Stickelberger and the Eigenvalue Theorem

IPM Mathematics Colloquium

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The Eigenvalue Theorem

Let F be algebraically closed and $I \subset F[x_1, \ldots, x_n]$ be an ideal with $\mathbf{V}(I) \subseteq F^n$ is finite (so I is zero dimensional). For $h \in F[x_1, \ldots, x_n]$, multiplication by h induces a linear map

$$m_h : F[x_1, \ldots, x_n]/I \longrightarrow F[x_1, \ldots, x_n]/I.$$

The Eigenvalue Theorem

The eigenvalues of m_h are $h(a)$ for $a \in \mathbf{V}(I)$.

I love this theorem because it reveals a wonderful link between polynomial algebra and linear algebra. To actually find solutions, note:

- The eigenvalues of m_{x_i} are the ith coordinates of the solutions.
- Since $m_{x_i} m_{x_j} = m_{x_i x_j} = m_{x_j x_i} = m_{x_j} m_{x_i}$, simultaneous eigenvectors exist. They give the solutions!

Recent work of Simon Telen explores how the Eigenvalue Theorem interacts with numerical algebraic geometry.

In this lecture, I will instead focus on a name attached to the theorem.

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Theorem 1. (Stickelberger Theorem)

Let $\mathbb{K} \subset \mathbb{F}$ be a field extension with $\mathbb{F}$ algebraically closed, $h \in \mathbb{K}[\underline{x}]$ and J be a zero dimensional ideal in $\mathbb{K}[\underline{x}]$. If $\mathcal{V}_{\mathbb{F}}(J) = \{\Delta_1, \ldots, \Delta_s\}$ are the zeros in $\mathbb{F}^n$ of J then there exists a basis of $\mathbb{F}[\underline{x}]/J$ such that the matrix of M_h, with respect to this basis, has the following block structure:

$$\begin{pmatrix} \mathbb{H}_1 & 0 & \ldots & 0 \\ 0 & \mathbb{H}_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \mathbb{H}_s \end{pmatrix} \quad \textit{where} \quad \mathbb{H}_i = \begin{pmatrix} h(\Delta_i) & \star & \ldots & \star \\ 0 & h(\Delta_i) & \ldots & \star \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & h(\Delta_i) \end{pmatrix}$$

The dimension of the i-th submatrix is equal to the multiplicity of Δ_i as a zero of the ideal J.

Notation: $M_h : \mathbb{F}[\underline{x}]/J \to \mathbb{F}[\underline{x}]/J$ is multiplication by h.

Sottile, 2002 in *Computations in Algebraic Geometry with Macaulay2*

Stickelberger's Theorem

Let $h \in A$ and m_h be as above. Then there is a one-to-one correspondence between eigenvectors v_ξ of m_h and roots ξ of I, the eigenvalue of m_h on v_ξ is the value $h(\xi)$ of h at ξ, and the multiplicity of this eigenvalue (on the eigenvector v_ξ) is the multiplicity of the root ξ.

Notation:

$A = k[X]/I$, where I is a zero-dimensional ideal in $k[X]$.

$m_h : A \to A$ is multiplication by $h \in A$.

Theorem 4.99 (Stickelberger)

For $f \in \overline{A}$, the linear map L_f has the following properties:

The trace of L_f is

$$\mathrm{Tr}(L_f) = \sum_{x \in \mathrm{Zer}(\mathcal{P};C^k)} \mu(x) f(x)$$

Return

The determinant of L_f is

$$\det(L_f) = \prod_{x \in \mathrm{Zer}(\mathcal{P};C^k)} f(x)^{\mu(x)}$$

Return

The characteristic polynomial $\chi(\mathcal{P}, f, T)$ of L_f is

$$\chi(\mathcal{P}, f, T) = \prod_{x \in \mathrm{Zer}(P;C^k)} (T - f(x))^{\mu(x)}$$

Notation: $\overline{A} = C[X_1, \ldots, X_k]/\mathrm{Ideal}(\mathcal{P}, C)$, C algebraically closed.

