \Re z \rightarrow x \Re z]\big) \vdash \neg \exists Z^{(1)} \forall x \big(Z_x \leftrightarrow \forall y [x \Re y \rightarrow \neg Z_y]\big)$$

Definition (YABLO System)

Let us call a directed graph $\langle A; R\rangle$ (with $R \subseteq A^2$) a Yablo system when $\neg \exists Z^{(1)} \forall x \big(Z_x \leftrightarrow \forall y [x R y \rightarrow \neg Z_y]\big)$.

example Any odd-cycle, such as $\langle \{a\}; \{a \Re a\}\rangle$. The Liar's Paradox

~~example~~ Any even-cycle, such as $\langle \{a, b\}; \{a \Re b \Re a\}\rangle$ (with $Z = \{a\}$).

Yablo's Paradox — 1st or 2nd Order?

The first-order condition $\forall x \exists y \big(x \Re y \wedge \forall z [y \Re z \rightarrow x \Re z]\big)$ (and many more weaker conditions) imply the Yablo-ness of the graph.

Theorem (Nonfirstorderizability of Yabloness)

The Yabloness $\neg\exists Z^{(1)} \forall x \big(Z_x \leftrightarrow \neg\exists y [x \Re y \wedge Z_y]\big)$ *is not equivalent to any first-order formula (in the language $\langle \Re \rangle$).*

https://en.wikipedia.org/wiki/Nonfirstorderizability

G. Boolos, To Be is To Be a Value of a Variable (or to be some values of some variables), *The Journal of Philosophy* 81:8 (1984) 430–449.

Geach-Kaplan sentence: some critics admire only one another

Yablo's Paradox — 1st or 2nd Order?

YABLONESS: $\neg\exists Z^{(1)}\forall x\big(Z_x\leftrightarrow\neg\exists y[x\Re y\wedge Z_y]\big)$

there is no group which contains all and only those whose no related one is (already) in the group

Theorem ((Very) Nonfirstorderizability of Non-YABLONESS)

The Non–YABLONESS $\exists Z^{(1)}\forall x\big(Z_x\leftrightarrow\neg\exists y[x\Re y\wedge Z_y]\big)$ *is not equivalent to any first-order* $\langle\Re\rangle$*-theory.*

Conjecture (Any Help is Appreciated!)

The YABLONESS $\neg\exists Z^{(1)}\forall x\big(Z_x\leftrightarrow\neg\exists y[x\Re y\wedge Z_y]\big)$ is not equivalent to any first-order $\langle\Re\rangle$-theory, either.

Yablo's Paradox — 1st or 2nd Order? or non?

Is That It?

L. M. Picollo, Yablo's Paradox in Second-Order Languages: Consistency and Unsatisfiability, *Studia Logica* 101:3 (2013) 601–617.

> *If we embrace the second-order notion of logical consequence we must subscribe to the idea that the second-order calculus is not powerful enough for representing Yablo's argument, and neither is the first-order calculus.*

Is there a better (or just another) logic that represents Yablo's Paradox (and his argument)?

Linear Temporal Logic

(Propositional) Linear Temporal Logic (LTL):

$\bigcirc$ **Next** $\Box$ **Always (from now on)**

Formulas: p (atomic) | $\neg\varphi$ | $\varphi_1 \wedge \varphi_2$ | $\varphi_1 \vee \varphi_2$ | $\varphi_1 \rightarrow \varphi_2$ | $\bigcirc\varphi$ | $\Box\varphi$

$\neg\bigcirc\varphi$: not in the next step φ
$\bigcirc\neg\varphi$: in the next step not φ

$\bigcirc\Box\varphi$: in the next time always (from then on) φ
$\Box\bigcirc\varphi$: always (from now on) in the next step φ

from the next step onward φ

LTL and YABLO's Paradox

YABLO's Paradox:

"everyone in an infinite linear row claims that
all the forthcoming ones are lying"

$\varphi \longleftrightarrow \Box\bigcirc\neg\varphi \qquad (\equiv \bigcirc\Box\neg\varphi) \quad (\equiv \Box\neg\bigcirc\varphi)$

"I will always deny all my future (from the next step onward) sayings"

"I will always deny whatever I will have said afterwards"

"All I will say from the next step on are lies!"

LTL—An Axiomatization

F. KRÖGER & S. MERZ, *Temporal Logic and State Systems* (Springer 2008).

Axioms:
- All the Propositional Tautologies

(LTL1) $\neg\bigcirc\varphi \longleftrightarrow \bigcirc\neg\varphi$

(LTL2) $\bigcirc(\varphi\rightarrow\psi) \longrightarrow (\bigcirc\varphi\rightarrow\bigcirc\psi)$

(LTL3) $\Box\varphi \longrightarrow \varphi\wedge\bigcirc\Box\varphi$

Rules: (MP)

$$\frac{\varphi,\quad \varphi\rightarrow\psi}{\psi}$$

(Next)

$$\frac{\varphi}{\bigcirc\varphi}$$

(Ind)

$$\frac{\varphi\rightarrow\bigcirc\varphi,\quad \varphi\rightarrow\psi}{\varphi\rightarrow\Box\psi}$$

Yablo's Paradox as an LTL–Theorem

A. Karimi & S. Salehi, Diagonal Arguments and Fixed Points, *Bulletin of the Iranian Mathematical Society*, to appear.

Theorem (Yablo's Paradox $\Longrightarrow$ Genuine Theorem)

(Propositional) Linear Temporal Logic $\vdash \neg\Box(\varphi \longleftrightarrow \Box\bigcirc\neg\varphi)$.

To Be Continued ...

In the Next Lecture

and at the Swamplandia 2016:

Ghent University, Belgium, May 30—June 1, 2016

www.swamplandia2016.ugent.be

Thank You!

Thanks to

The Participants .... For Listening ···

and

The Organizers — For Taking Care of Everything ···

SAEEDSALEHI.ir