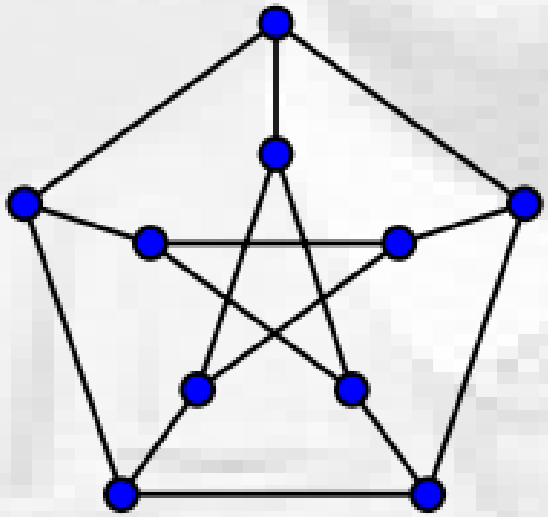
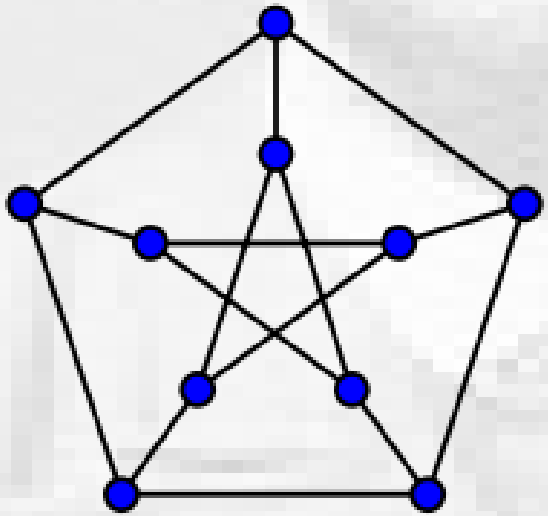
Julius Petersen, His Graph & Its Homeomorphs and Amallamorphs



Hossein Shahmohamad
School of Mathematical Sciences, R.I.T
Rochester, New York
June 9, 2011

Julius Petersen, His Graph & Its Homeomorphs and Amallamorphs

(A 50 minutes Crash Course in Graph Theory)



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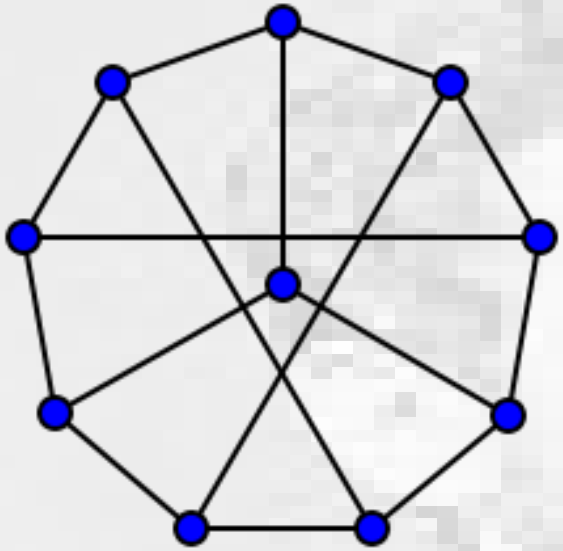
JULIUS PETERSEN

A Graph For All Seasons



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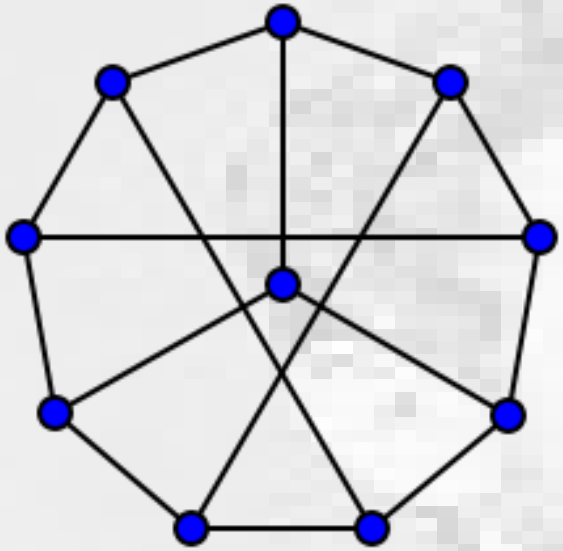

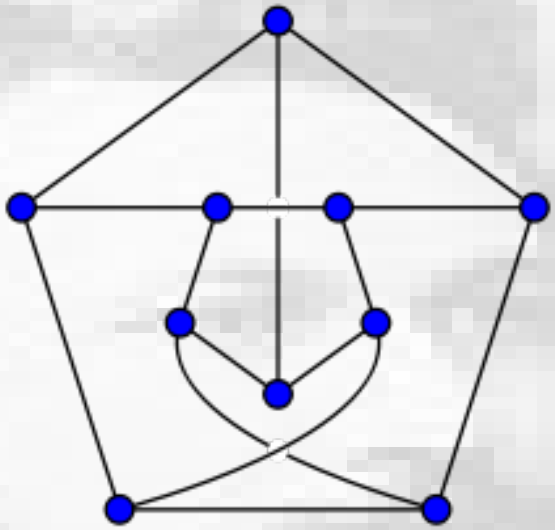

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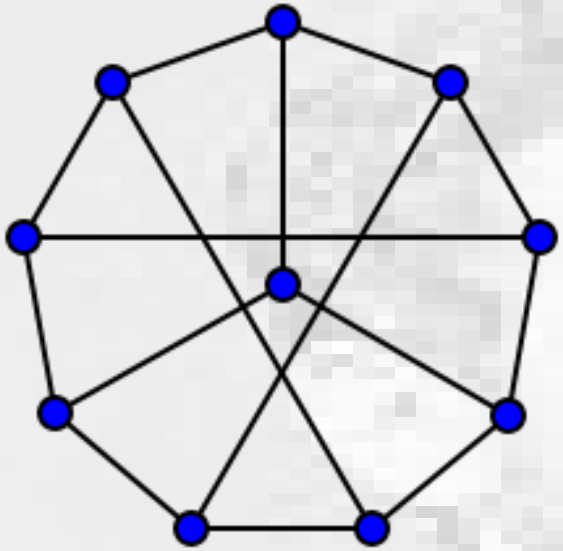

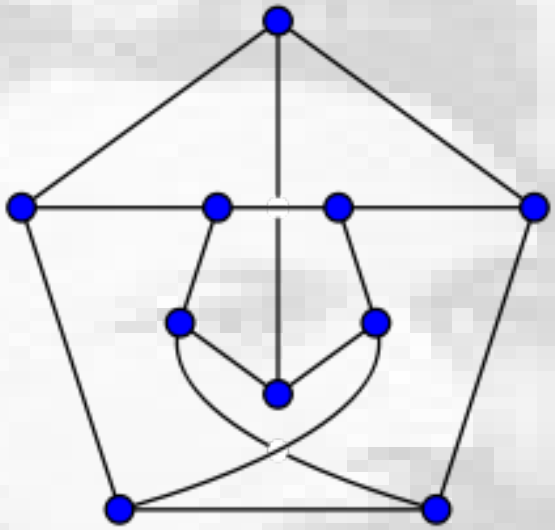

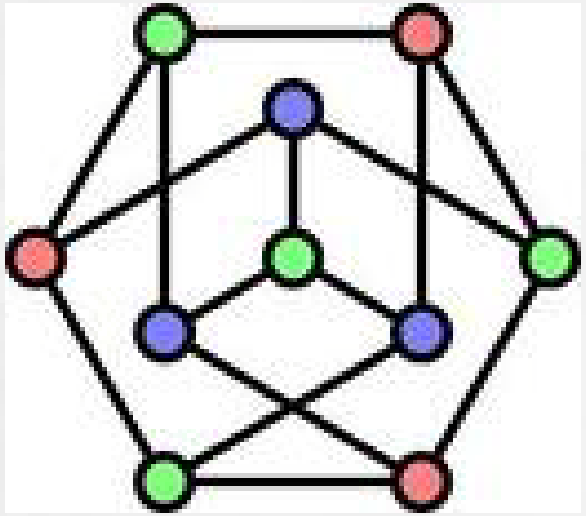

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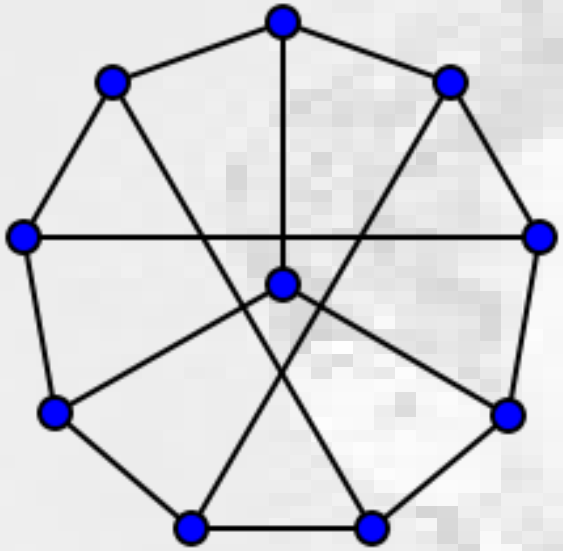

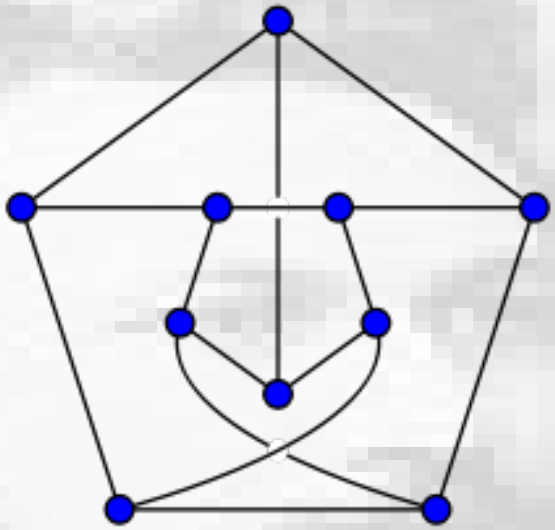

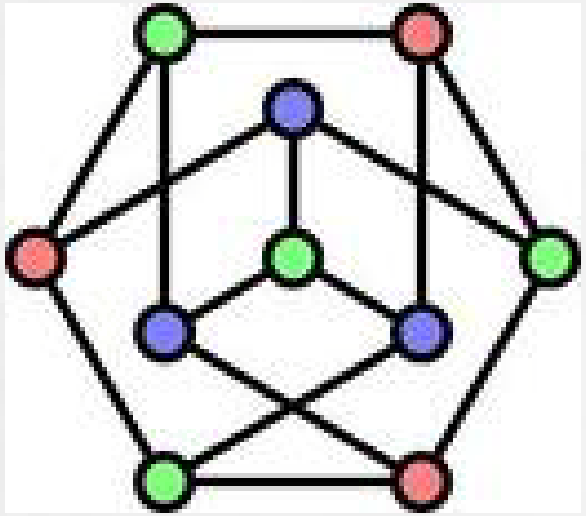

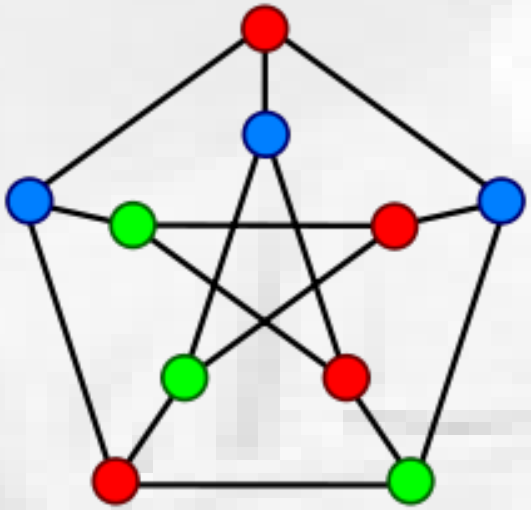

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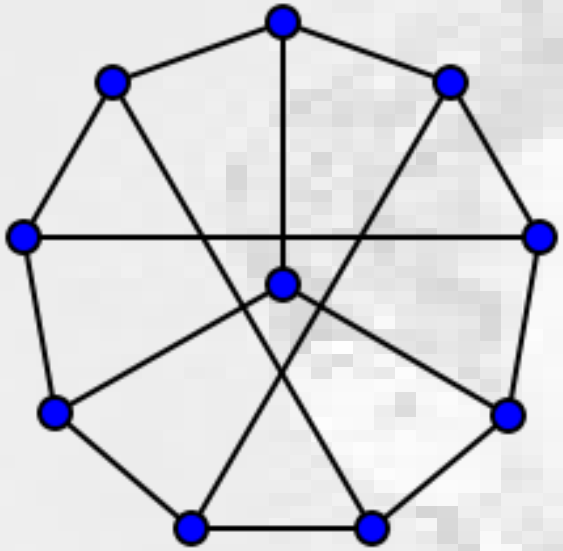

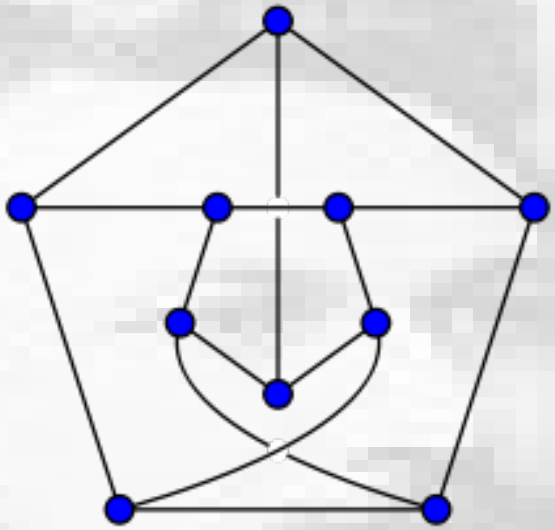

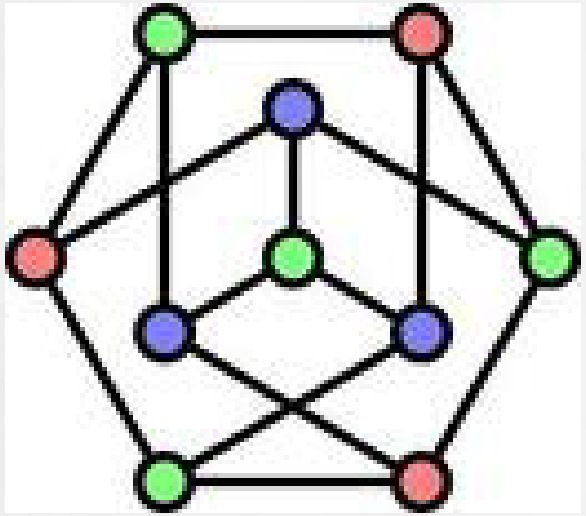

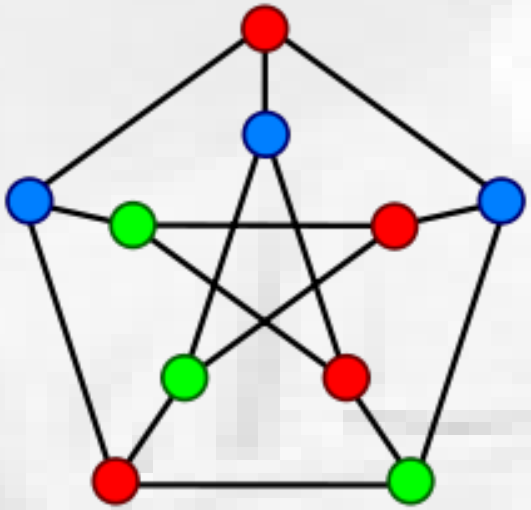

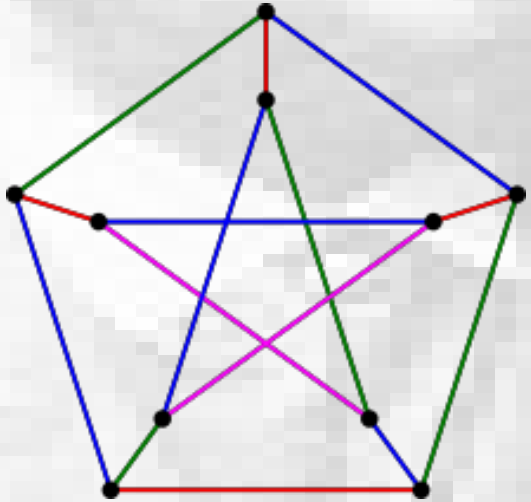

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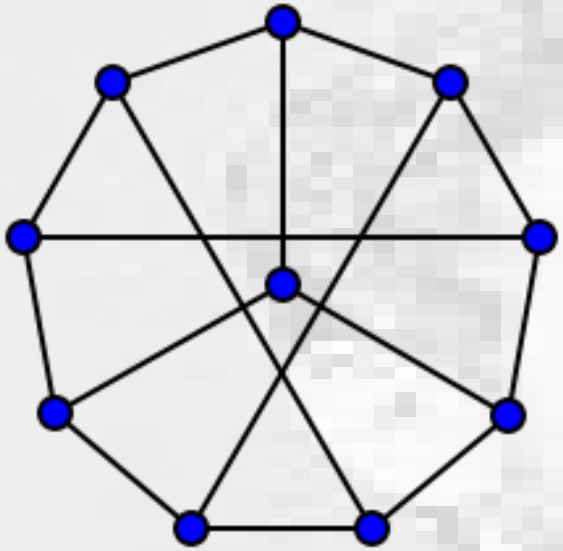
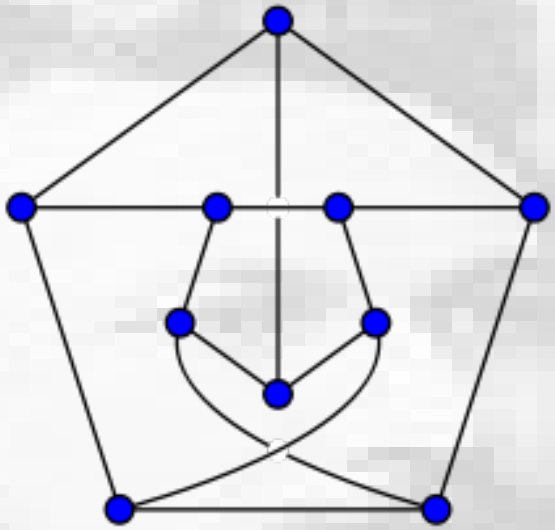
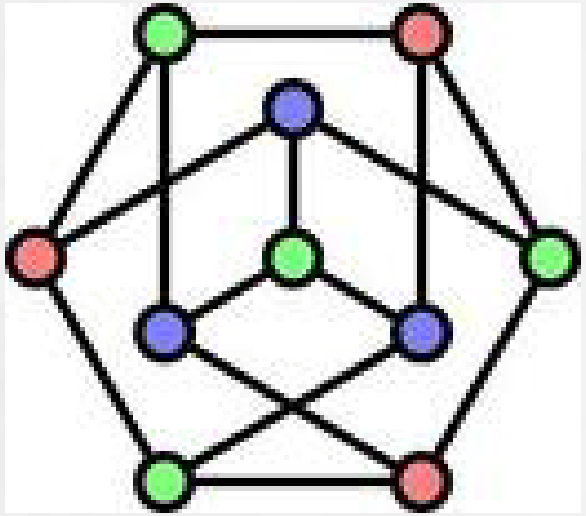
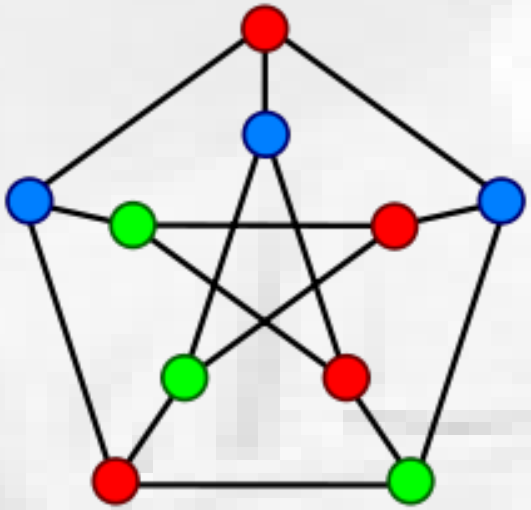
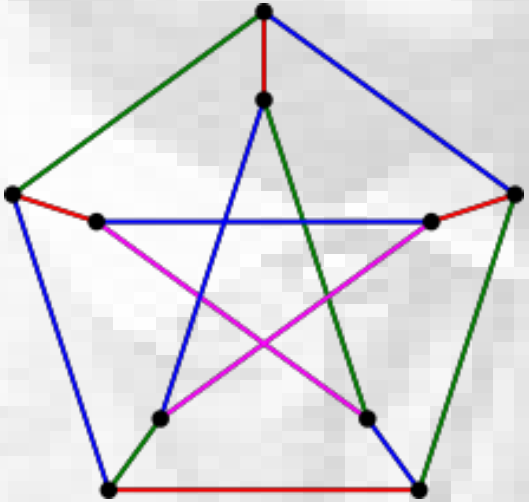
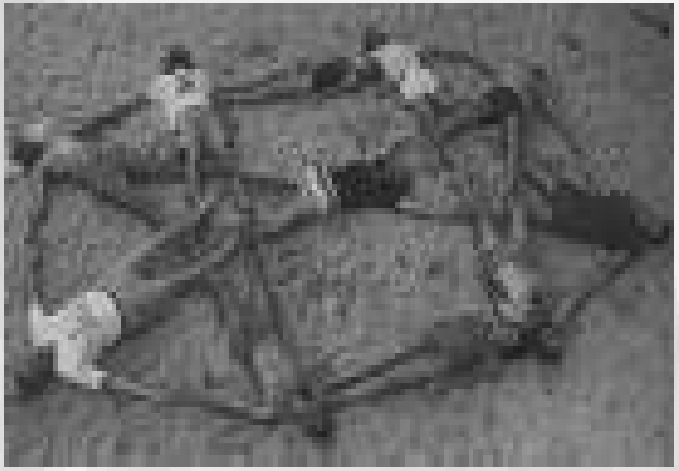




