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Julius Petersen, His Graph & Its Homeomorphs and Amallamorphs
(A 50 minutes Crash Course in Graph Theory)

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A Graph For All Seasons
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[Diagram of various graphs]
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Donald Knuth states that the Petersen graph is "a remarkable configuration that serves as a counterexample to many optimistic predictions about what might be true for graphs in general."
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The Petersen graph is an undirected graph with order $n=10$ vertices and size $m=15$ edges. It is a small graph that serves as a useful example and counterexample for many problems in graph theory.
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The Petersen graph is an undirected graph with order $n=10$ vertices and size $m=15$ edges. It is a small graph that serves as a useful example and counterexample for many problems in graph theory.

The Petersen graph is the smallest bridgeless cubic graph with no three-edge-coloring. It has girth 5.
The Colorful Life of Petersen
The Colorful Life of Petersen

- 1839  Birth
The Colorful Life of Petersen

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- 1871  Doctoral Dissertation
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- 1884 Tait’s loss becomes J.P.’s gain
• In proving the 4-Color Conjecture, Tait assumes the statement: “Every cubic graph is 1-factorable”.

![Graph Diagram]
• In proving the 4-Color Conjecture, Tait assumes the statement: “Every cubic graph is 1-factorable”.

• Petersen used the following graph to prove that Tait’s assumption was incorrect.
• In proving the 4-Color Conjecture, Tait assumes the statement: “*Every cubic graph is 1-factorable*”.

• Petersen used the following graph to prove that Tait’s assumption was *incorrect*.
• In proving the 4-Color Conjecture, Tait assumes the statement: “Every cubic graph is 1-factorable”.

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• In proving the 4-Color Conjecture, Tait assumes the statement: “*Every cubic graph is 1-factorable*”.

• Petersen used the following graph to prove that Tait’s assumption was *incorrect*.
• Indeed, J.P. proves the following results which is now known as the **Petersen Theorem**:
  
  Every bridgeless cubic graph has a **1-factor** and a **2-factor**.
The Colorful Life of Petersen

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• 1871  Doctoral Dissertation
• 1878  Sylvester Invents the word “Graff”
• 1879  Oh no! Accusation of Plagiarism
• 1884  Tait’s loss becomes J.P.’s gain
• 1886  Kempe’s drawing of Petersen Graph
The Colorful Life of Petersen
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• 1886  Kempe’s drawing of Petersen Graph
• 1889  Sylvester meets J.P. on Cayley’s error
The Colorful Life of Petersen

• 1890  Hilbert’s Finite Basis & age of invariants
The Colorful Life of Petersen

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The Colorful Life of Petersen

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The Colorful Life of Petersen
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The Colorful Life of Petersen

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- 1891  First major paper in Graph Theory
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- 1900  Ouch ! Petersen is fired from job
- 1908  Stroke
The Colorful Life of Petersen

- 1890: Hilbert’s Finite Basis & age of invariants
- 1891: First major paper in Graph Theory
- 1898: Petersen debuts his “Baby”
- 1900: Ouch! Petersen is fired from job
- 1908: Stroke
- 1910: Death
The Colorful Life of Petersen

• 1936  Denes Konig discovery of 1891
The Colorful Life of Petersen

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The Colorful Life of Petersen

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• 1990  A centennial gift for J.P. in Denmark
The Colorful Life of Petersen

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• 1936  First Graph Theory text arrives
• 1990  A centennial gift for J.P. in Denmark
• 2003  Many equivalences of Petersen Graph are discovered
The Colorful Petersen Graph

An Unforgettable Construction for $P$

• Consider $S = \{1,2,3,4,5\}$
• Let $V(P) =$ 2-element subsets of $S$
• Let $E(P) =$ disjoint 2-subsets of $S$
The Colorful Petersen Graph

- P is Non-bipartite
The Colorful Petersen Graph

- P is Non-bipartite
- P is Vertex transitive
The Colorful Petersen Graph

• P is Non-bipartite
• P is Vertex transitive
• P is Edge Transitive
The Colorful Petersen Graph

- P is Non-bipartite
- P is Vertex transitive
- P is Edge Transitive
- P has n-gons for n=5,6,8,9 \((12 \ C_5)\)
The Colorful Petersen Graph

- P is Non-bipartite
- P is Vertex transitive
- P is Edge Transitive
- P has n-gons for n=5,6,8,9 \((12 \ C_5)\)
- Aut(P) has order 120
The Colorful Petersen Graph

- P is Non-bipartite
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- P has n-gons for n=5,6,8,9 (12 $C_5$)
- $\text{Aut}(P)$ has order 120
- Eigenvalues of $P$ are $3,-2,-2,-2,-2,1,1,1,1,1,1$
The Colorful Petersen Graph