Goal of This Lecture

- The last three slides gave versions of a result called "Stickelberger's Theorem".
- The statement of the result was slightly different in each case.
- One feature they have in common: **the authors <u>never</u> cite a specific paper of Stickelberger!**

Three Questions to Answer

- What did Stickelberger really do?
- How does it relate to "Stickelberger's Theorem"?
- Why are his papers never cited?

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Ludwig Stickelberger

- Born in Switzerland in 1850.
- PhD in Berlin in 1874 under Weierstrass and Kummer.
- University of Freiburg 1879–1919.
- Published 12 papers (4 with Frobenius) + one posthumous.
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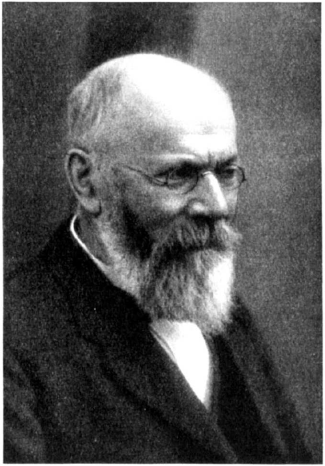
Stickelberger's Obituary, 1937

Ludwig Stickelberger.

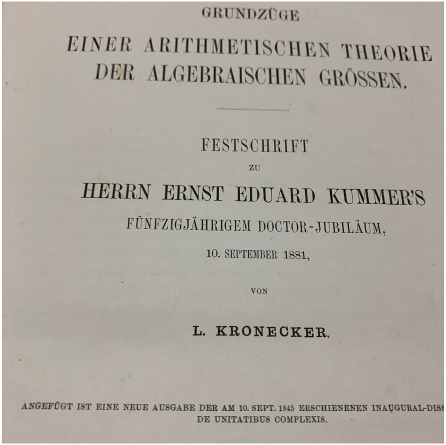
Von LOTHAR HEFFTER in Freiburg i. B.

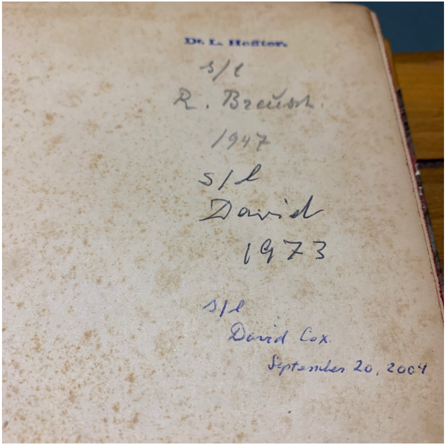
Mit einem Bildnis.



Ludwig Stickelberger entstammte einem alten, teils in der Schweiz, teils im Breisgau ansässigen alemannischen Geschlecht und wurde am 18. Mai 1850 zu Buch im Kanton Schaffhausen als Sohn eines Pfarrers geboren. Er hatte mit 17 Jahren die Schule durchlaufen, studierte in Heidelberg und Berlin und erwarb an letzterer Universität 1874 die Doktorwürde, um sich noch im selben Jahr am Polytechnikum in Zürich zu habilitieren. 1879 wurde er auf Veranlassung von Lüroth, der seine Bedeutung erkannt hatte, als a. o. Professor nach Freiburg i. B. berufen, erhielt hier 1896

My Connection to Heffter





Stickelberger's Theorem, 1890

Consider the cyclotomic extension $\mathbb{Q} \subseteq \mathbb{Q}(\zeta_m)$.

- Let $G = \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^\times$.
- $[a] \in (\mathbb{Z}/m\mathbb{Z})^\times$ gives $\sigma_a \in G$ satisfying $\sigma_a(\zeta_m) = \zeta_m^a$.

Definitions

- The Stickelberger element:

$$\theta = \frac{1}{m} \sum_{\gcd(a,m)=1} a \cdot \sigma_a^{-1} \text{ in the group ring } \mathbb{Q}[G]$$

- The Stickelberger ideal:

$$I = (\theta\mathbb{Z}[G]) \cap \mathbb{Z}[G] \subseteq \mathbb{Z}[G]$$

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The Stickelberger ideal I annihilates the class group of $\mathbb{Q}(\zeta_m)$.