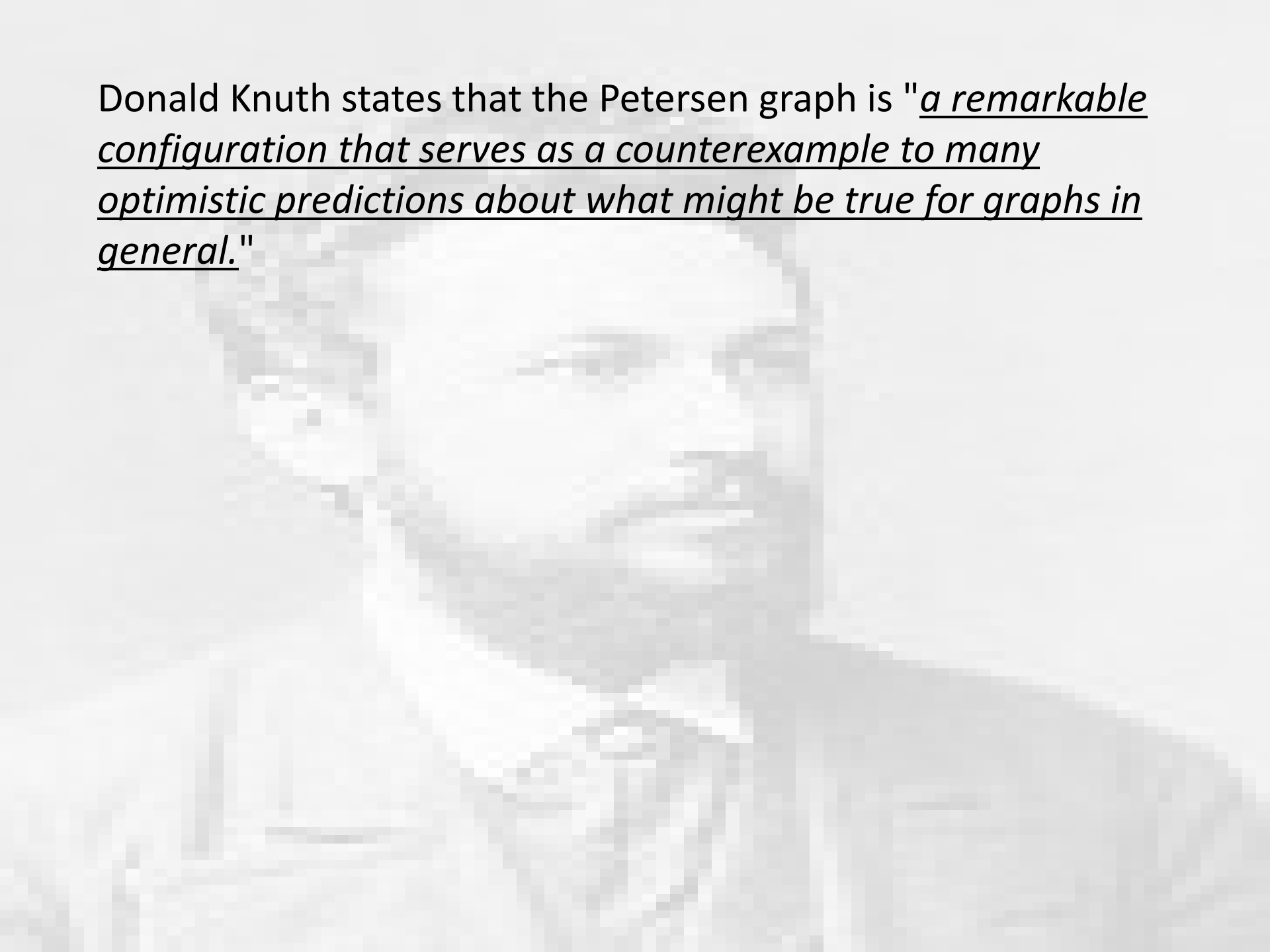








Donald Knuth states that the Petersen graph is "*a remarkable configuration that serves as a counterexample to many optimistic predictions about what might be true for graphs in general.*"



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The Petersen graph is an undirected graph with order n=10 vertices and size m=15 edges. It is a small graph that serves as a useful example and counterexample for many problems in graph theory.

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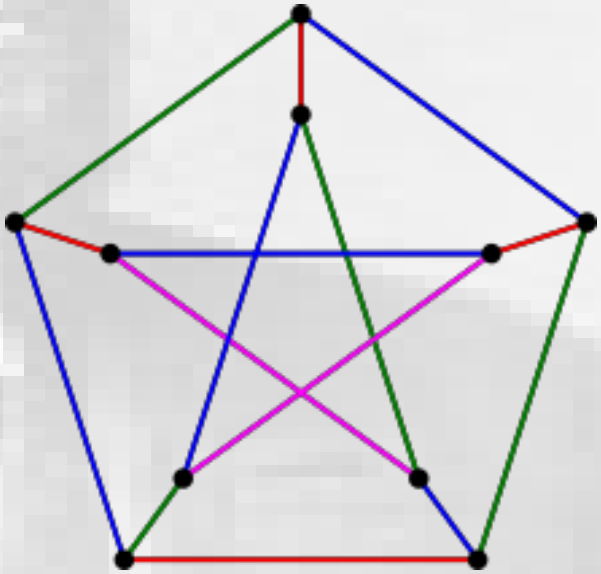
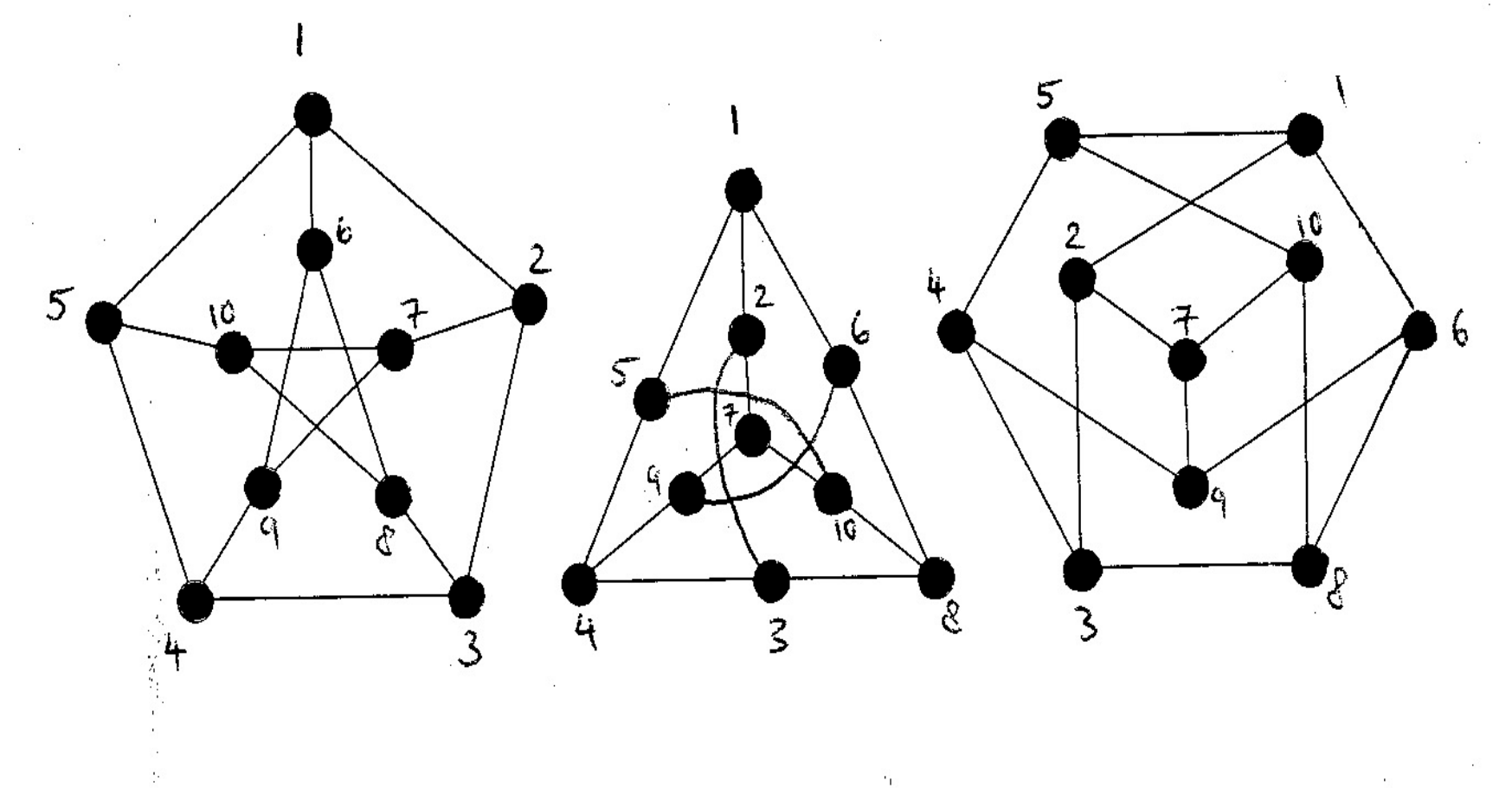
The Petersen graph is an undirected graph with order n=10 vertices and size m=15 edges. It is a small graph that serves as a useful example and counterexample for many problems in graph theory.

The Petersen graph is the smallest bridgeless cubic graph with no three-edge-coloring. It has girth 5.



The Colorful Life of Petersen



The Colorful Life of Petersen

- 1839 Birth

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- 1839 Birth
- 1871 Doctoral Dissertation

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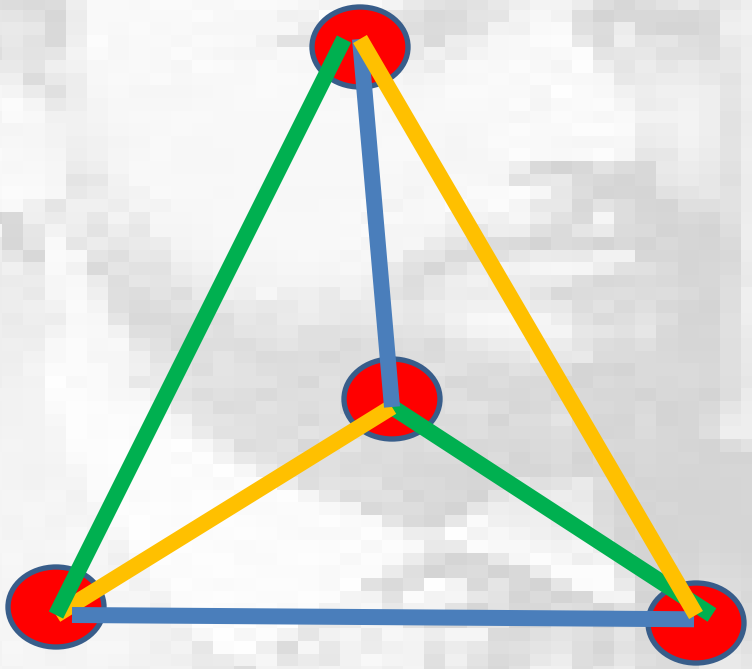
- 1884 Tait’s loss becomes J.P.’s gain

- In proving the 4-Color Conjecture, Tait assumes the statement: "*Every cubic graph is 1-factorable*".



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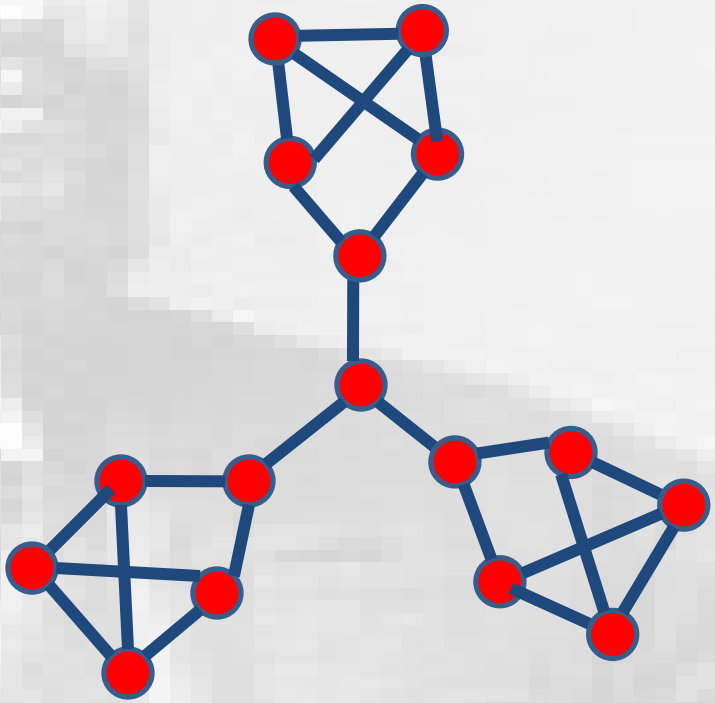

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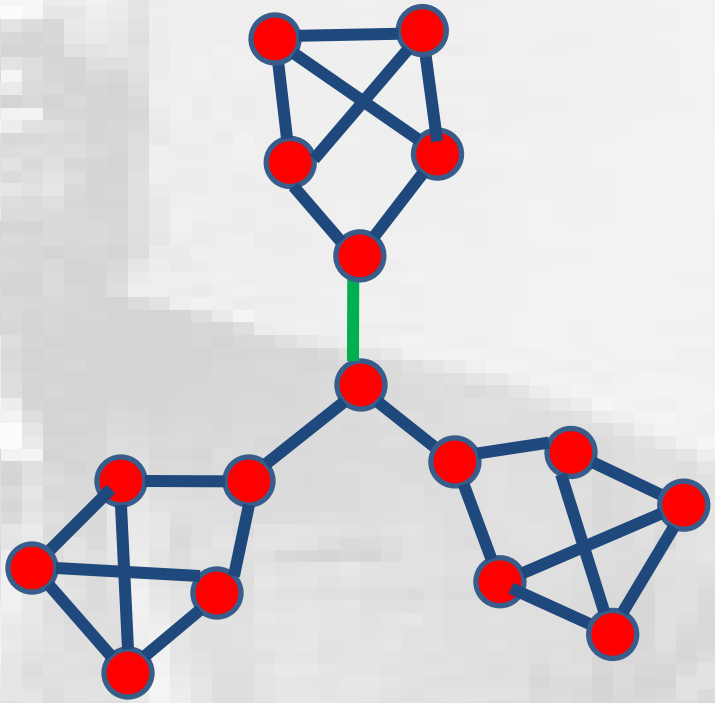

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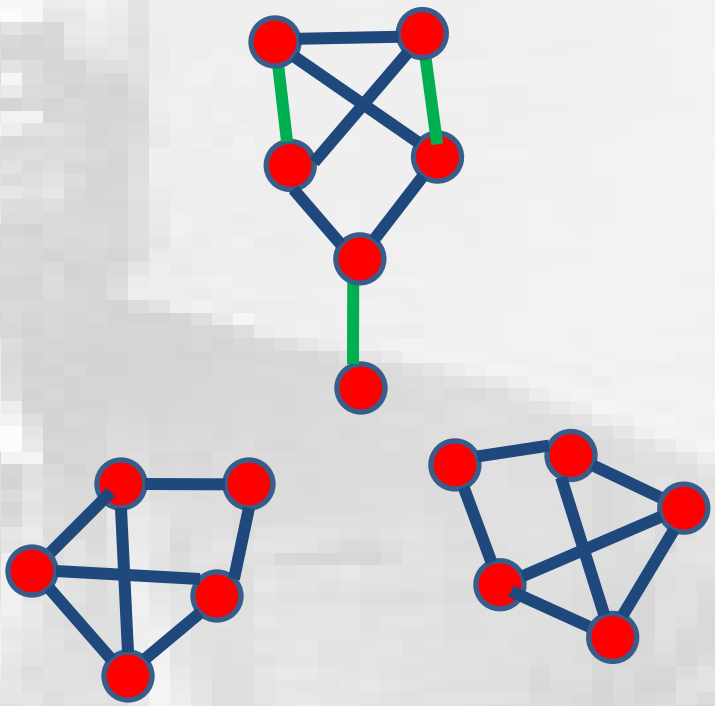
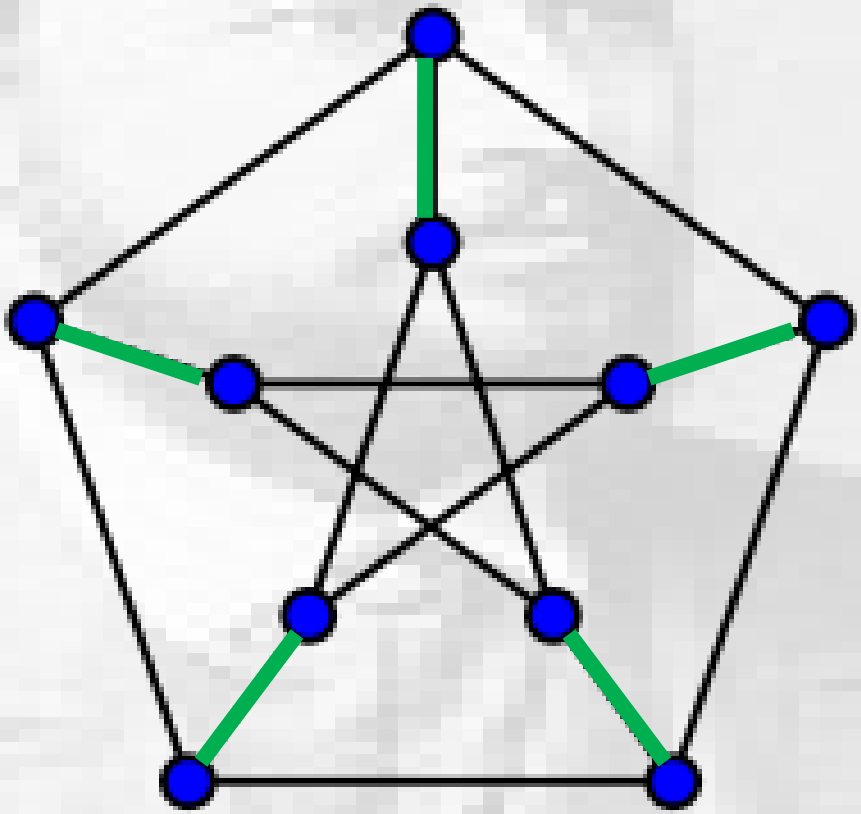


- Indeed, J.P. proves the following results which is now known as the Petersen Theorem:

  Every bridgeless cubic graph has a **1-factor** and a **2-factor**.