- P is Non-bipartite
- P is Vertex transitive
- P is Edge Transitive
- P has n-gons for n=5,6,8,9 \((12 \ C_5)\)
- \(\text{Aut}(P)\) has order 120
- Eigenvalues of P are 3,-2,-2,-2,-2,1,1,1,1,1
- P is Hypo-Hamiltonian, \textit{not} Hamiltonian and \textit{not} Eulerian.
The Colorful Petersen Graph

• P is a nice “cage” for all crazy graph theorists.
Figure 3.3 The (4, 6)-cage

Biggs’ unique (4, 6) cage

\[ f(4, 6) = 26 \]
Robertson's unique 
(4,5) Cage 
$f(4,5) = 19$
Figure 3.9  The 7-cage and 8-cage

Tutte-Coxeter's
Unique (3,8) Cage

$\text{f}(3,8) = 30$

McGee's
unique (3,7) Cage

$\text{f}(3,7) = 24$
(5,5) Cage
\( f(5,5) = 30 \)

**Figure 3.6** The (4, 5)-cage and (5, 5)-cage

\( |V| = 11 \)

A \( \{3, 4\} \)-regular graph with girth 4.

Is it a \( (3, 4; 4) \) Cage? 

**Figure 7.9** The Herschel graph
Heawood's unique (3,6) Cage
\[ f(3,6) = 14 \]

Petersen's unique (3,5) Cage
\[ f(2 \leq r) = 10 \]

A 4-regular graph with girth > 7, |V| = 13.
Is this a (4,7)?
The Colorful Petersen Graph

• P is a nice “cage” for all crazy graph theorists.
• P is non-planar. However, on the Projective Plane, P shares a bed with Dodecahedron.
The Colorful Petersen Graph

6 crossings

5 crossings

3 crossings

2 crossings
The Colorful Petersen Graph

Dodecahedron

Dodecahedron embedded on the Projective Plane
The Colorful Petersen Graph

• P is a nice “cage” for all crazy graph theorists.
• P is non-planar. However, on the Projective Plane, P shares a bed with Dodecahedron.
• P is a “Strongly Regular Graph” in Town.
The Colorful Petersen Graph

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\[ P = \text{SRG}(10,3,0,1) \]
The Colorful Petersen Graph

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- P is a “Strongly Regular Graph” in Town.
- P is not a “Line Graph”.
The Colorful Petersen Graph

- P is a nice “cage” for all crazy graph theorists.
- P is non-planar. However, on the Projective Plane, P shares a bed with Dodecahedron.
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- P is “Path Perfect”.
The Colorful Petersen Graph

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• P is non-planar. However, on the Projective Plane, P shares a bed with Dodecahedron.
• P is a “Strongly Regular Graph” in Town.
• P is not a “Line Graph”.
• P is “Path Perfect”.
• P is an induced subgraph of Clebsch Graph.
Clebsch Graph
Clebsch Graph
The Colorful Petersen Graph

• $P$ is the complement of the line graph of $K_5$
The Colorful Petersen Graph

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The Colorful Petersen Graph

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The Colorful Petersen Graph

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• P has a 1-factor and two 2-factors.
The Colorful Petersen Graph

- $P$ is the complement of the line graph of $K_5$.
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- $P$ is the smallest vertex-transitive graph that is not a Cayley graph.
The Colorful Petersen Graph

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• P has crossing number 2.
• P is embeddable on the Torus.
• P has a 1-factor and two 2-factors.
• P is the smallest vertex-transitive graph that is not a Cayley graph.
• The Petersen graph has chromatic number 3 and chromatic index 4.
The Colorful Petersen Graph

• $P + P + P$ is not equal to $K_{10}$. 
The Colorful Petersen Graph

- $P + P + P$ is not equal to $K_{10}$.
- $P$ has 2000 spanning trees.
The Colorful Petersen Graph

- P+P+P is not equal to $K_{10}$.
- P has 2000 spanning trees.
- Petersen has infinitely many cousins.
The Colorful Petersen Graph

- P+P+P is not equal to $K_{10}$.
- P has 2000 spanning trees.
- Petersen has infinitely many cousins.
- P, $C_5$, $G(50)$ and possibly $SRG(3250, 57,0,1)$ are extremal graphs. P is one of 4 graphs with girth 5 or more and $1/2n(n-1)^{0.5}$ edges.
These 3 homeomorphs are all the same (isomorphic) graph.
How many $K_4$-homeomorphs of order 2 are there? Only these 3.
How many $K_4$-homeomorphs of order 13 are there?
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Will any of these (>500) graphs on 13+4=17 vertices be chromatically-equivalent?
These 3 amallamorphs are all the same (*isomorphic*) graph.
How many $K_4$-amallamorphs of order 2 are there?
Only these 3.
How many $K_4$-amallamorphs of order 13 are there?
How many $K_4$-amallamorphs of order 13 are there?

