Bounds on point-conic incidences over finite fields and applications

FGC-IPM Number Theory Seminar

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Joint work with Thang Pham and Audie Warren
(arXiv:2111.04072)

Institute for Research in Fundamental Sciences (IPM)

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- $\mathbb{F}_q$ is a finite field of order $q$ and characteristic $p$. 

$\mathbb{F}_d$ will be the $d$-dimensional projective space over $\mathbb{F}$.

$P$ will be a set of points, $L$ a set of lines and $C$ a set of curves (generally conics) over $\mathbb{F}_d$ (generally $d = 2$).

Bold lowercase letters, e.g. $p$, will be points.

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Definition (Incidences)

Let $\mathbb{F}$ be a field. Given a finite set of points $\mathcal{P}$ and a finite set of algebraic curves $\mathcal{C}$, we denote the number of incidences between $\mathcal{P}$ and $\mathcal{C}$ by

$$I(\mathcal{P}, \mathcal{C}) = |\{(p, C) \in \mathcal{P} \times \mathcal{C} : p \in C\}|.$$
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Let $\phi \in \mathbb{F}[x, y, z]$ be a homogeneous polynomial of degree 2. Then the curve $C = \{x \in \mathbb{F}P^2 : \phi(x) = 0\}$ is called a conic.
**Objective**

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This talk will be mostly about upper bounds on $I(\mathcal{P}, \mathcal{C})$, where $\mathcal{P} \subset \mathbb{F}P^2_p$ and $\mathcal{C}$ is a set of irreducible conics over $\mathbb{F}_p$. 
Theorem (Kővári-Sós-Turán)

Suppose that the incidence graph on $P \times C$ (over $\mathbb{F}^2$) contains no copy of $K_{s,t}$, i.e. for any $s$ points in $P$ there are fewer than $t$ curves in $C$ incident to it, then

$$I(P, C) \ll t^{1/s} |P| |C|^{1-1/s} + s|C|.$$  

The roles of $P$ and $C$ and respectively $s$ and $t$ may be reversed.


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Writing $(\mathcal{P}, \mathcal{C}, I)$ for the incidence graph on $\mathcal{P}$ and $\mathcal{C}$, we have:

<table>
<thead>
<tr>
<th>$\mathcal{C}$</th>
<th>$(\mathcal{P}, \mathcal{C}, I)$ has no $K_{2,2}$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Lines</td>
<td>$K_{2,2}$</td>
<td>$K_{2,2}$</td>
</tr>
<tr>
<td>General irreducible conics</td>
<td>$K_{5,2}$</td>
<td>$K_{2,5}$</td>
</tr>
<tr>
<td>Circles and parabolas</td>
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</tr>
</tbody>
</table>
Two lines over the plane $\mathbb{F}^2$ meet in at most one point. Also for any two points, there is at most one line passing both. So

$$I(\mathcal{P}, \mathcal{L}) \ll \min\{|\mathcal{L}|^{1/2}|\mathcal{P}| + |\mathcal{L}|, |\mathcal{P}|^{1/2}|\mathcal{L}| + |\mathcal{P}|\}.$$
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- An irreducible conic is determined uniquely by five points (with no three collinear). Also by Bézout’s theorem, any two distinct conics meet in at most four distinct points. So

$$I(\mathcal{P}, C) \ll \min\{|\mathcal{P}|C|^{4/5} + |C|, |\mathcal{P}|^{1/2}|C| + |\mathcal{P}|\}.$$
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\[ I(P, C) \ll \min\{|P||C|^{4/5} + |C|, |P|^{1/2}|C| + |P|\}. \]

- Any two parabolas or circles meet in at most two points. Also they are determined uniquely by three (non-collinear) points. So

\[ I(P, C) \ll \min\{|P||C|^{2/3} + |C|, |P|^{1/2}|C| + |P|\}. \]
Incidences: some well-known results

Over $\mathbb{R}^2$: (This was extended to $\mathbb{C}^2$ by Toth in 2015.)

Theorem (Szemerédi-Trotter (1983))

For finite sets of points $\mathcal{P}$ and lines $\mathcal{L}$ over $\mathbb{R}^2$, we have

$$I(\mathcal{P}, \mathcal{L}) \ll |\mathcal{P}|^{2/3} |\mathcal{L}|^{2/3} + |\mathcal{P}| + |\mathcal{L}|.$$

Over $\mathbb{F}_p^2$: (Various explicit forms, based on the same principal ideas, appeared later.)