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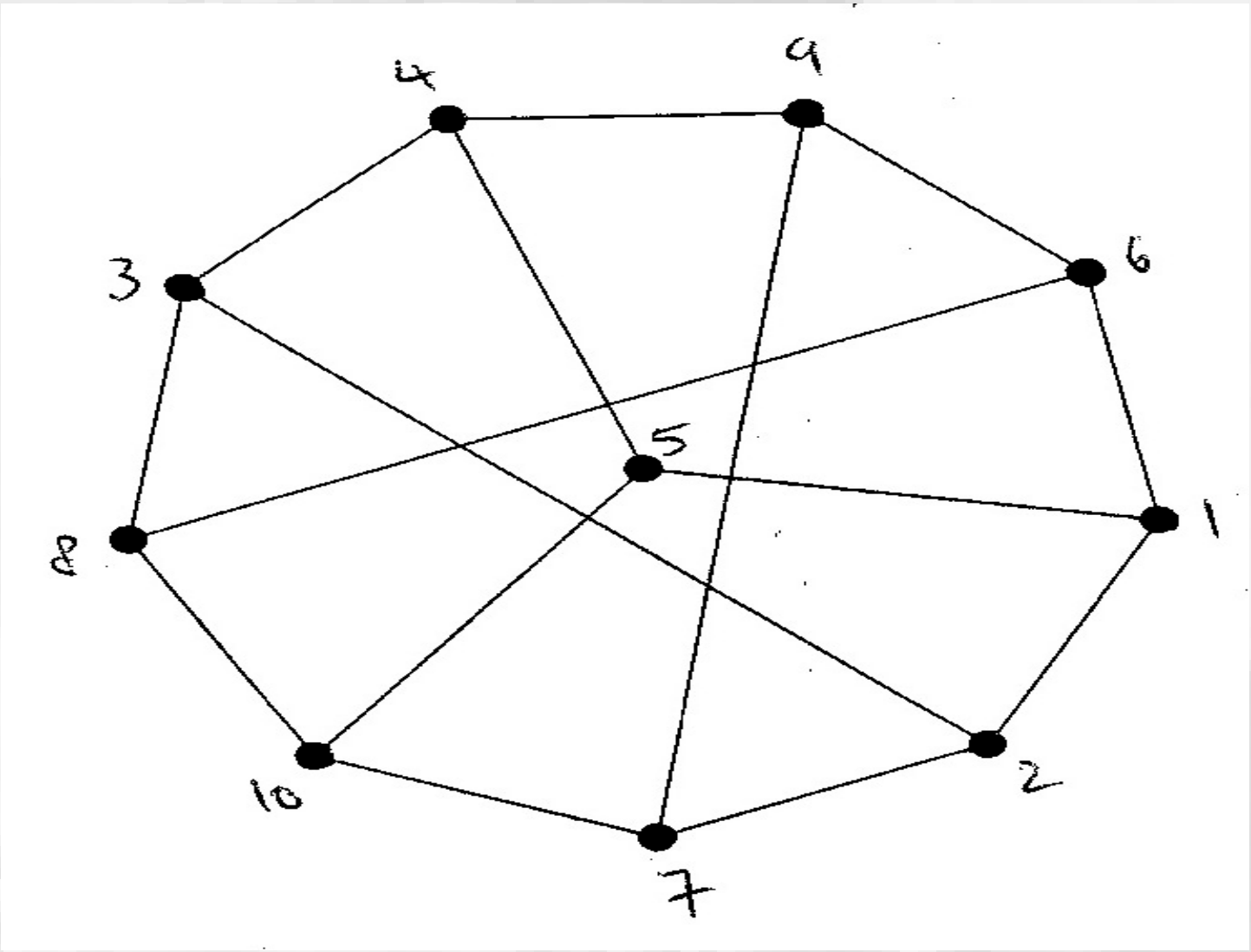

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- 1884 Tait’s loss becomes J.P.’s gain
- 1886 Kempe’s drawing of Petersen Graph
- 1889 Sylvester meets J.P. on Cayley’s error

The Colorful Life of Petersen

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The Colorful Life of Petersen

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- 1891 First major paper in Graph Theory

The Colorful Life of Petersen

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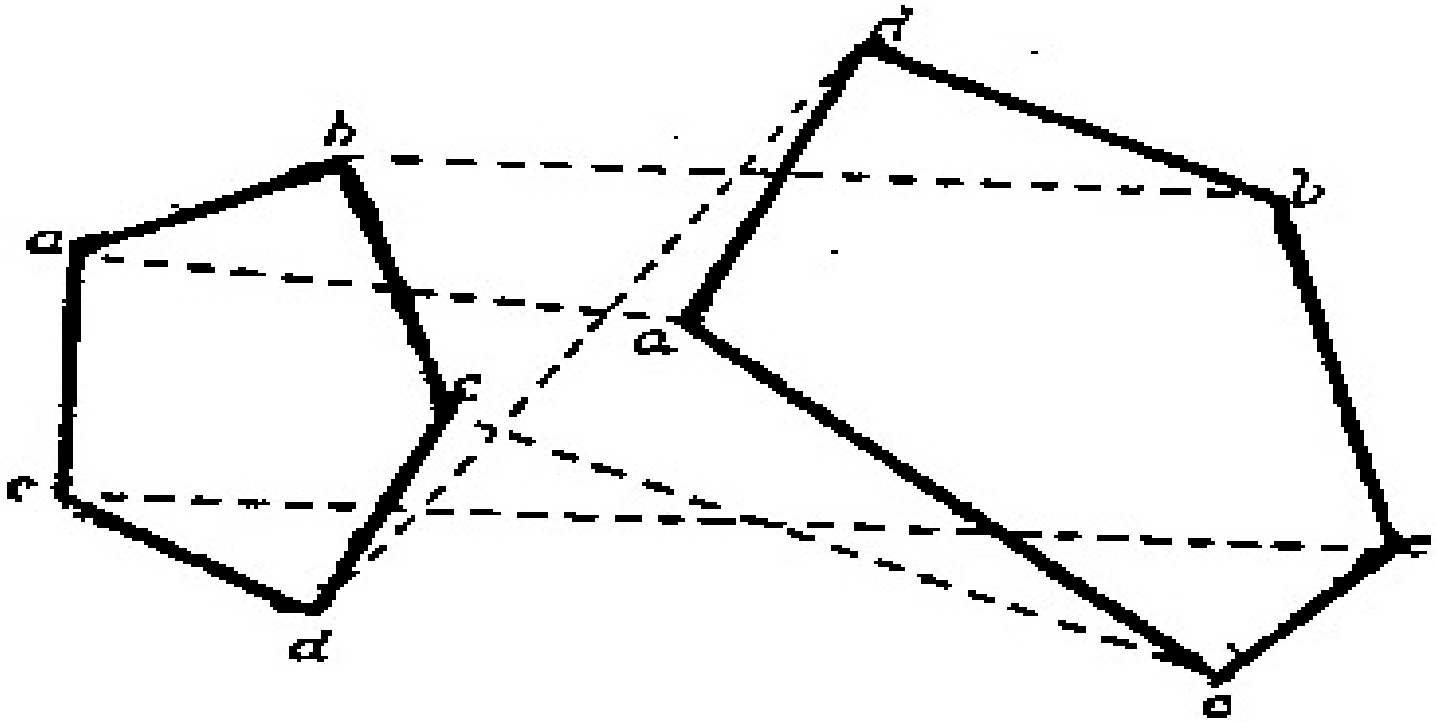
- 1898 Petersen debuts his "Baby"

The Colorful Life of Petersen



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- 1910 Death

The Colorful Life of Petersen

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The Colorful Life of Petersen

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The Colorful Life of Petersen

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The Colorful Life of Petersen

- 1936 Denes Konig discovery of 1891
- 1936 First Graph Theory text arrives
- 1990 A centennial gift for J.P. in Denmark
- 2003 Many equivalences of Petersen Graph are discovered

The Colorful Petersen Graph

An Unforgettable Construction for P

- Consider S={1,2,3,4,5}
- Let V(P)=2-element subsets of S
- Let E(P)=disjoint 2-subsets of S