Will any of these (>1000) graphs on 13+4=17 vertices be flow-equivalent?
How many $P$-homeomorphs of order 4 are there?
Theorem *(Shahmohamad & Whitehead)*: The following 18 pairs of P homeomorphs are chromatically equivalent for every choice of positive integers $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$: 
<table>
<thead>
<tr>
<th></th>
<th>(k, m, n, o, p, q, r, s, t, u, v, w, x, y, z)</th>
<th>(k, m, n, o, p, q, r, s, t, u, v, w, x, y, z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(b, a, e, d, h, a, c, b, g, f, c, a, b, a, b)</td>
<td>(b, a, g, d, h, a, c, b, e, f, c, a, b, a, b)</td>
</tr>
<tr>
<td>2</td>
<td>(b, a, c, a, b, g, e, f, d, h, b, a, c, a, b)</td>
<td>(b, a, c, a, b, g, e, d, f, h, b, a, c, a, b)</td>
</tr>
<tr>
<td>3</td>
<td>(b, d, e, g, a, c, a, f, h, b, a, c, b, a)</td>
<td>(b, g, e, d, a, c, a, h, f, b, a, c, a, b)</td>
</tr>
<tr>
<td>4</td>
<td>(e, c, a, d, c, a, d, g, b, f, d, c, a, b)</td>
<td>(e, c, a, d, c, a, d, f, b, g, d, c, a, b)</td>
</tr>
<tr>
<td>5</td>
<td>(d, a, b, f, e, b, c, c, d, g, a, d, b, a, c)</td>
<td>(d, a, b, g, e, b, c, c, d, f, a, d, b, a, c)</td>
</tr>
<tr>
<td>6</td>
<td>(b, b, c, d, g, c, b, e, a, a, d, c, f, d)</td>
<td>(b, b, c, d, g, c, b, f, a, a, d, c, e, d)</td>
</tr>
<tr>
<td>7</td>
<td>(c, d, b, e, c, c, g, a, a, d, f, b, a, b, d)</td>
<td>(c, d, b, g, c, c, e, a, a, d, f, b, a, b, d)</td>
</tr>
<tr>
<td>8</td>
<td>(b, 1, b, e, f, a, 1, g, a, c, 1, a, b, d, 1)</td>
<td>(b, 1, b, e, f, a, 1, d, a, c, 1, a, b, g, 1)</td>
</tr>
<tr>
<td>9</td>
<td>(d, a, 3, 1, a, a, 1, c, 2, 3, b, 2, 2, 2)</td>
<td>(d, a, 3, 1, a, a, 1, b, 2, 3, c, 2, 2, 2)</td>
</tr>
<tr>
<td>10</td>
<td>(1, 1, a, 1, 1, 1, b, 1, a, 1, a, 1, a, a)</td>
<td>(1, 1, a, 1, 1, 1, 1, a, a, 1, b, a, a)</td>
</tr>
<tr>
<td>11</td>
<td>(1, 2, 1, a, 1, b, 1, 1, 2, 1, 2, 3, 2, 2, 1)</td>
<td>(1, 2, 1, b, 1, a, 1, 1, 2, 1, 2, 3, 2, 2, 1)</td>
</tr>
<tr>
<td>12</td>
<td>(1, 2, a, 1, a, 2, a, 2, 2, 3, 2, 2, 1, 3, 1)</td>
<td>(1, 2, 2, 1, a, 3, a, 2, 2, 2, 2, a, 1, 3, 1)</td>
</tr>
<tr>
<td>13</td>
<td>(1, 1, 1, 1, a, 1, 1, 2, 1, 2, 3, 1, 1, 3, 2)</td>
<td>(1, a, 2, 1, 1, 1, 1, 1, 2, 1, 2, 3, 1, 3)</td>
</tr>
<tr>
<td>14</td>
<td>(1, 1, 3, 2, 1, 2, 1, 1, 2, 3, 1, 1, 4, 1)</td>
<td>(1, 1, 3, 1, 1, 1, 1, 1, 2, 4, 2, 3, 2, 1)</td>
</tr>
<tr>
<td>15</td>
<td>(1, 1, 3, 1, 1, 2, 1, 3, 2, 2, 3, 1, 2, 1)</td>
<td>(1, 2, 3, 1, 1, 2, 1, 3, 2, 2, 1, 1, 1, 3, 1)</td>
</tr>
<tr>
<td>16</td>
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<td>(1, 1, 2, 1, 1, 2, 1, 4, 2, 2, 1, 3, 1, 2, 1)</td>
</tr>
<tr>
<td>17</td>
<td>(1, 1, 3, 1, 1, 2, 1, 1, 1, 2, 3, 1, 1, 3, 2)</td>
<td>(1, 1, 2, 1, 1, 2, 1, 3, 1, 1, 1, 3, 1, 3, 2)</td>
</tr>
<tr>
<td>18</td>
<td>(1, 1, 3, 2, 1, 2, 1, 1, 2, 2, 1, 3, 1, 2, 1)</td>
<td>(1, 1, 3, 1, 1, 1, 1, 2, 2, 2, 3, 1, 2, 1)</td>
</tr>
</tbody>
</table>
The Colorful Petersen Graph

Petersen Giraffe