Spectral invariants and noncommutative geometry

Masoud Khalkhali
IPM, June-July 2011

Introducing metric notions such as volume and curvature into noncommutative geometry to a large degree depends, among other things, on ideas and techniques from spectral geometry. Spectral geometry is the mathematics of extracting geometric and topological information about a manifold from the spectrum of operators like Laplacian or Dirac. In this series of lectures we shall first see how in classical Riemannian geometry, as well as in quantum mechanics, starting with the celebrated Weyl's law and the correspondence principle, the trace of the heat kernel of the Laplacian on a closed Riemannian manifold can encode important geometric and topological informations through its short time and long time asymptotic behavior. Thanks to Tauberian type theorems, there is a more or less equivalent formulation of these results in terms of residues and special values of the spectral zeta functions. I shall then explain how some of these spectral and functional analytic methods carry over to the noncommutative world, thanks to Alain Connes' notion of spectral triples and his trace theorem. Finally I shall outline a recent proof (based on my joint work with F. Fathizadeh) of the Gauss-Bonnet theorem for the noncommutative two torus. This extends and refines a recent work of Connes and Tretkoff on the same subject.

Course outline: the following topics will be covered
1. Pseudodifferential operators on closed manifolds,
2. Finiteness and regularity results for elliptic PDE's,
3. Hodge decomposition theorem,
4. Asymptotic expansion for the heat kernel: Seeley-DeWitt-Gilkey coefficients, Weyl's law, scalar curvature,
5. Lefschetz principle and applications: a local formula for the index, Atiyah-Bott fixed point formula,
6. Spectral zeta functions and their residues,
7. Wodzicki residue and the Dixmier trace,
8. Spectral triples and Connes' trace theorem; the spectral action principle,
9. Gauss-Bonnet theorems in noncommutative geometry.

References:
1. P. Gilkey: Invariance theory, the heat equation, and the Atiyah-Singer

index theorem.
2. A. Connes: Noncommutative Geometry, Academic Press, 1994.
3. A. Connes and M. Marcolli: Noncommutative geometry, quantum fields and motives, 2008.
4. A. Connes and P. Tretkoff: The Gauss-Bonnet Theorem for the noncommutative two torus.
5. M. Khalkhali and F. Fathizadeh: The Gauss-Bonnet theorem for noncommutative two tori with a general conformal structure.

Lectures: 3 lectures per week; 2 hours each. Time and place to be announced.