{1,2}


{1,2}
{2,3}


{1,2}
{3,4}
{2,3}


{1,2}
{4,5}
{3,4}
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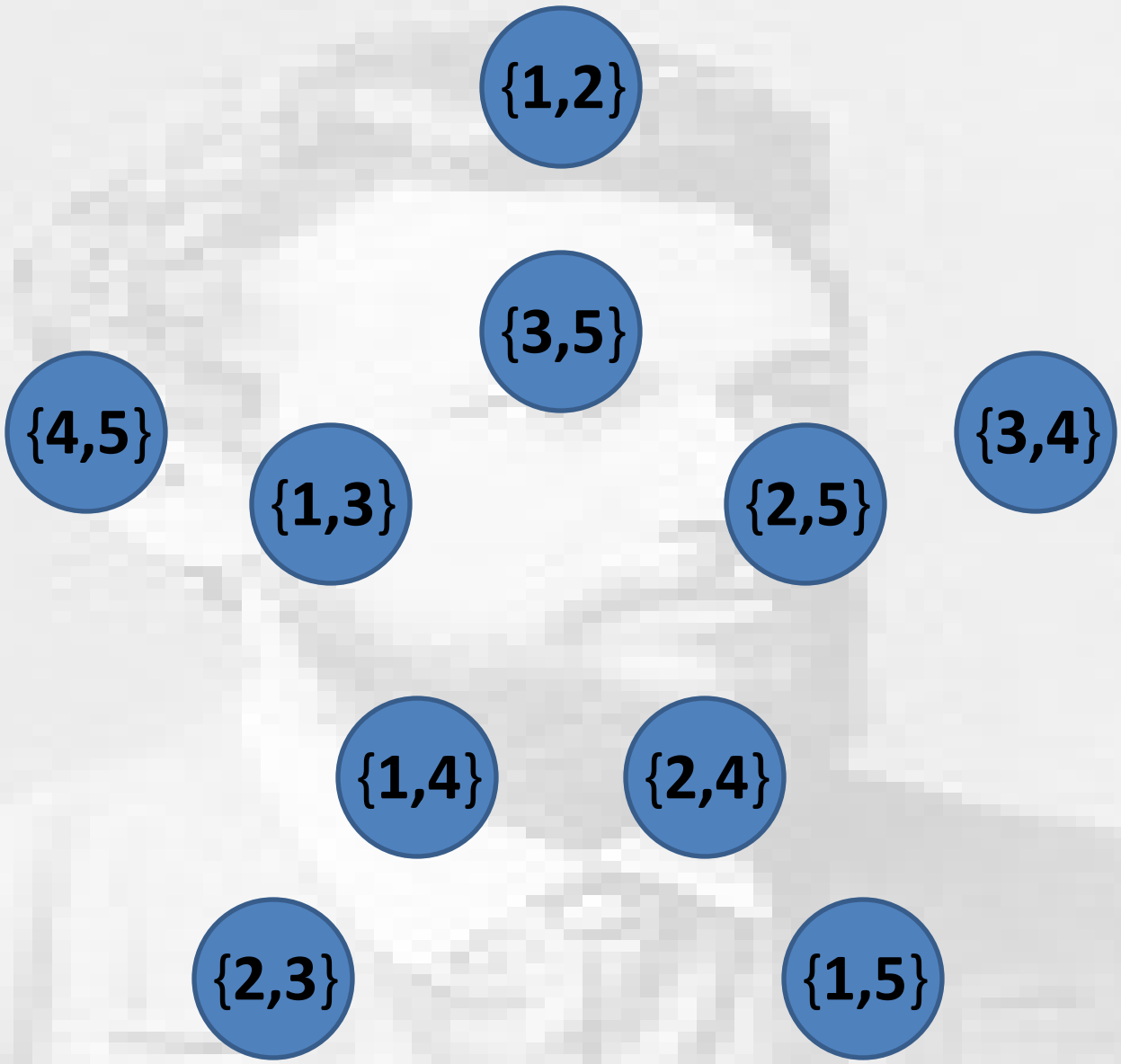
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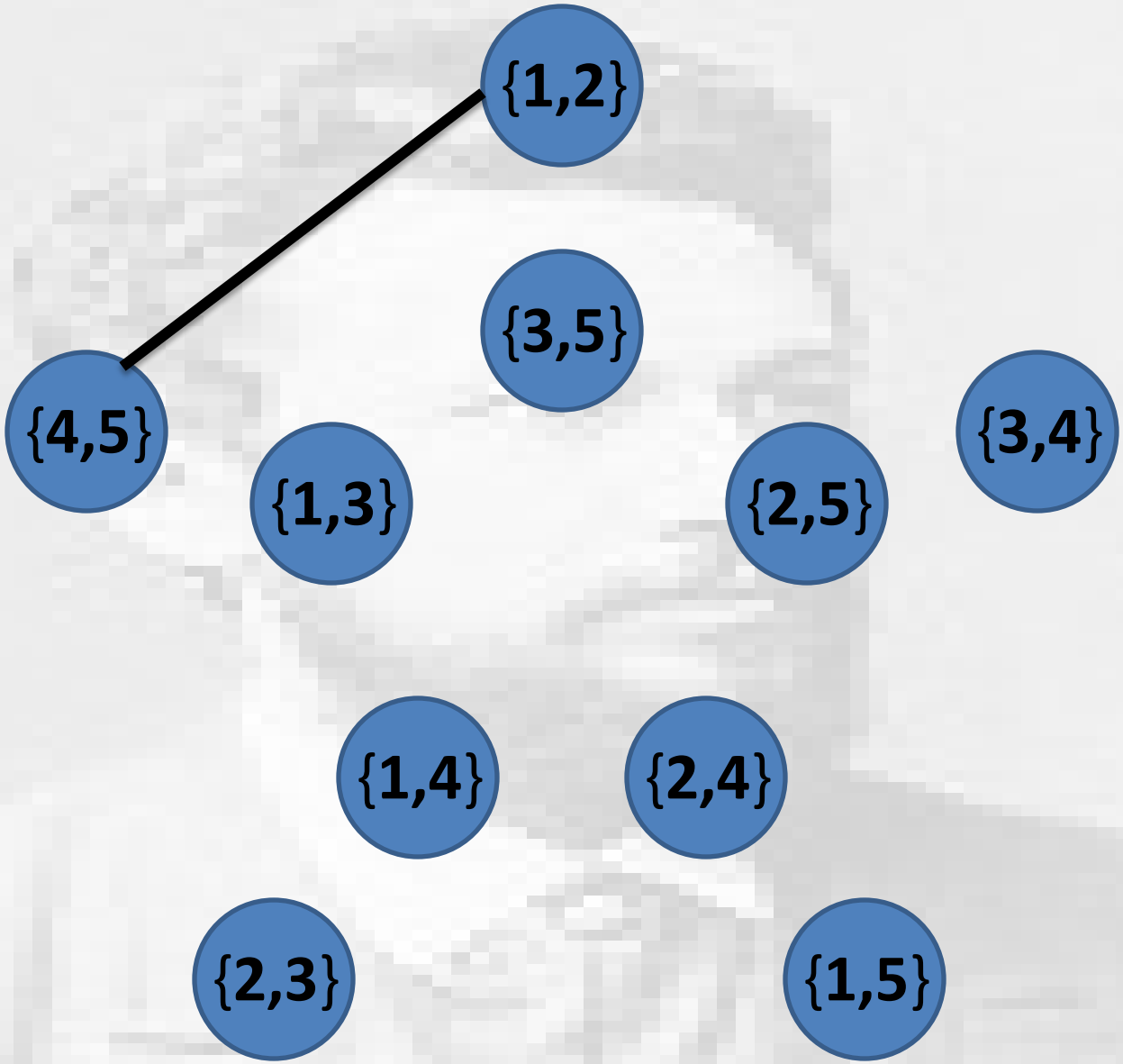
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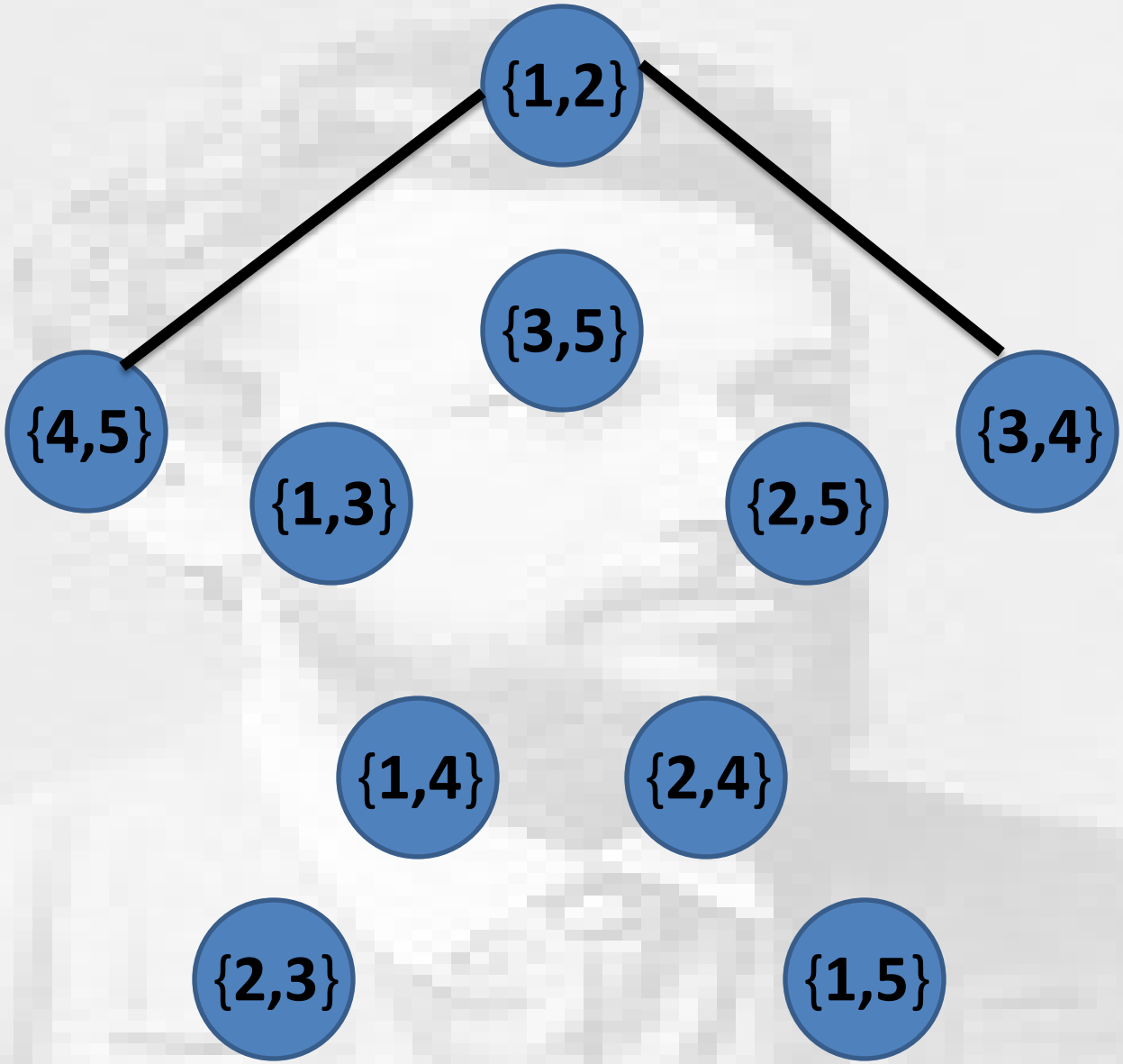
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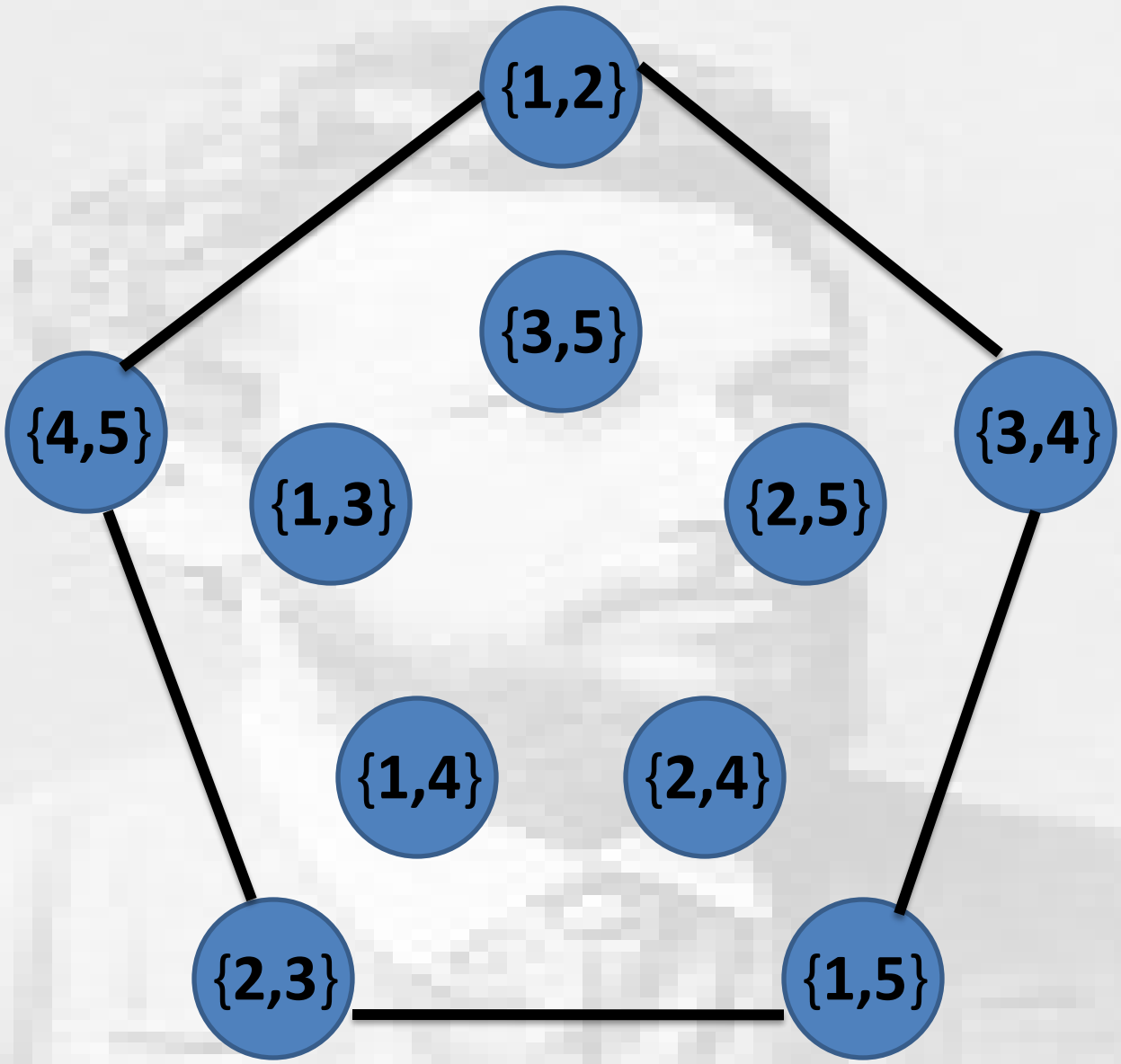
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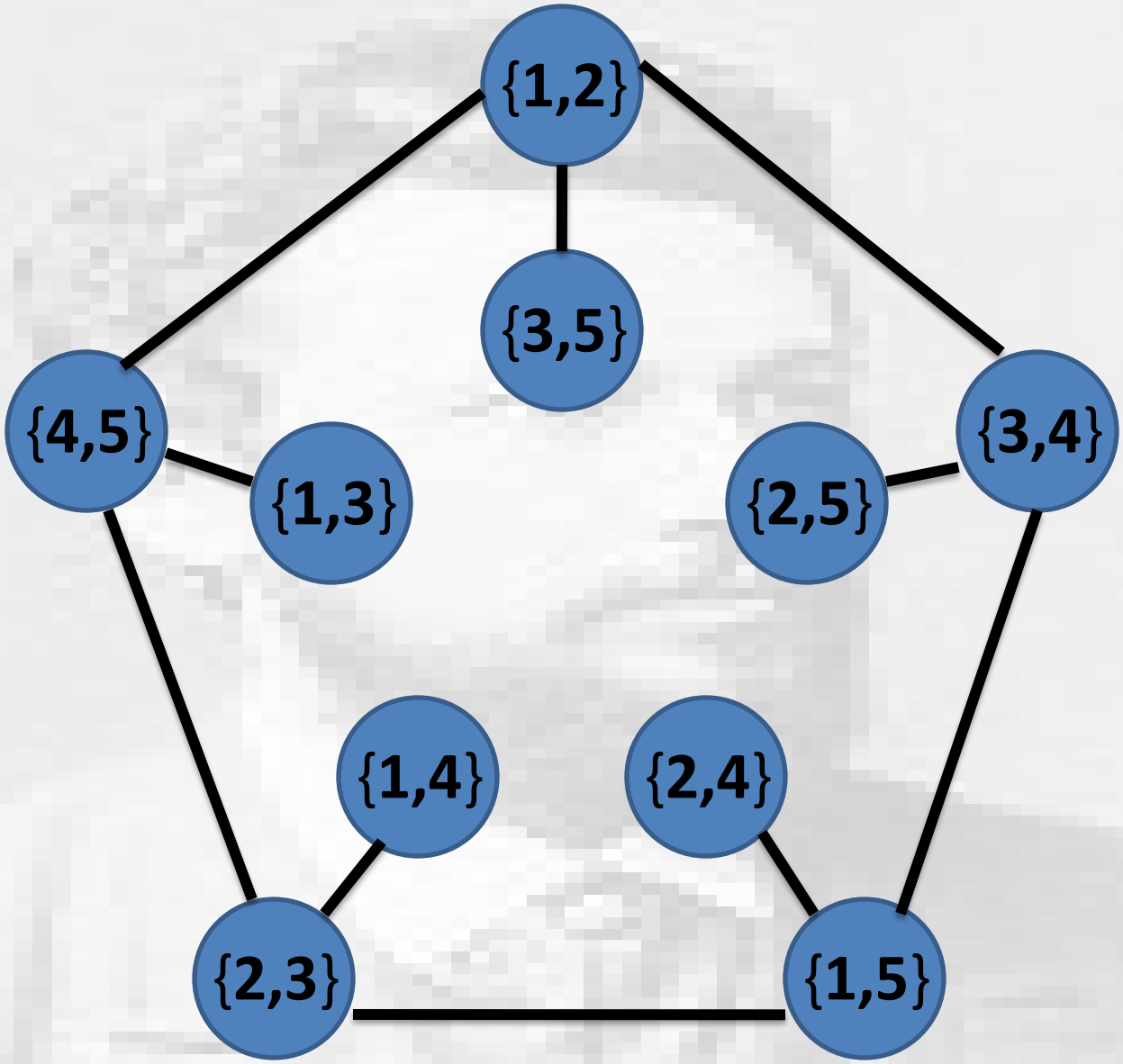
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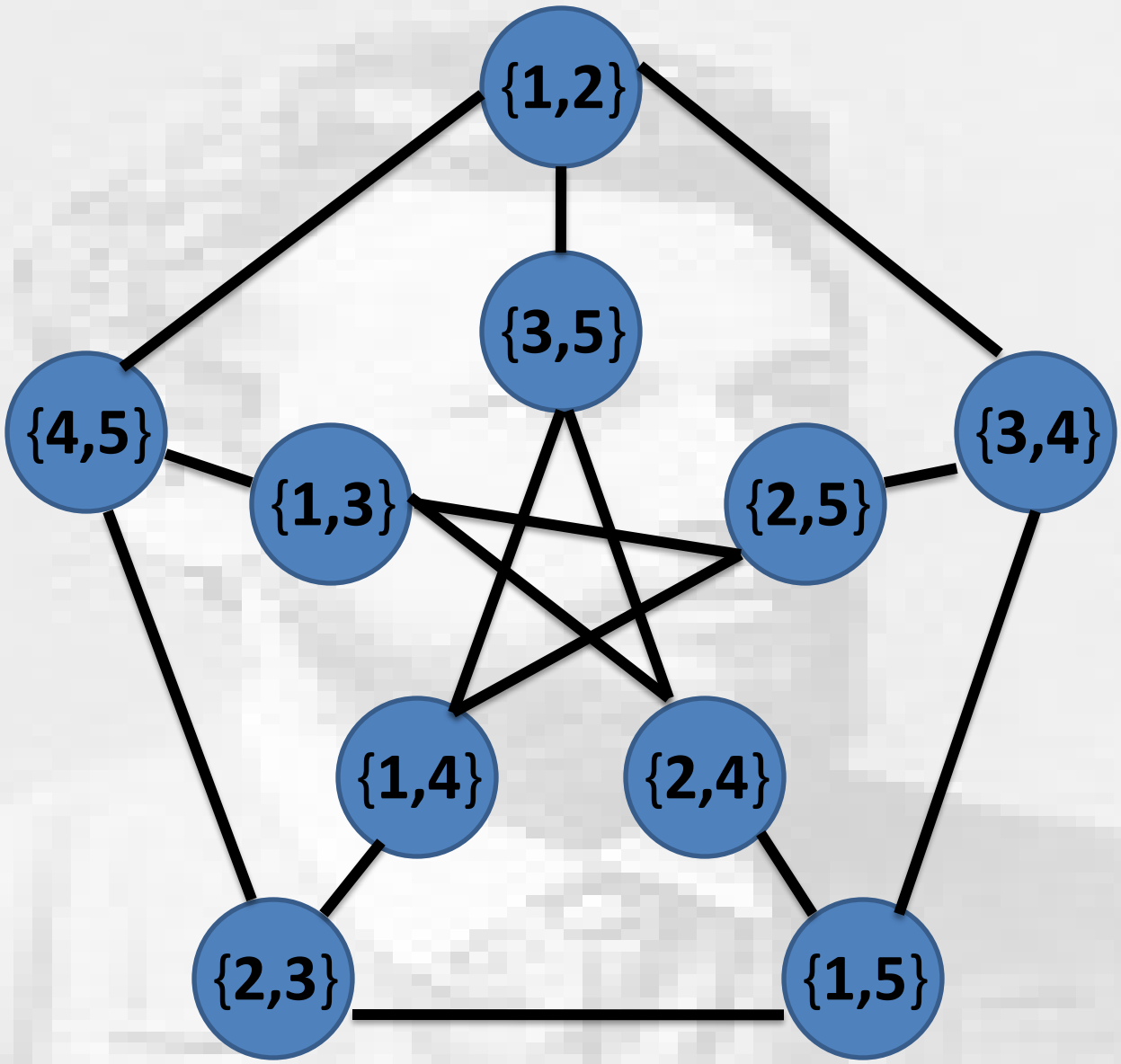
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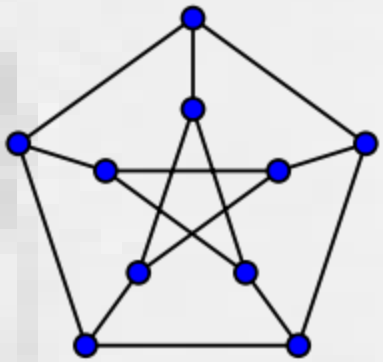
The Colorful Petersen Graph

- P is Non-bipartite

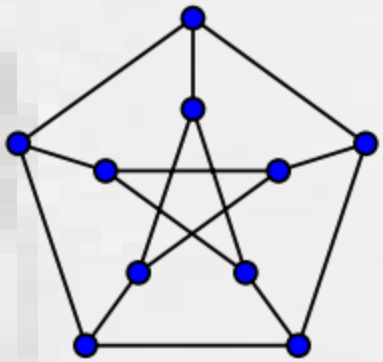
The Colorful Petersen Graph

- P is Non-bipartite
- P is Vertex transitive



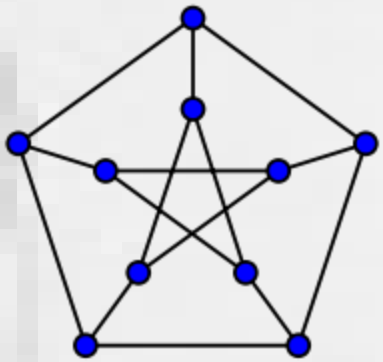
The Colorful Petersen Graph

- P is Non-bipartite
- P is Vertex transitive
- P is Edge Transitive



The Colorful Petersen Graph

- P is Non-bipartite
- P is Vertex transitive
- P is Edge Transitive
- P has n-gons for n=5,6,8,9 (12 C_5)



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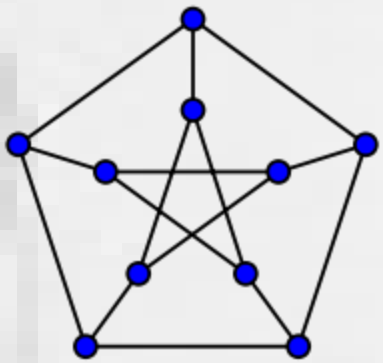

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- P has n-gons for n=5,6,8,9 (12 C_5)
- Aut(P) has order 120

The Colorful Petersen Graph

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The Colorful Petersen Graph

- P is Non-bipartite
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- P is Edge Transitive
- P has n-gons for n=5,6,8,9 (12 C_5)
- Aut(P) has order 120
- Eigenvalues of P are 3,-2,-2,-2,-2,1,1,1,1,1
- P is Hypo-Hamiltonian, *not* Hamiltonian and *not* Eulerian.

The Colorful Petersen Graph

- P is a nice “cage” for all crazy graph theorists.

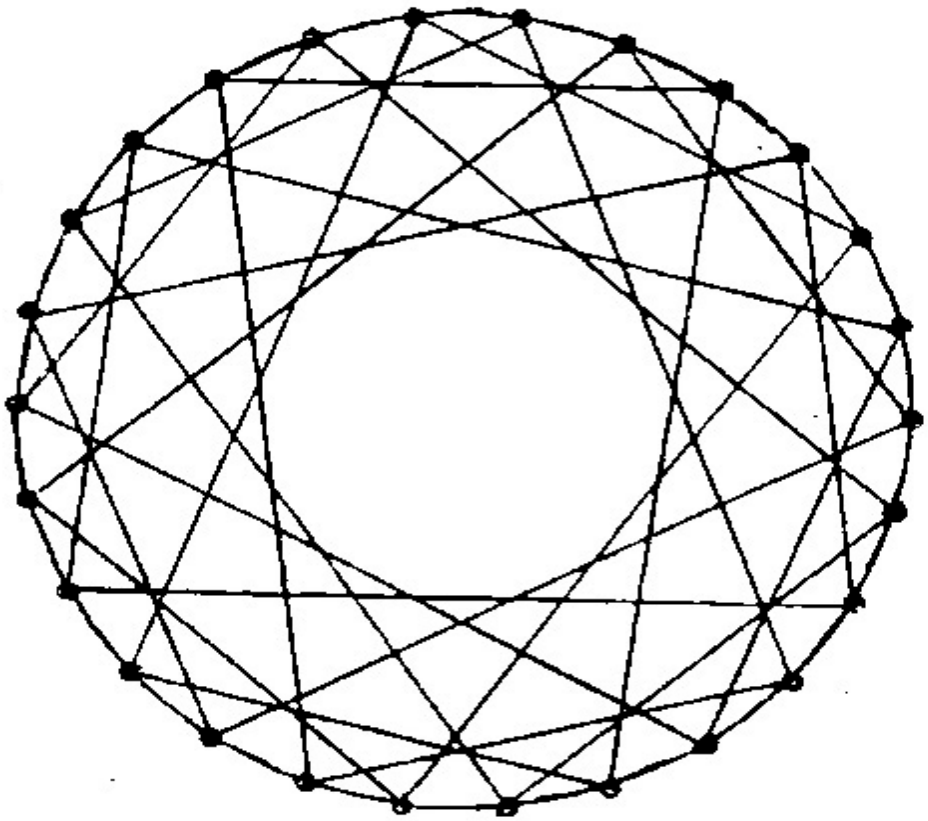
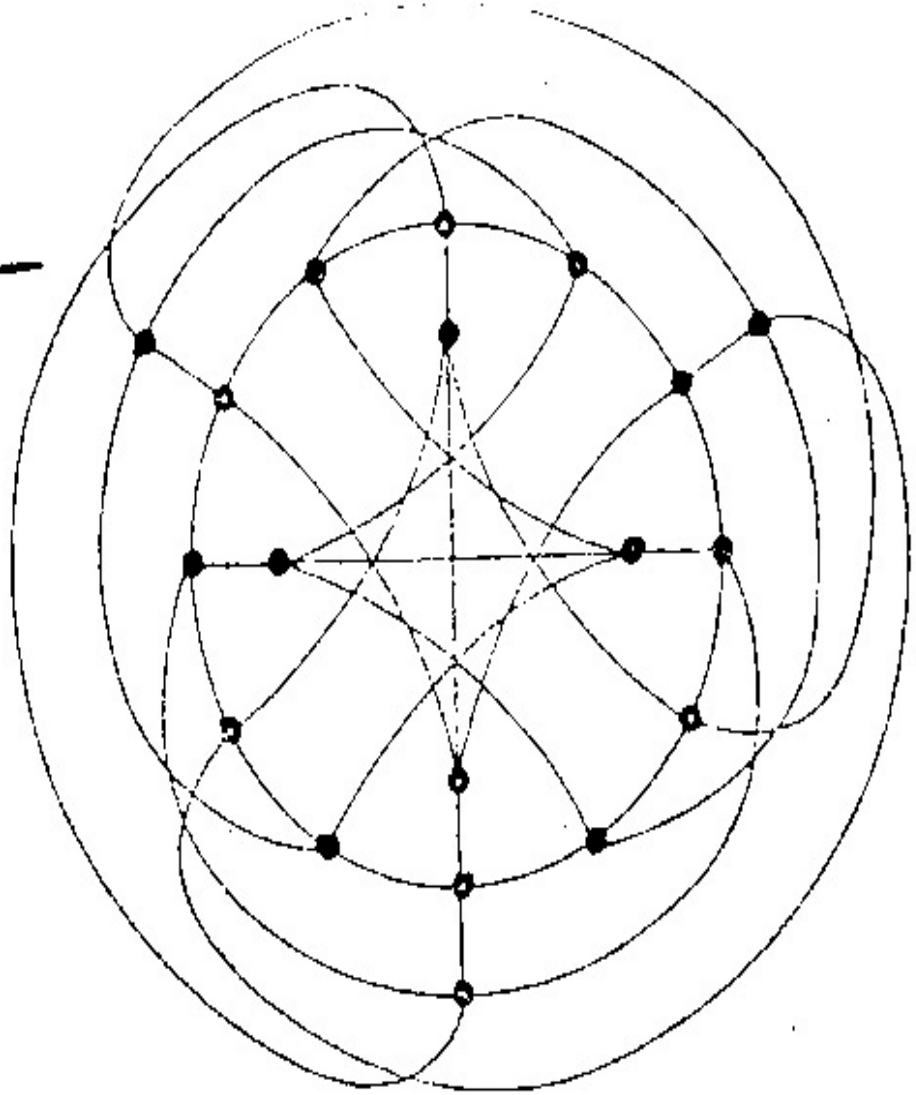