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Stickelberger's 1897 Work on Traces

Verhandlungen des ersten internationalen Mathematiker-Kongresses

In a number field Ω, let $\mathfrak{a}, \mathfrak{b}, \ldots$ be ideals of $\mathfrak{o}$ containing a prime p. Then $\mathfrak{o}/\mathfrak{a}, \mathfrak{o}/\mathfrak{b}, \ldots$ are vector spaces over $\mathbb{F}_p$.

I. Sind $\mathfrak{a}$, $\mathfrak{b}$ theilerfremd und bedeuten $A(\omega)$, $B(\omega)$, $C(\omega)$ die Spuren einer Zahl ω in bezug auf $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{a}\mathfrak{b} = \mathfrak{c}$, so ist

$$A(\omega) + B(\omega) \equiv C(\omega) \pmod{p}.$$

II. Ist $\mathfrak{a}$ die k^{te} Potenz eines Primideals $\mathfrak{p}$, so besteht zwischen den zugehörigen Spuren $A(\omega)$, $P(\omega)$ die Relation

$$A(\omega) \equiv kP(\omega) \pmod{p}.$$

III. Sind $\mathfrak{p}_1, \mathfrak{p}_2, \cdots \mathfrak{p}_m$ die verschiedenen in p aufgehenden Primideale des Körpers, und ist

$$\mathfrak{o}p = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_m^{e_m}, \tag{2}$$

so ist die absolute Spur

$$S(\omega) \equiv e_1P_1(\omega) + e_2P_2(\omega) + \cdots + e_mP_m(\omega) \pmod{p}. \tag{3}$$

Stickelberger trace formula

Stickelberger's 1897 Work on Traces

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Stickelberger's 1897 Paper was on Discriminants

The title of the paper is

Über eine neue Eigenschaft der Diskriminanten algebraischer Zahlkörper

The Discriminant of a Number Field $\mathbb{Q} \subseteq \Omega$ of Degree n

Let $b_1, \ldots, b_n$ be an integral basis of the ring $\mathcal{O}$ of algebraic integers of Ω. Then

$$D = \det(\mathrm{Tr}(b_i b_j)).$$

Here is a result Stickelberger proves using his trace formula:

Theorem

If an odd prime p does not divide D, then the Legendre symbol $\left(\frac{D}{p}\right)$ satisfies

$$\left(\frac{D}{p}\right) = (-1)^{n-m},$$

where $p\mathcal{O} = \mathfrak{p}_1 \cdots \mathfrak{p}_m$ is the prime factorization in $\mathcal{O}$.

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A Modern Version of the Trace Formula

In the Stickelberger Trace formula, $\mathcal{O}/p\mathcal{O}$ is a finite-dimensional $\mathbb{F}_p$-algebra. In 1988, Scheja and Storch generalized his trace formula to a finite-dimensional commutative algebra A over a field F.

Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be the maximal ideals of A. The localizations $A_{\mathfrak{m}_i}$ have residue fields $L_i \simeq A/\mathfrak{m}_i$ and satisfy

$$A \simeq \prod_{i=1}^r A_{\mathfrak{m}_i}.$$

- For each i, define λ_i by $\dim_F A_{\mathfrak{m}_i} = \lambda_i [L_i : F]$.
- $\alpha \in A$ gives F-linear multiplication maps $m_\alpha : A \to A$ and $m_\alpha : L_i \to L_i$.

Stickelberger Trace Formula (Scheja and Storch, 1988)

Assume that L_i is a separable extension of F for $1 \le i \le r$. Then for $\alpha \in A$, the multiplication maps m_α satisfy

$$\mathrm{Tr}_A(m_\alpha) = \sum_{i=1}^r \lambda_i \mathrm{Tr}_{L_i}(m_\alpha).$$

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From the Trace Formula to the Eigenvalue Theorem

Assume F is algebraically closed of characteristic 0 and $A = F[x_1, \ldots, x_n]/I$, $I \subseteq F[x_1, \ldots, x_n]$ zero-dimensional ideal. Then:

- Solutions $a \in \mathbf{V}(I) \longleftrightarrow$ maximal ideals $\mathfrak{m}_a \subseteq A$.
- $A/\mathfrak{m}_a \simeq F$ via $g \mapsto g(a)$.