Theorem (Bourgain-Katz-Tao (2004))

For any point set $\mathcal{P}$ and any line set $\mathcal{L}$ in $\mathbb{F}_p^2$ with $|\mathcal{P}| = |\mathcal{L}| = N = p^\alpha$, $0 < \alpha < 2$, we have

$$I(\mathcal{P}, \mathcal{L}) \ll N^{3/2 - \epsilon},$$

where $\epsilon = \epsilon(\alpha) > 0$. 
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Given finite sets of points $\mathcal{P}$ and lines $\mathcal{L}$ over $\mathbb{F}^2$, with
$|\mathcal{L}|^{7/8} \ll |\mathcal{P}| \ll |\mathcal{L}|^{8/7}$, if $\text{char}(\mathbb{F}) = p > 0$, suppose
$|\mathcal{P}|^{13} |\mathcal{L}|^{-2} \ll p^{15}$. Then

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- For $|\mathcal{P}| = |\mathcal{L}| = N \ll p^{15/11}$, the above result gives
  $$I(\mathcal{P}, \mathcal{L}) \ll N^{3/2-1/30} = N^{4/3+2/15}$$ (the bound $N^{4/3}$ would be tight).
Incidences: some well-known results

Point-line incidences for “large sets” in $\mathbb{F}_q$:

**Theorem (Vinh (2011))**

*Given finite sets of points $\mathcal{P}$ and lines $\mathcal{L}$ over $\mathbb{F}_q^2$, we have*

$$\left| l(\mathcal{P}, \mathcal{L}) - \frac{|\mathcal{P}||\mathcal{L}|}{q} \right| \leq \sqrt{q |\mathcal{P}||\mathcal{L}|}.$$
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**More generally:**

**Theorem (Vinh (2011))**

*Let $\mathcal{P}$ be a set of points and $\mathcal{H}$ be a set of hyperplanes in $\mathbb{F}_q^d$. The number of incidences between $\mathcal{P}$ and $\mathcal{H}$ satisfies*

$$
\left| I(\mathcal{P}, \mathcal{H}) - \frac{|\mathcal{P}||\mathcal{H}|}{q} \right| \leq \sqrt{q^{d-1}|\mathcal{P}||\mathcal{H}|}.
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Recall a Möbius transformation is the mapping

\[ f(x) = \frac{ax + b}{cx + d}, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{F}_p). \]

- Bourgain (2012) proved the first (non-quantitative) bound between points and Möbius transformations over \( \mathbb{F}_p \).
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- Shkredov (2021) proved an explicit bound on incidences between points and hyperbolae \((x - a)(y - b) = 1\) with \((a, b)\) coming from Cartesian products.

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Let $C$ be parabolas $C_{a,b}$ of the form $y = x^2 + ax + b$. 

This idea was due to Pham, Vinh and de Zeeuw (2018).
Point-conic incidences: warm up

- Let $C$ be parabolas $C_{a,b}$ of the form $y = x^2 + ax + b$.
- Apply the mapping $\phi : \mathbb{F}_q^2 \rightarrow \mathbb{F}_q^2$, defined by $(x, y) \mapsto (x, y - x^2)$ to the plane.

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So \( I(\mathcal{P}, \mathcal{C}) = I(\phi(\mathcal{P}), \mathcal{L}) \) where \( |\mathcal{L}| = |\mathcal{C}| \) and \( |\mathcal{P}| = |\phi(\mathcal{P})| \). This idea was due to Pham, Vinh and de Zeeuw (2018).
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Theorem (M., Pham and Warren (2021+))

For any set $C$ of irreducible conics in $\mathbb{F}_p^2$, and any set of points $\mathcal{P} \subseteq \mathbb{F}_p^2$ with $|\mathcal{P}| \ll p^{15/13}$, we have

$$I(\mathcal{P}, C) \ll |\mathcal{P}|^{23/27} |C|^{23/27} + |\mathcal{P}|^{13/9} |C|^{12/27} + |C|.$$
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- A much better bound is proved when $\mathcal{P}$ is a Cartesian product.
Theorem (M., Pham and Warren (2021+))

Let $\mathcal{P} \subseteq \mathbb{F}_p^2$, with $|\mathcal{P}| \ll p^{15/13}$ and let $\mathcal{C}$ be either a set of
- circles (in which case, suppose $p \equiv 3 \pmod{4}$), or
- parabolas of the form $y = ax^2 + bx + c$, or
- hyperbolas of the form $(x - a)(y - b) = c$.

Then

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- A much better bound is proved when $\mathcal{P}$ is a Cartesian product.
The key observation for circles etc.: If we fix a point $q \in P$, then the incidence structure arising from $P$ and the curves through $q$ resembles that of a point-line one.

Specifically, we bound the number of $k$-rich curves: curves in $C$ containing at least $k$ points of $P$. This is converted to a bound on $I(P, C)$. 

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- **The key observation for general conics:** If we fix two points \( q_1, q_2 \in \mathcal{P} \), then the incidence structure arising from \( \mathcal{P} \) and the curves through both points resembles that of a point-Möbius transformations one.
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- Based on these, initially we show that there cannot be too many such curves with too many points on them.
The overall strategy

- **The key observation for circles etc.:** If we fix a point $q \in \mathcal{P}$, then the incidence structure arising from $\mathcal{P}$ and the curves through $q$ resembles that of a point-line one.

- **The key observation for general conics:** If we fix two points $q_1, q_2 \in \mathcal{P}$, then the incidence structure arising from $\mathcal{P}$ and the curves through both points resembles that of a point-Möbius transformations one.

- Based on these, initially we show that there cannot be too many such curves with too many points on them.