Figure 3.5 The (4, 6)-cage

Biggs' unique (4,6) Cage

$f(4,6) = \underline{26}$

Robertson' ←
unique
(4,5) Cage
$f(4,5) = 19$



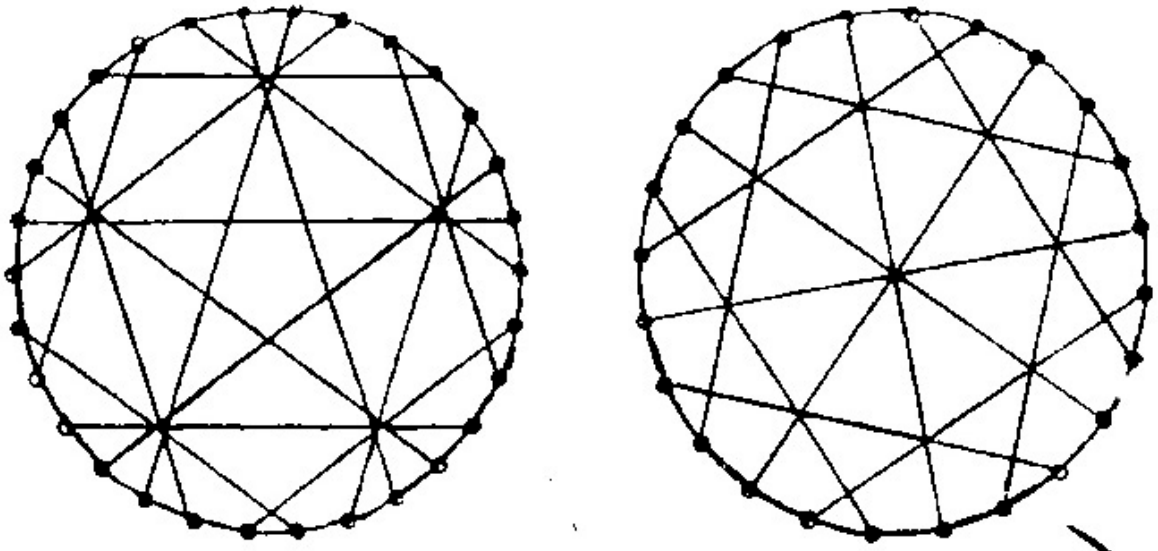


Figure 3.9 *The 7-cage and 8-cage*

Tutte-Coxeter's
Unique (3,8) Cage
$f(3,8) = 30$

McGee's
Unique (3,7) Cage
$f(3,7) = 24$

(5,5) Cage

$f(5,5) = \underline{30}$

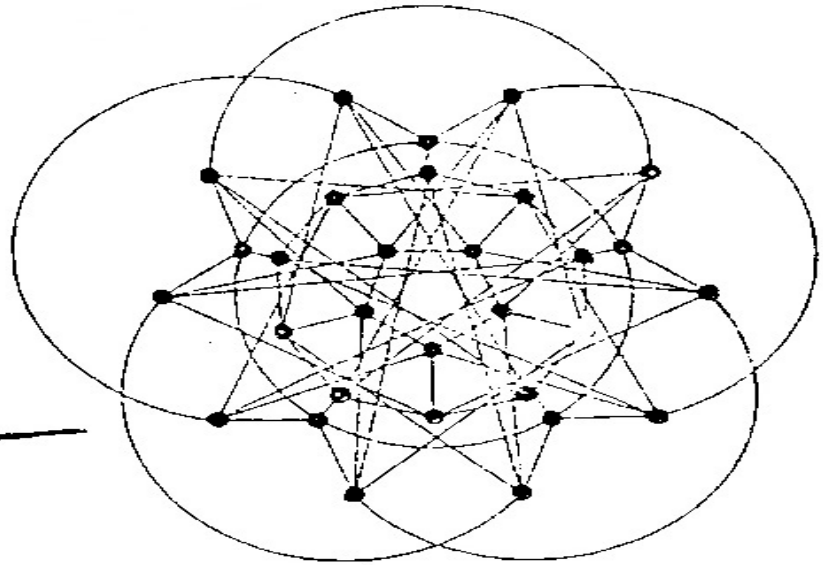


Figure 3.6 *The (4, 5)-cage and (5, 5)-cage*

$|V| = 11$

A $\{3,4\}$-regular graph with girth 4.

Is it a $(3,4;4)$ Cage !?

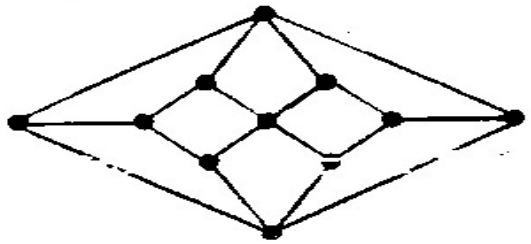


Figure 7.9 *The Herschel graph*

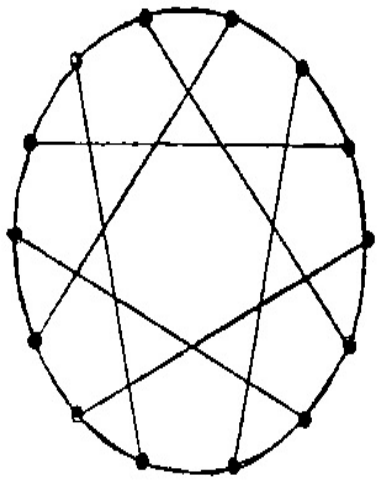


Figure 3.8 *The Heawood graph: the unique 6-cage*

↓

Heawood's unique (3,6) Cage

$f(3,6) = 14$

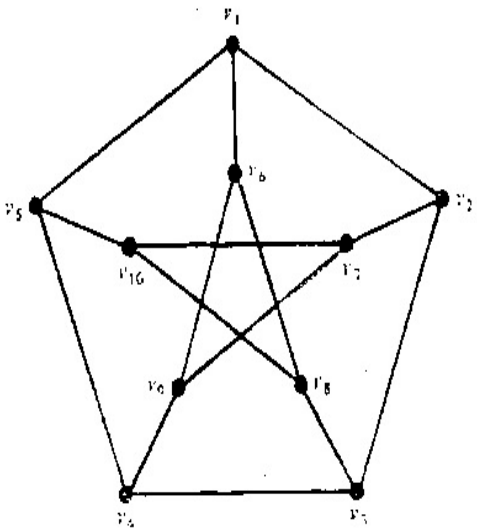


Figure 3.7 *The Petersen graph*

↓

Petersen's unique (3,5) Cage

$f(3,5) = 10$

G:

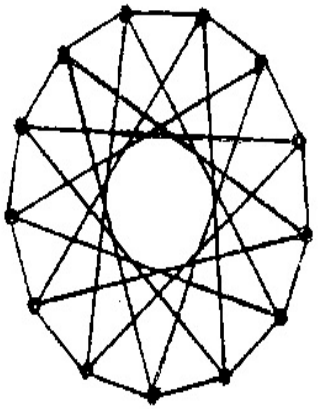


Figure 12.1 *An extremal graph showing* $r(3,5) \geq 14$

↓

A 4-regular graph with girth = ?, $|V| = 13$

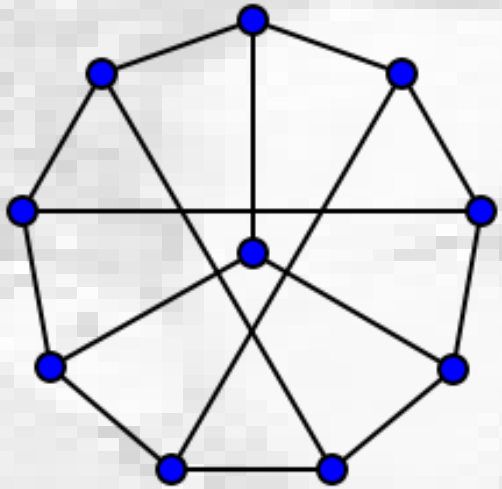
Is this a (4, ?)

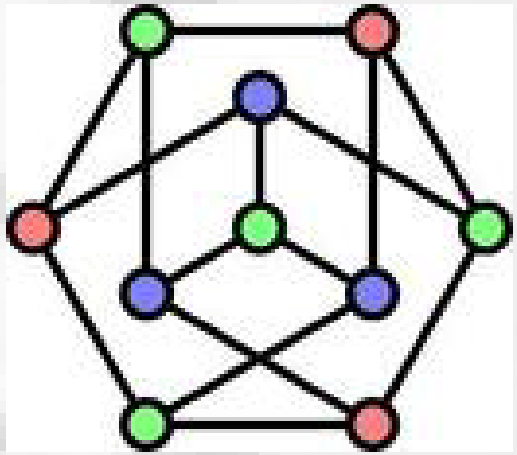
The Colorful Petersen Graph

- P is a nice “cage” for all crazy graph theorists.
- P is non-planar. However, on the Projective Plane, P shares a bed with Dodecahedron.

The Colorful Petersen Graph

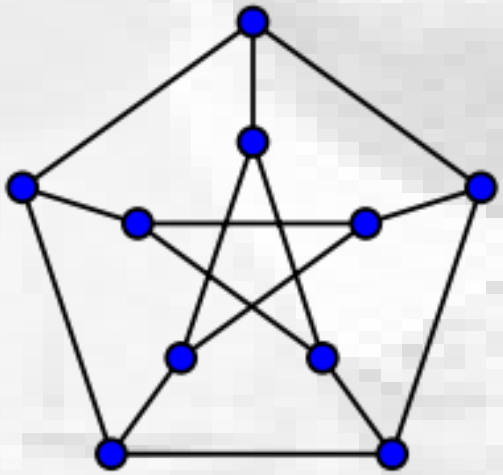
6 crossings

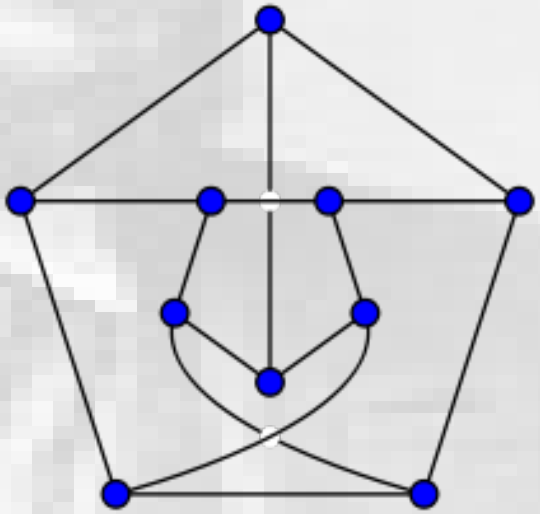




3 crossings

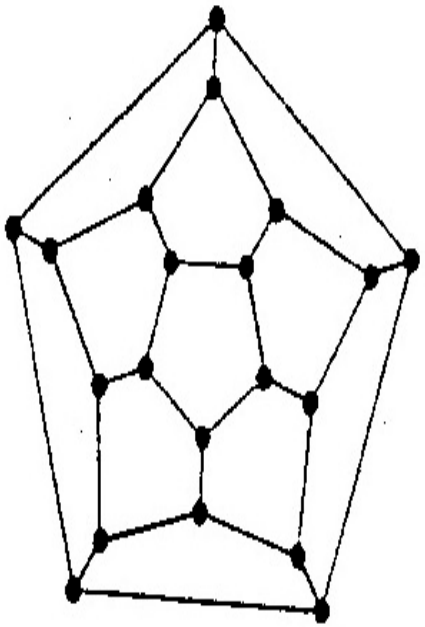
5 crossings

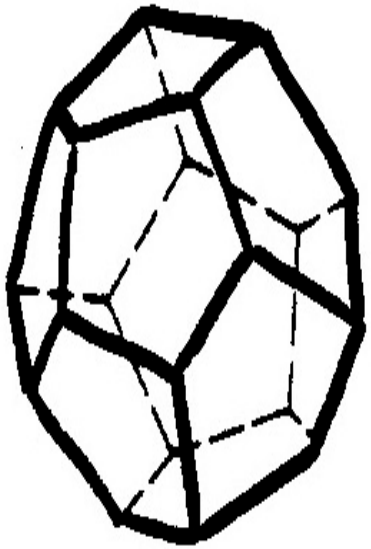




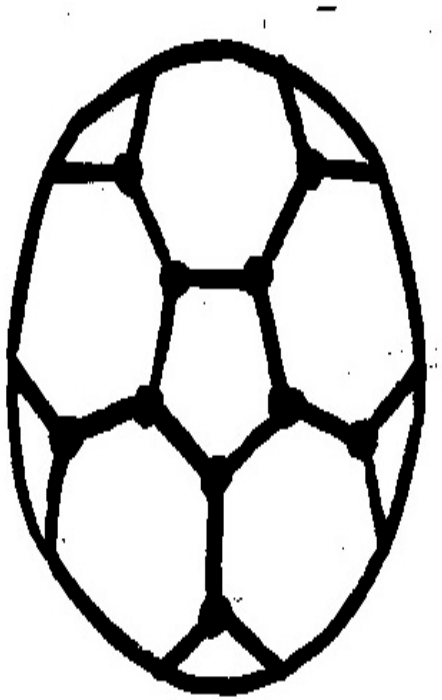
2 crossings

The Colorful Petersen Graph





Dodecahedron



Dodecahedron embedded on the Projective Plane

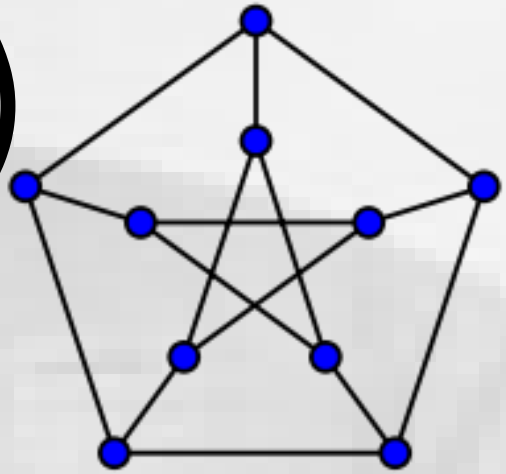
The Colorful Petersen Graph

- P is a nice “cage” for all crazy graph theorists.
- P is non-planar. However, on the Projective Plane, P shares a bed with Dodecahedron.
- P is a “Strongly Regular Graph” in Town.

The Colorful Petersen Graph

- P is a nice “cage” for all crazy graph theorists.
- P is non-planar. However, on the Projective Plane, P shares a bed with Dodecahedron.
- P is a “Strongly Regular Graph” in Town.

$$P = SRG(10,3,0,1)$$



The Colorful Petersen Graph

- P is a nice “cage” for all crazy graph theorists.
- P is non-planar. However, on the Projective Plane, P shares a bed with Dodecahedron.
- P is a “Strongly Regular Graph” in Town.
- P is not a “Line Graph”.

The Colorful Petersen Graph

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- P is non-planar. However, on the Projective Plane, P shares a bed with Dodecahedron.
- P is a “Strongly Regular Graph” in Town.
- P is not a “Line Graph”.
- P is “Path Perfect”.