Then $m_f(1) = f \mapsto f(a) \in F$, so that $\mathrm{Tr}_F(m_f) = f(a)$. Hence:

Corollary of the Stickelberger Trace Formula

When F is algebraically closed, $\mathrm{Tr}_A(m_f) = \sum_{a \in \mathbf{V}(I)} \mu(a) f(a)$.

We have seen this before. Furthermore:

TFAE

1. For every $f \in F[x_1, \ldots, x_n]$, $\mathrm{Tr}(m_f) = \sum_{a \in \mathbf{V}(I)} \mu(a) f(a)$.
2. For every $f \in F[x_1, \ldots, x_n]$, $\det(m_f - x\,I) = \prod_{a \in \mathbf{V}(I)} (f(a) - x)^{\mu(a)}$.

Note: (1) implies (2) by applying (1) to f^ℓ, $\ell \geq 0$ & the Newton identities.

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When F is algebraically closed, $\mathrm{Tr}_A(m_f) = \sum_{a\in\mathbf{V}(I)} \mu(a)f(a)$.

We have seen this before. Furthermore:

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1. For every $f \in F[x_1,\ldots,x_n]$, $\mathrm{Tr}(m_f) = \sum_{a\in\mathbf{V}(I)} \mu(a)f(a)$.
2. For every $f \in F[x_1,\ldots,x_n]$, $\det(m_f - x\,I) = \prod_{a\in\mathbf{V}(I)} (f(a)-x)^{\mu(a)}$.

Note: (1) implies (2) by applying (1) to f^ℓ, $\ell \geq 0$ & the Newton identities.

From the Trace Formula to the Eigenvalue Theorem

Assume F is algebraically closed of characteristic 0 and $A = F[x_1, \ldots, x_n]/I$, $I \subseteq F[x_1, \ldots, x_n]$ zero-dimensional ideal. Then:

- Solutions $a \in \mathbf{V}(I) \longleftrightarrow$ maximal ideals $\mathfrak{m}_a \subseteq A$.
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An Application over $\mathbb{R}$

When $A = \mathbb{R}[x_1, \dots, x_n]/I$, maximal ideals $\mathfrak{m}_i$ come in two flavors:

r_1 ideals with $L_i \simeq \mathbb{R} \longleftrightarrow$ real solution

r_2 ideals with $L_i \simeq \mathbb{C} \longleftrightarrow$ complex conjugate pair of solutions

The trace form of A is the quadratic form $Q_A(f) = \mathrm{Tr}_A(m_{f^2})$. The Stickelberger Trace Formula implies

$$Q_A = \textstyle\sum_{i=1}^r \lambda_i Q_{L_i}.$$

Using bases $\{1\}$ of $\mathbb{R} \subseteq \mathbb{R}$ and $\{1, \sqrt{-1}\}$ of $\mathbb{R} \subseteq \mathbb{C}$, one computes

$$\text{matrix of } Q_{L_i} = \begin{cases} (1) & L_i = \mathbb{R} \quad \text{(happens } r_1 \text{ times)} \\ \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} & L_i = \mathbb{C} \quad \text{(happens } r_2 \text{ times).} \end{cases}$$

Since $\lambda_i > 0$ for all i, the signature of Q_A is

$$\boxed{\sigma(Q_A) = (r_1 + r_2) - r_2 = r_1 = \#\text{real solutions}}$$

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- 1981 Lazard publishes Résolutions des systèmes d'équations algébriques. In the zero-dimensional case, he gives an algorithm to compute

$$\prod_{a \in \mathbf{V}(I)} L(a)^{\mu(a)},$$

Familiar

where $L = U_0 + U_1 x_1 + \cdots + U_n x_n$ for new variables $U_0, \ldots, U_n$.

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