The Colorful Petersen Graph

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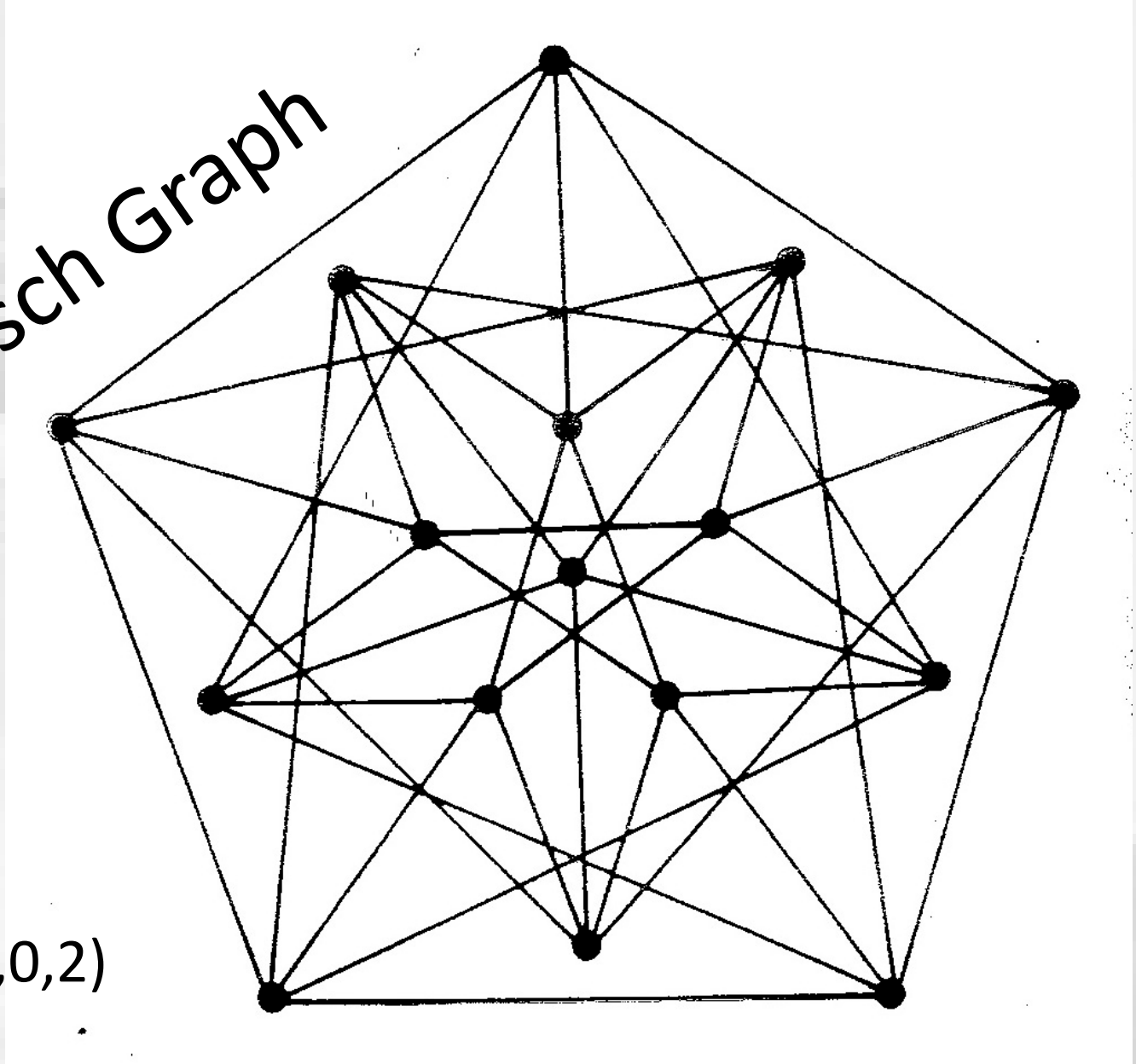
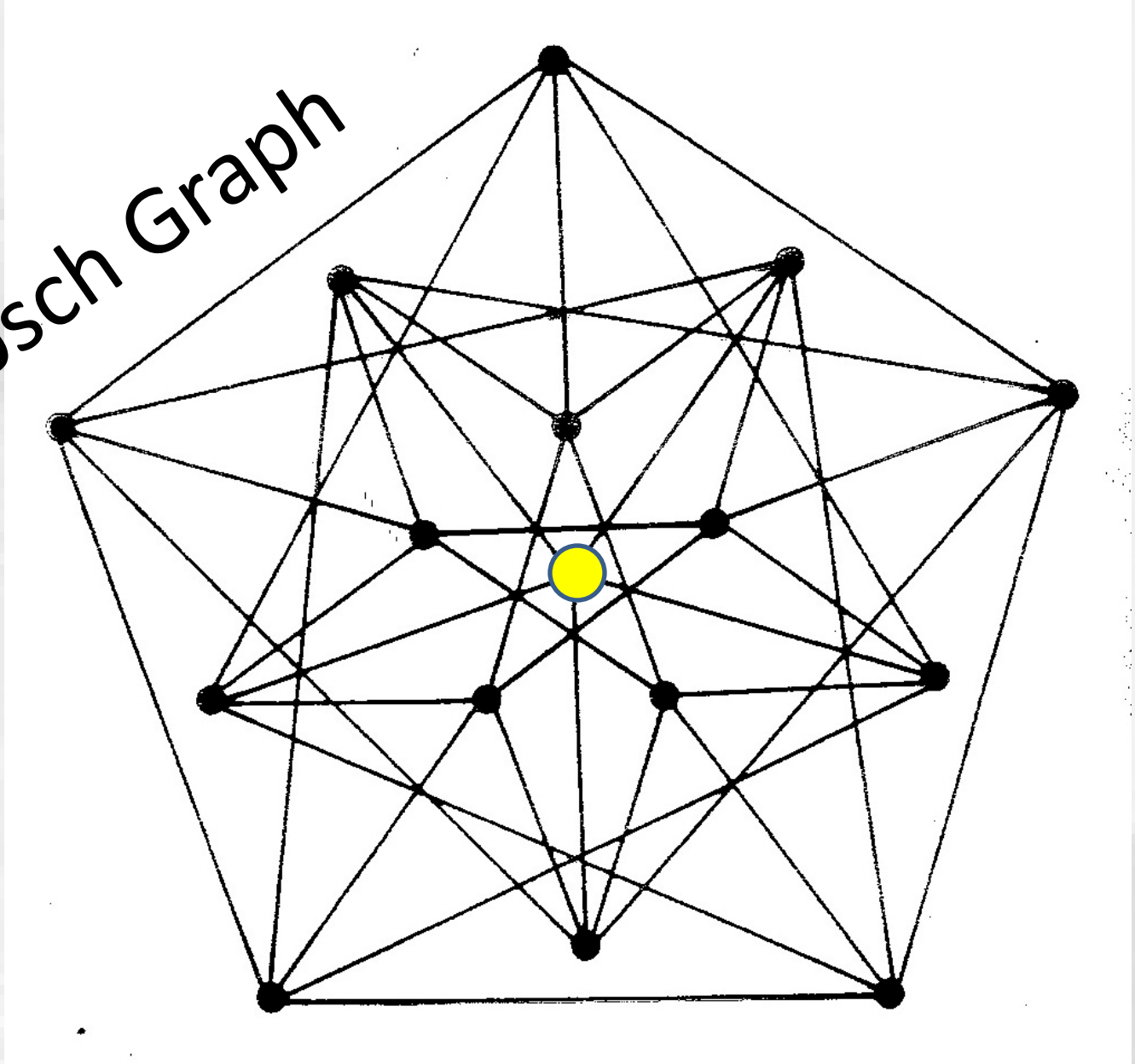
- P is non-planar. However, on the Projective Plane, P shares a bed with Dodecahedron.
- P is a “Strongly Regular Graph” in Town.
- P is not a “Line Graph”.
- P is “Path Perfect”.
- P is an induced subgraph of Clebsch Graph.

Clebsch Graph

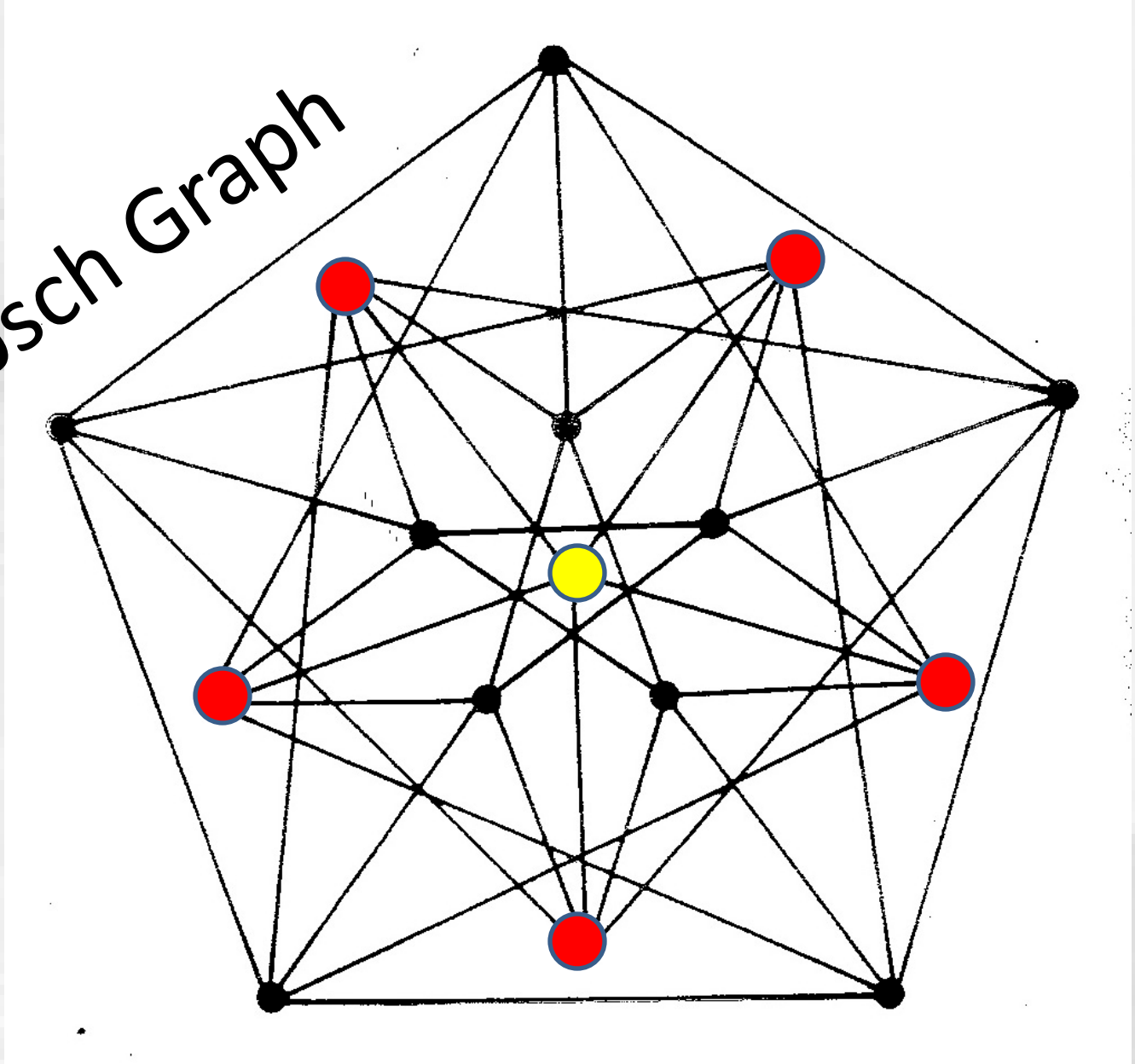


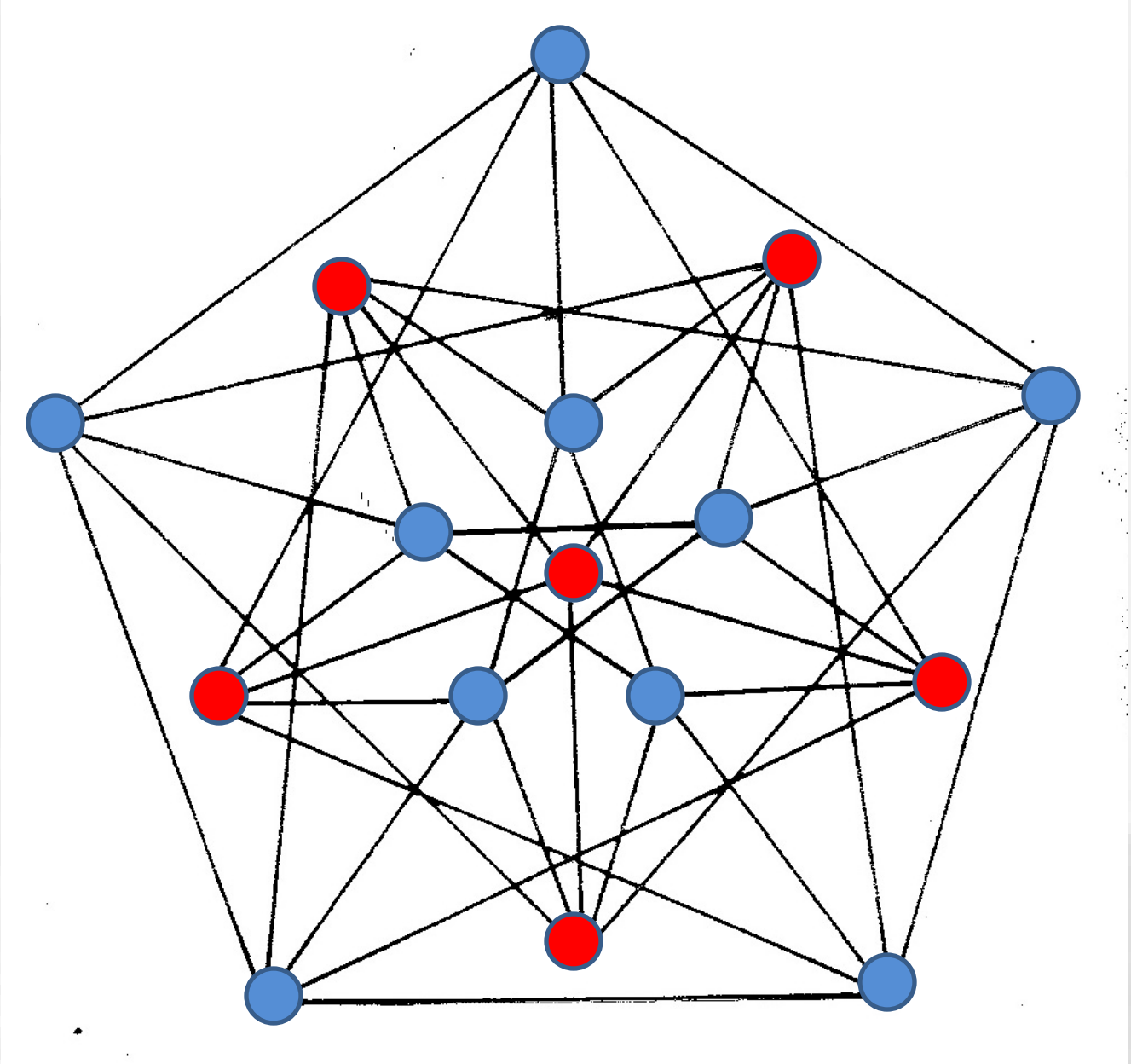
SRG(16,5,0,2)

Clebsch Graph



Clebsch Graph





The Colorful Petersen Graph

- P is the complement of the line graph of K_5

The Colorful Petersen Graph

- P is the complement of the line graph of K_5
- P has crossing number 2.

The Colorful Petersen Graph

- P is the complement of the line graph of K_5
- P has crossing number 2.
- P is embeddable on the Torus.

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- P is embeddable on the Torus.
- P has a 1-factor and two 2-factors.

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- P is embeddable on the Torus.
- P has a 1-factor and two 2-factors.
- P is the smallest vertex-transitive graph that is not a Cayley graph.

The Colorful Petersen Graph

- P is the complement of the line graph of K_5
- P has crossing number 2.
- P is embeddable on the Torus.
- P has a 1-factor and two 2-factors.
- P is the smallest vertex-transitive graph that is not a Cayley graph.
- The Petersen graph has chromatic number 3 and chromatic index 4.

The Colorful Petersen Graph

- P+P+P is not equal to K_{10} .

The Colorful Petersen Graph

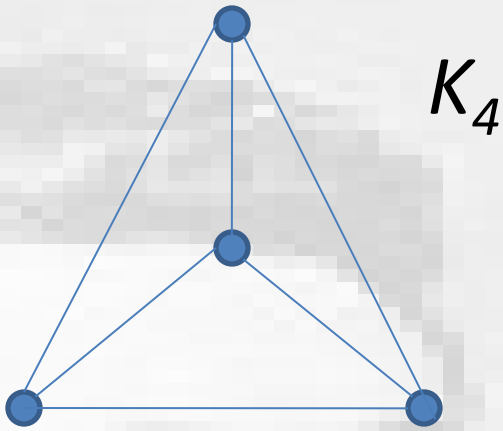
- P+P+P is not equal to K_{10} .
- P has 2000 spanning trees.

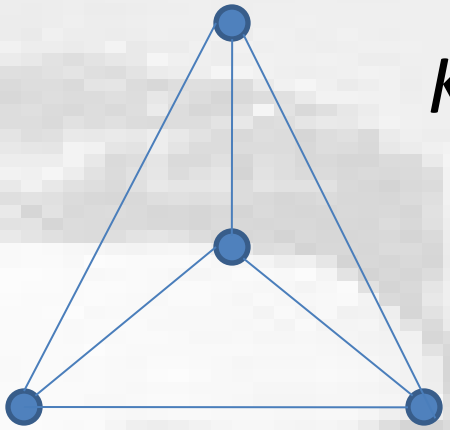
The Colorful Petersen Graph

- P+P+P is not equal to K_{10} .
- P has 2000 spanning trees.
- Petersen has infinitely many cousins.

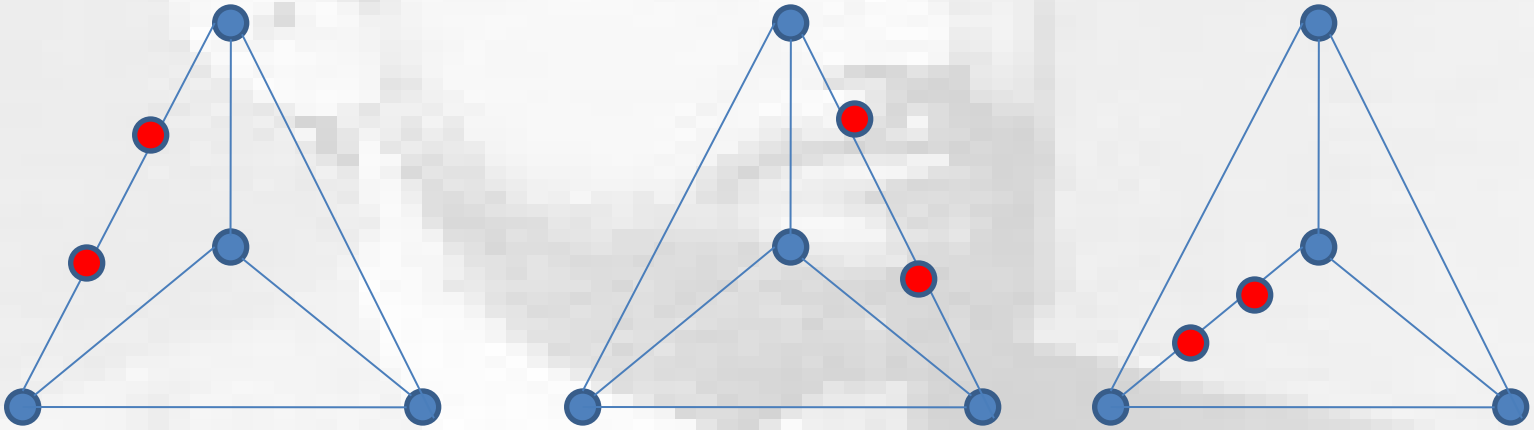
The Colorful Petersen Graph

- P+P+P is not equal to K_{10} .
- P has 2000 spanning trees.
- Petersen has infinitely many cousins.
- P, C_5 ,G(50) and possibly SRG(3250, 57,0,1) are extremal graphs. P is one of 4 graphs with girth 5 or more and $1/2n(n-1)^{0.5}$ edges.

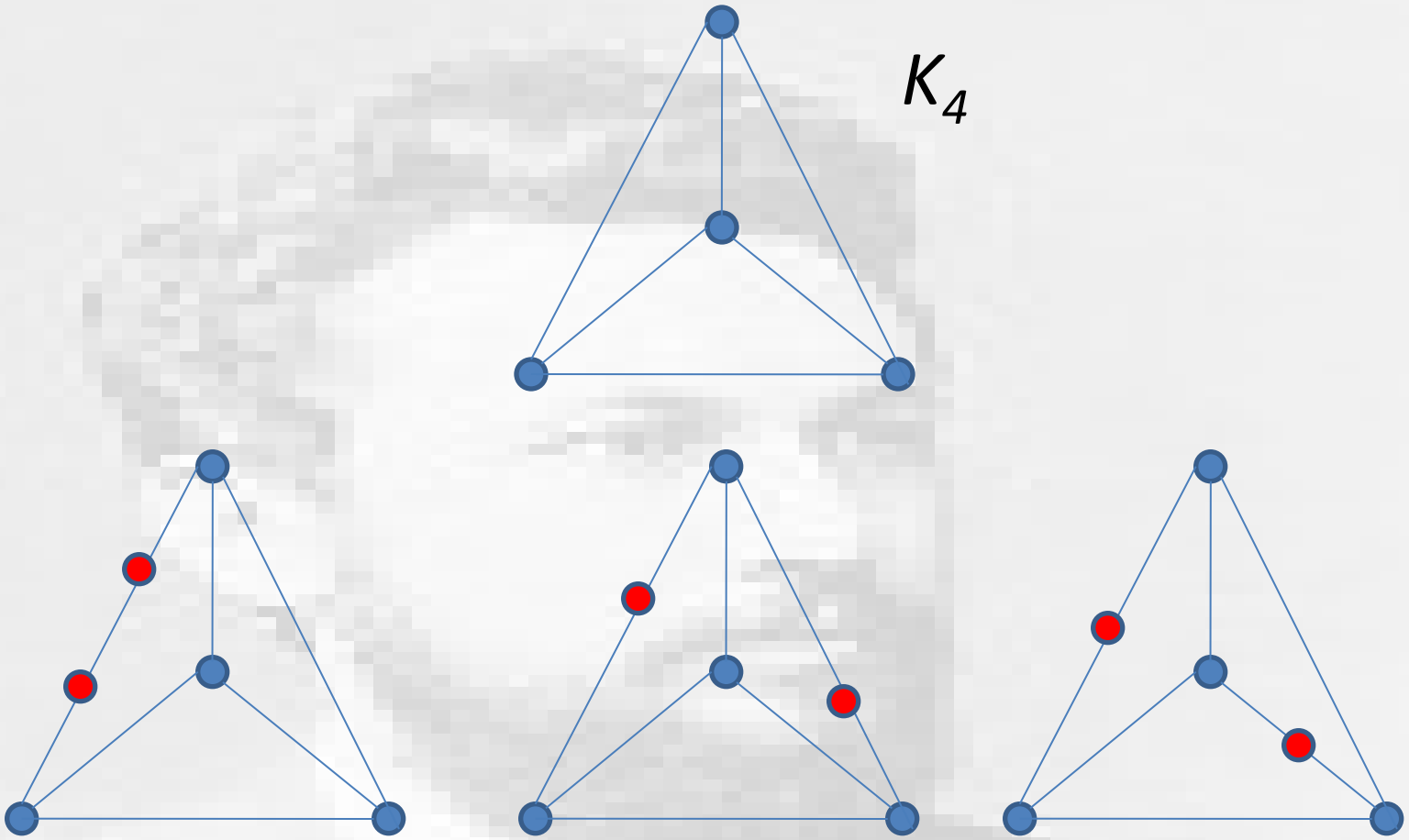

K_4



K_4

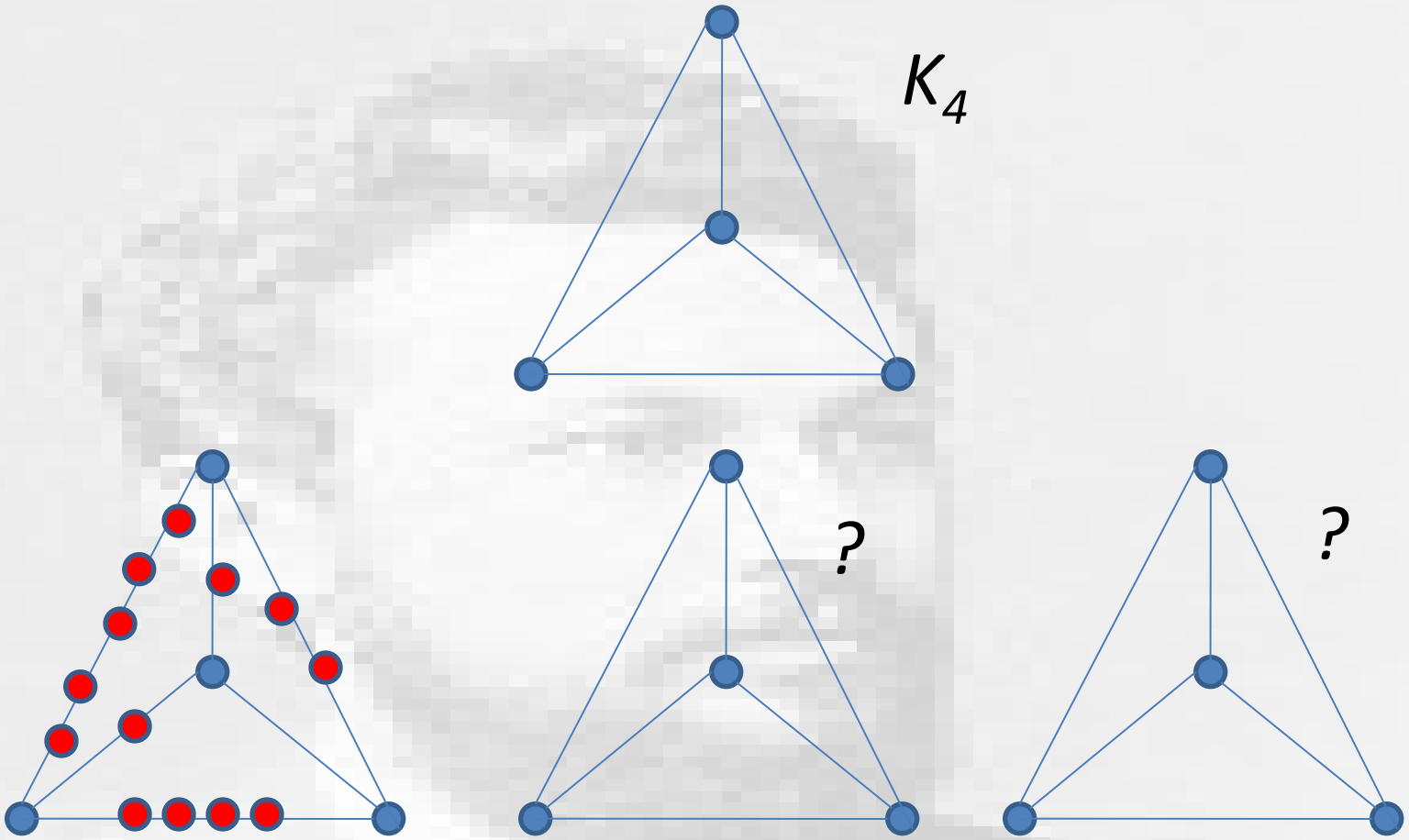


These 3 homeomorphs are all the same (*isomorphic*) graph.

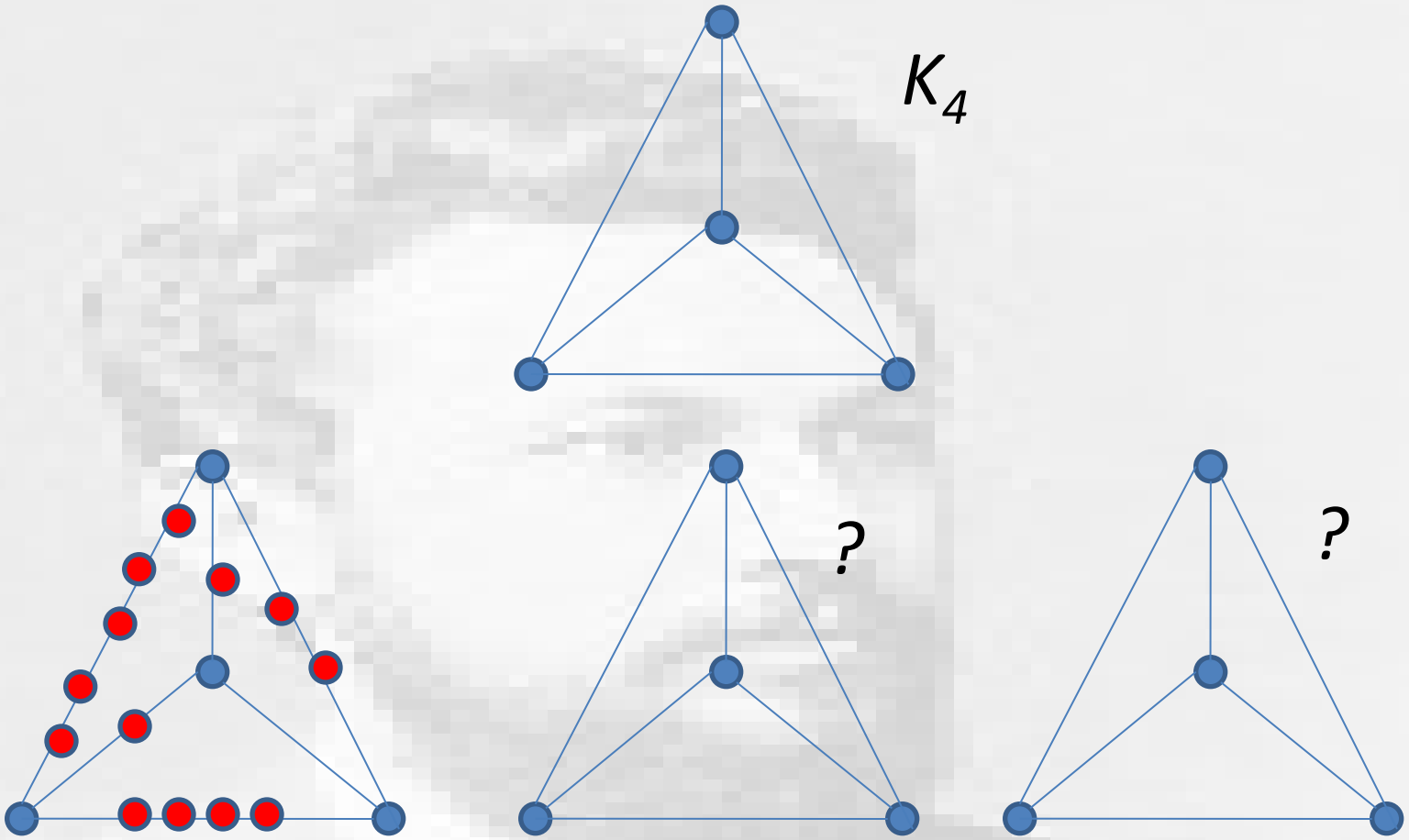


How many K_4-homeomorphs of order 2 are there?

Only these 3.

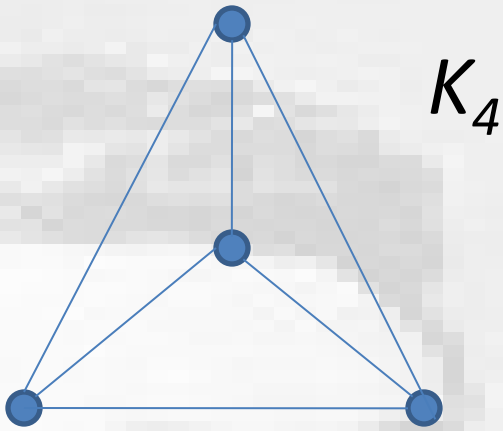


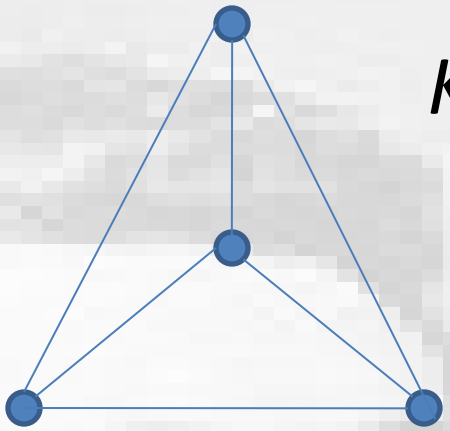
How many K_4-homeomorphs of order 13 are there?



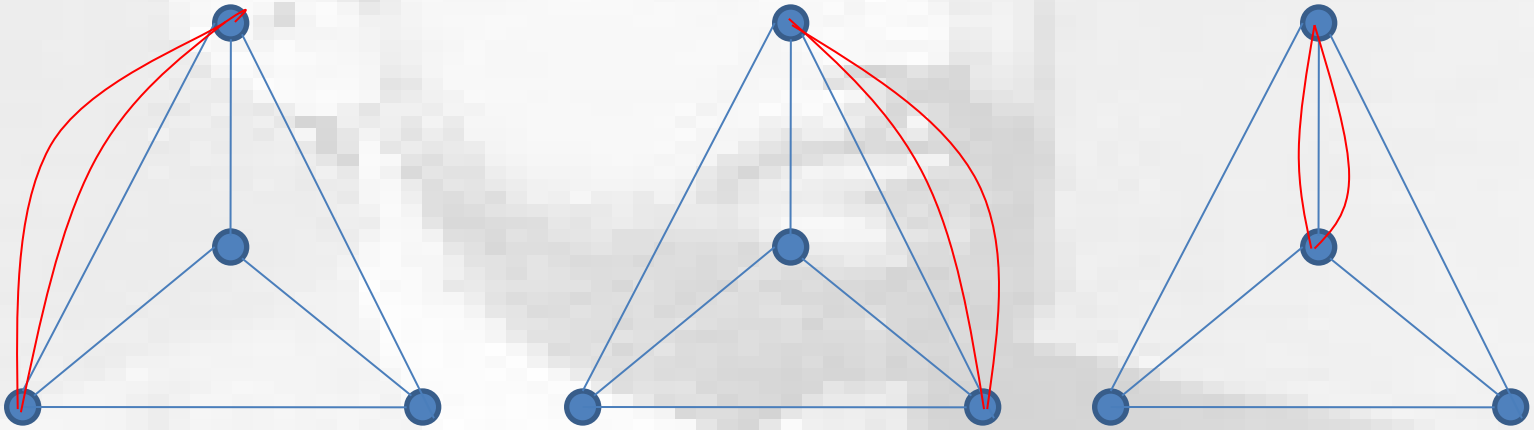
How many K_4-homeomorphs of order 13 are there?

Will any of these (>500) graphs on 13+4=17 vertices be chromatically-equivalent?

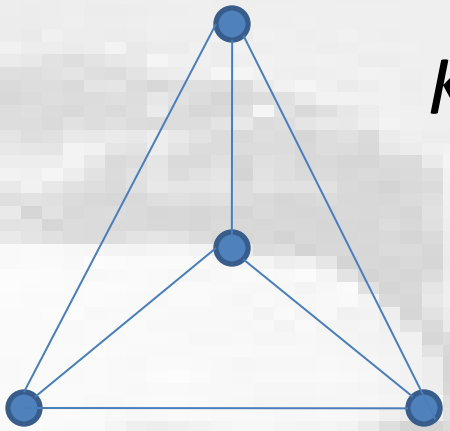

K_4

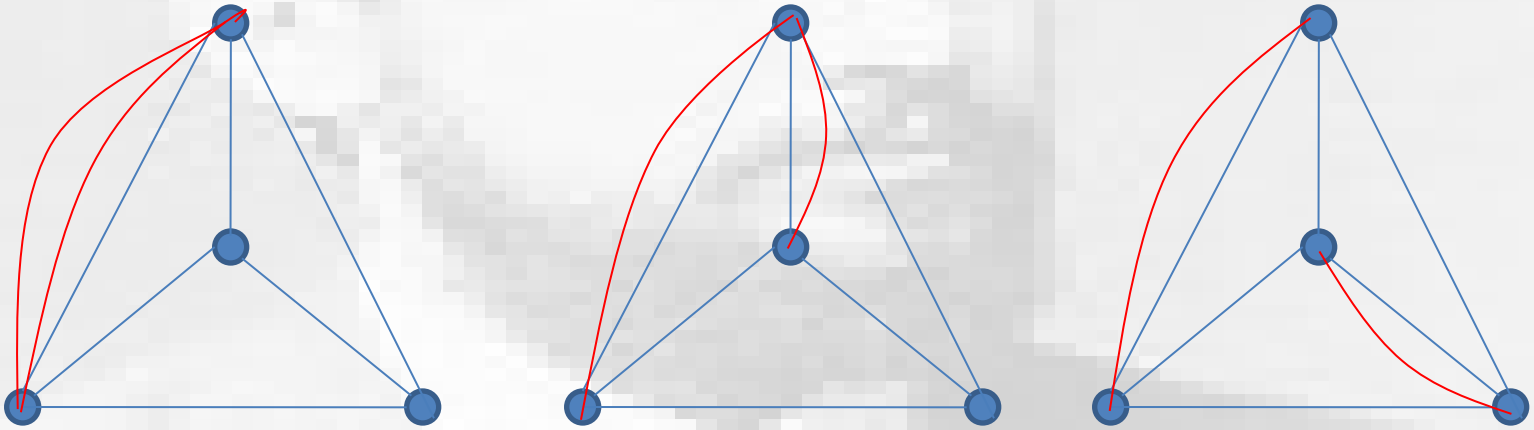


K_4



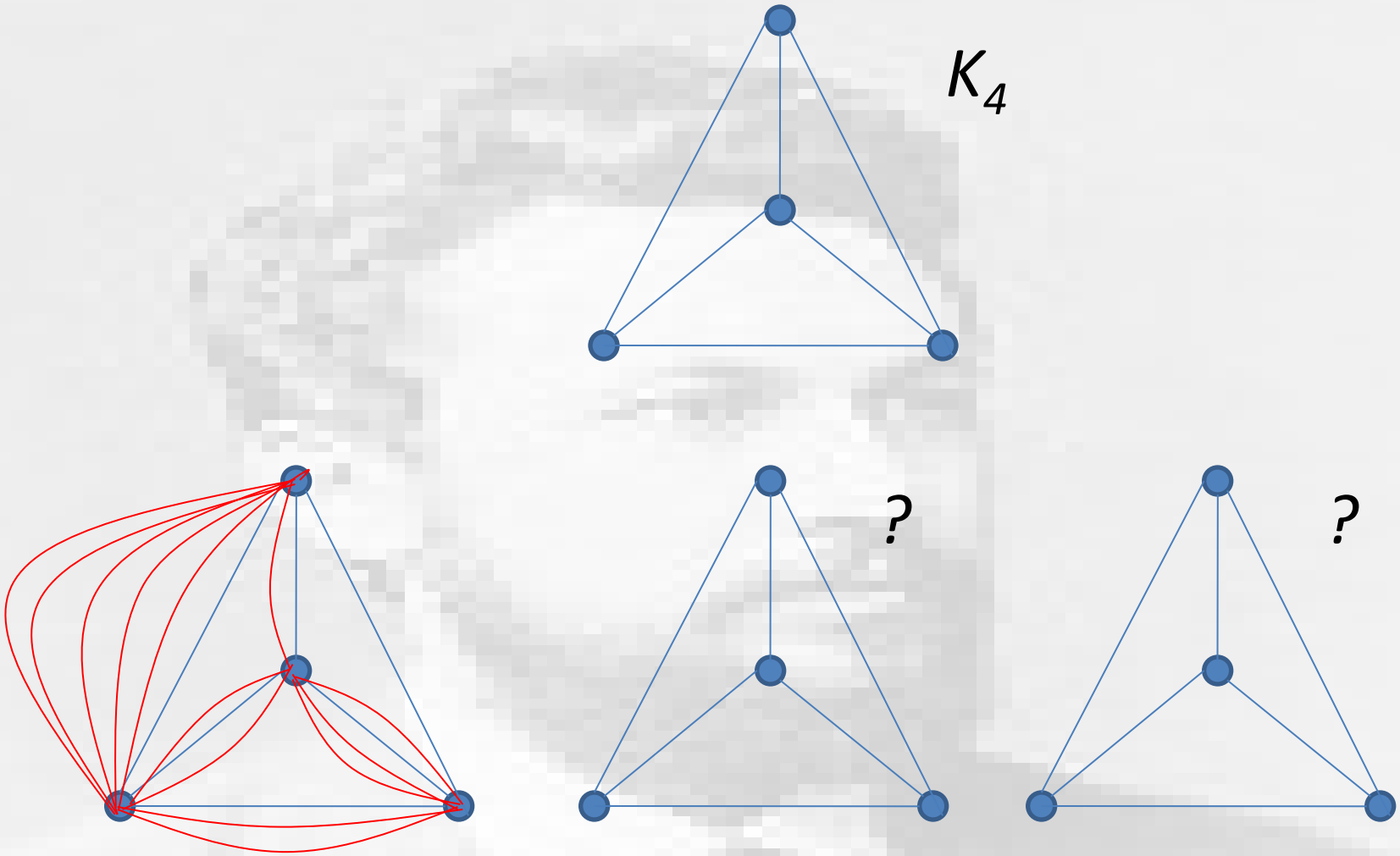
These 3 amallamorphs are all the same (*isomorphic*) graph.



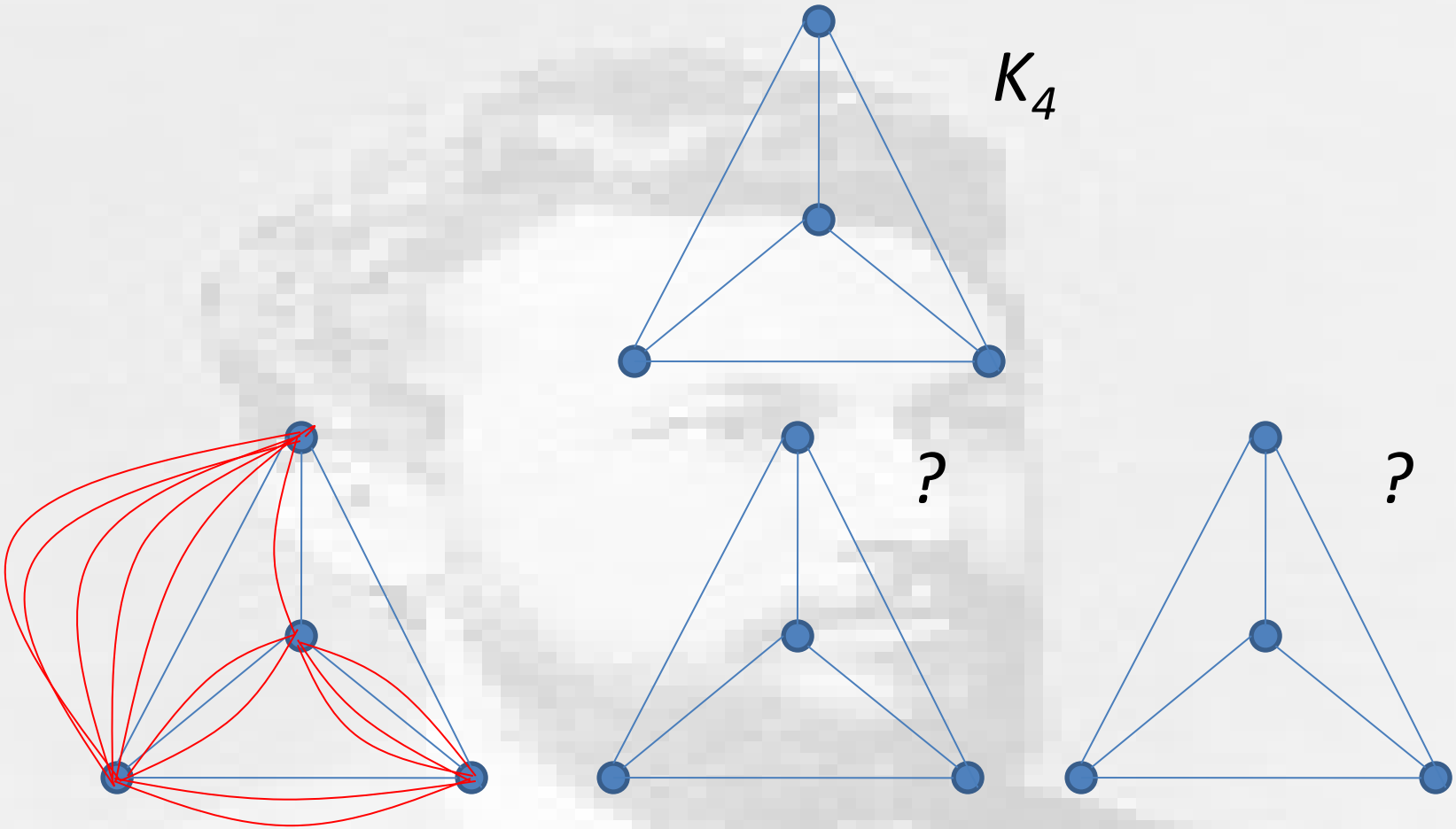


How many K_4-amallamorphs of order 2 are there?

Only these 3.

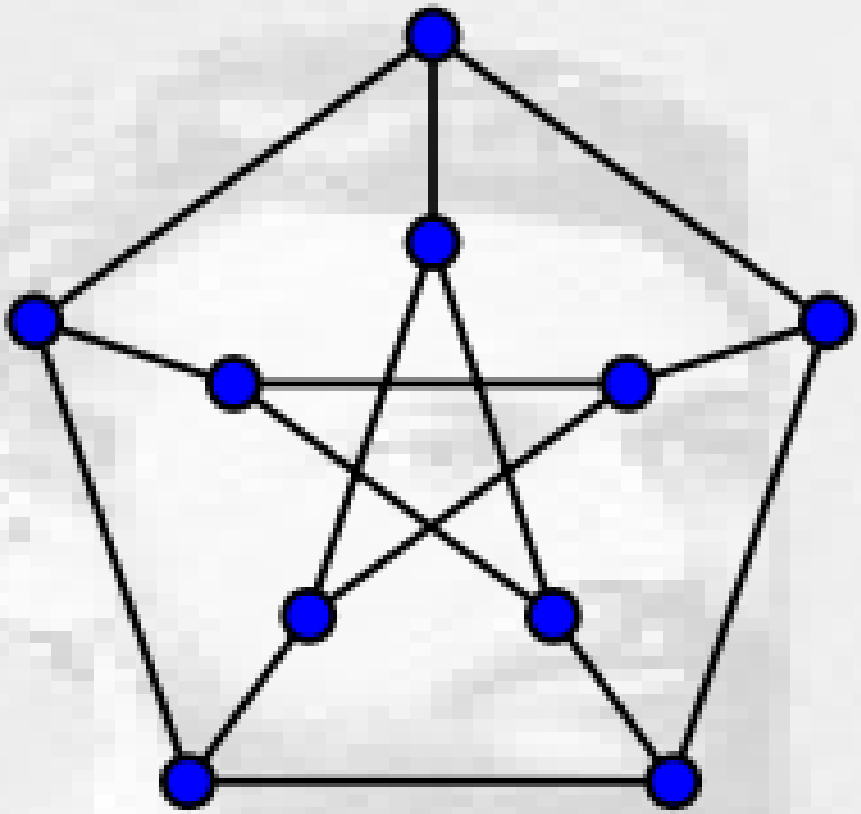


How many K_4-amallamorphs of order 13 are there?

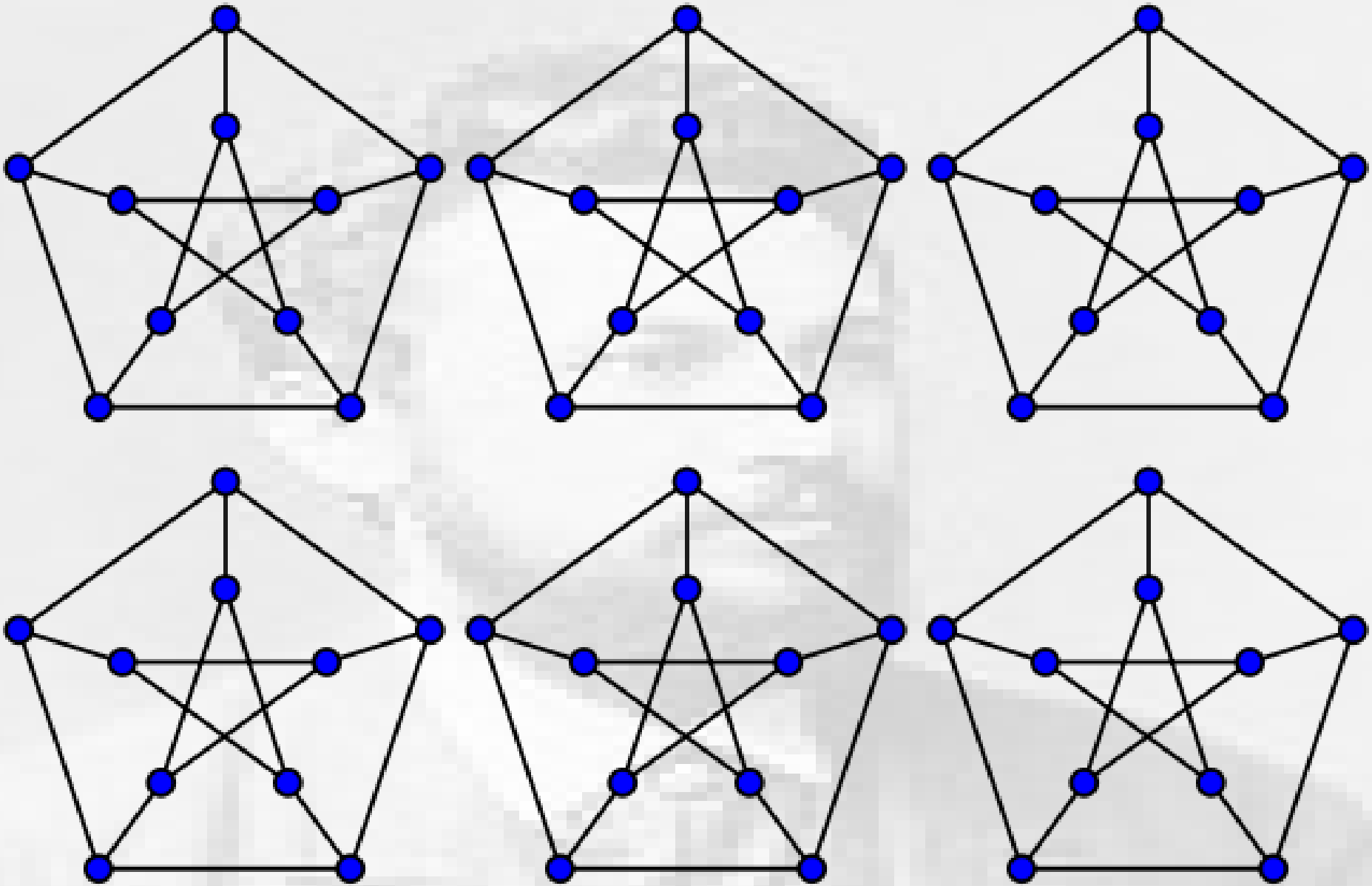


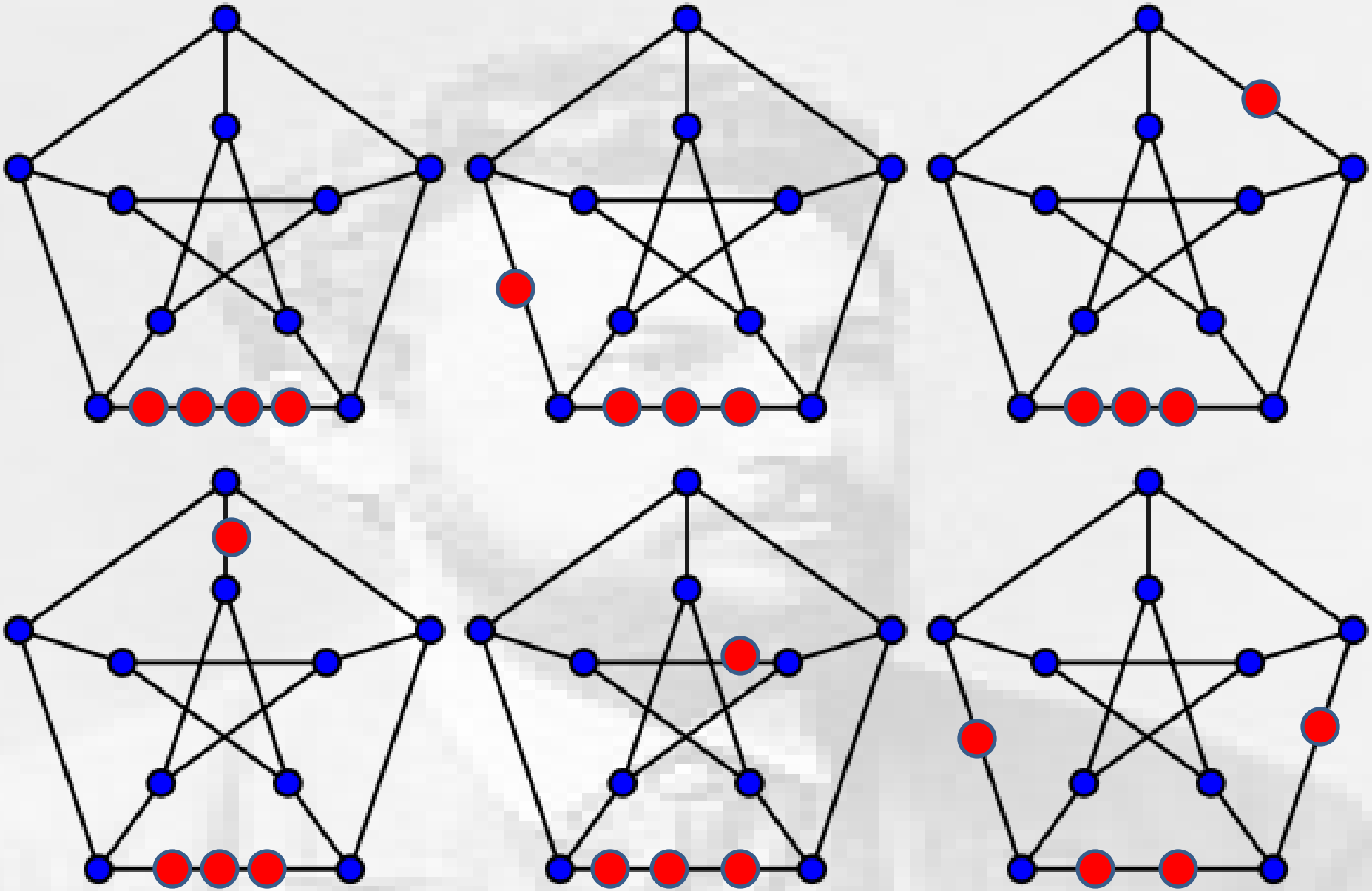
How many K_4-amallamorphs of order 13 are there?

Will any of these (>1000) graphs on 13+4=17 vertices be flow-equivalent?



How many *P*-homeomorphs of order 4 are there?

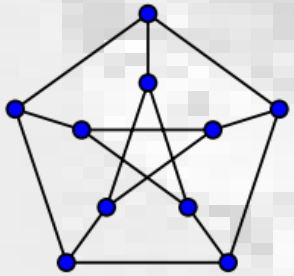


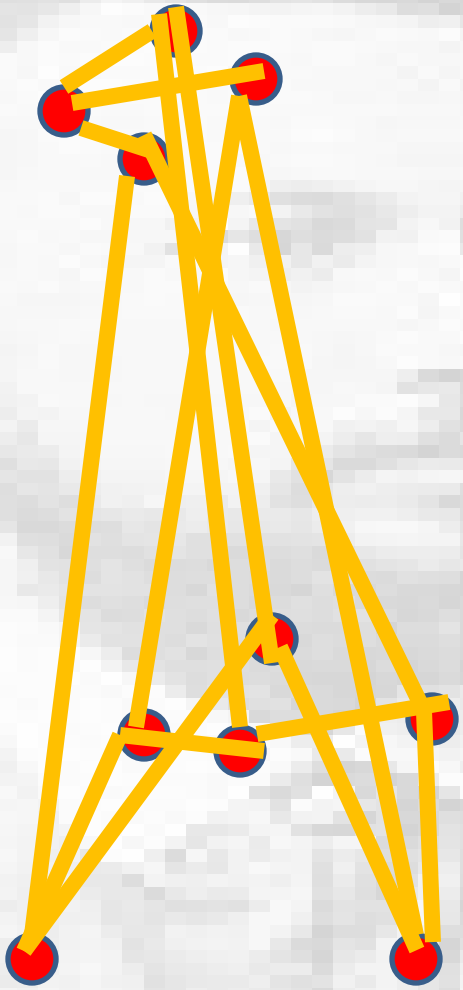


Theorem(*Shahmohamad & Whitehead*):
The following 18 pairs of P homeomorphs are chromatically equivalent for every choice of positive integers a, b, c, d, e, f, g, h:

	(k,m,n,o,p,q,r,s,t,u,v,w,x,y,z)	(k,m,n,o,p,q,r,s,t,u,v,w,x,y,z)
1	(b,a,e,d,h,a,c,b,g,f,c,a,b,a,b)	(b,a,g,d,h,a,c,b,e,f,c,a,b,a,b)
2	(b,a,c,a,b,g,e,f,d,h,b,a,c,a,b)	(b,a,c,a,b,g,e,d,f,h,b,a,c,a,b)
3	(b,d,e,g,a,c,a,f,h,b,b,a,c,b,a)	(b,g,e,d,a,c,a,h,f,b,b,a,c,b,a)
4	(e,c,a,d,c,a,d,g,b,f,d,c,a,b,b)	(e,c,a,d,c,a,d,f,b,g,d,c,a,b,b)
5	(d,a,b,f,e,b,c,c,d,g,a,d,b,a,c)	(d,a,b,g,e,b,c,c,d,f,a,d,b,a,c)
6	(b,b,c,d,g,c,b,e,a,a,a,d,c,f,d)	(b,b,c,d,g,c,b,f,a,a,a,d,c,e,d)
7	(c,d,b,e,c,c,g,a,a,d,f,b,a,b,d)	(c,d,b,g,c,c,e,a,a,d,f,b,a,b,d)
8	(b,1,b,e,f,a,1,g,a,c,1,a,b,d,1)	(b,1,b,e,f,a,1,d,a,c,1,a,b,g,1)
9	(d,a,3,1,a,a,a,1,c,2,3,b,2,2,2)	(d,a,3,1,a,a,a,1,b,2,3,c,2,2,2)
10	(1,1,a,1,1,1,1,b,1,a,1,a,1,a,a)	(1,1,a,1,1,1,1,1,1,a,a,1,b,a,a)
11	(1,2,1,a,1,b,1,1,2,1,2,3,2,2,1)	(1,2,1,b,1,a,1,1,2,1,2,3,2,2,1)
12	(1,2,a,1,a,2,a,2,2,3,2,2,1,3,1)	(1,2,2,1,a,3,a,2,2,2,2,a,1,3,1)
13	(1,1,1,1,a,1,1,2,1,2,3,1,1,3,2)	(1,a,2,1,1,1,1,1,1,2,1,2,3,1,3)
14	(1,1,3,2,1,2,1,1,2,3,1,1,1,4,1)	(1,1,3,1,1,1,1,1,1,2,4,2,3,2,1)
15	(1,1,3,1,1,2,1,3,2,2,3,1,2,1,1)	(1,2,3,1,1,2,1,3,2,2,1,1,1,3,1)
16	(1,1,2,1,1,2,1,2,3,2,4,1,2,1,1)	(1,1,2,1,1,2,1,4,2,2,1,3,1,2,1)
17	(1,1,3,1,1,2,1,1,1,2,3,1,1,3,2)	(1,1,2,1,1,2,1,3,1,1,1,3,1,3,2)
18	(1,1,3,2,1,2,1,1,2,2,1,3,1,2,1)	(1,1,3,1,1,1,1,2,2,2,3,1,2,2,1)

The Colorful Petersen Graph





Petersen Giraffe