- This easily leads to a similar statement about all of $\mathcal{C}$. Specifically, we bound the number of *k-rich curves*: curves in $\mathcal{C}$ containing at least $k$ points of $\mathcal{P}$. 
The overall strategy

- **The key observation for circles etc.:** If we fix a point \( q \in \mathcal{P} \), then the incidence structure arising from \( \mathcal{P} \) and the curves through \( q \) resembles that of a point-line one.

- **The key observation for general conics:** If we fix two points \( q_1, q_2 \in \mathcal{P} \), then the incidence structure arising from \( \mathcal{P} \) and the curves through both points resembles that of a point-Möbius transformations one.

- Based on these, initially we show that there cannot be too many such curves with too many points on them.

- This easily leads to a similar statement about all of \( \mathcal{C} \). Specifically, we bound the number of \emph{k-rich curves}: curves in \( \mathcal{C} \) containing at least \( k \) points of \( \mathcal{P} \).

- This is converted to a bound on \( I(\mathcal{P}, \mathcal{C}) \).
We write $C_k$ for the set of $k$-rich curves in $C$ (w.r.t. $\mathcal{P}$), i.e.

$$C_k = \{ C \in C : |C \cap \mathcal{P}| \geq k \}.$$
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Our strategy (for circles etc.) is to first bound the number of $k$-rich curves through some fixed arbitrary $q$:

$$C_{q,k} = \{ C \in C : q \in C \quad \text{and} \quad |C \cap \mathcal{P}| \geq k \}.$$
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Our strategy (for circles etc.) is to first bound the number of $k$-rich curves through some fixed arbitrary $q$:

$$C_{q,k} = \{ C \in C : q \in C \text{ and } |C \cap \mathcal{P}| \geq k \}.$$

Then note that

$$|C_k| \ll \frac{1}{k} \sum_{q \in \mathcal{P}} |C_{q,k}|.$$
Clearly

$$I(\mathcal{P}, \mathcal{C}) = \sum_{k=1}^{\lvert \mathcal{P} \rvert} \lvert \mathcal{C} = k \rvert k,$$

which we split as

$$\sum_{k \leq \Delta} \lvert \mathcal{C} = k \rvert k + \sum_{k > \Delta} \lvert \mathcal{C} = k \rvert k,$$

which is bounded by

$$\ll \Delta \lvert \mathcal{C} \rvert + \sum_{i} \sum_{\mathcal{C} \in \mathcal{C}_2} \Delta \leq \lvert \mathcal{C} \cap \mathcal{P} \rvert < 2^i + 1 \Delta (2^i \Delta)$$

Applying a bound on $$\lvert \mathcal{C} \rvert$$ and optimizing for $$\Delta$$ results in an incidence bound.
Clearly

\[ I(\mathcal{P}, \mathcal{C}) = \sum_{k=1}^{\mathcal{P}} |C_{=k}| k, \]

which we split as

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Clearly

\[ I(\mathcal{P}, \mathcal{C}) = \sum_{k=1}^{\frac{|\mathcal{P}|}{k=1}} |\mathcal{C}|_{k} k, \]

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which is bounded by

\[ \ll \Delta |\mathcal{C}| + \sum_{i} \sum_{\mathcal{C} \in \mathcal{C}} (2^{i} \Delta) \]

\[ 2^{i} \Delta \leq |\mathcal{C} \cap \mathcal{P}| < 2^{i+1} \Delta \]
From $k$-rich curves to incidences

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and further by

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and further by

$$\Delta |\mathcal{C}| + \sum_i |\mathcal{C}_{2^i \Delta}| (2^i \Delta).$$

Applying a bound on $|\mathcal{C}_k|$ and optimizing for $\Delta$ results in an incidence bound.
Bounding $|C_{q,k}|$ (for circles)

Fix $q \in \mathcal{P}$ and let $C$ be a circle through it:

$$(x, y) \in \mathbb{F}^2_q : \quad (x - c)^2 + (y - d)^2 = r \quad \text{for some} \quad c, d, r \in \mathbb{F}_q.$$
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- By a translation, we assume $q = (0, 0)$ and so $C = C_{a,b}$ must take the form

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- Suppose $C$ is $k$-rich and so the points 
  \[(\alpha_1, \beta_1), \ldots, (\alpha_{k-1}, \beta_{k-1}) \in \mathcal{P} \setminus \{(0, 0)\} \text{ lie on } C.\]
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- We associate each of the points to a line $l_{\alpha_i, \beta_i}$ of the form

  \[(X, Y) \in \mathbb{F}_q^2 : -2\alpha_i X - 2\beta_i Y + \alpha_i^2 + \beta_i^2 = 0.\]
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- By assuming $q \equiv 3 \pmod{4}$, $-1$ is a non-square and these lines are defined without multiplicity.
We have transformed our point-circle incidence relation to a line-point one. I.e. $C_{a,b} \mapsto (a, b)$ and $(\alpha_i, \beta_i) \mapsto l_{\alpha_i, \beta_i}$. So a $k$-rich circle is now a $k$-rich point (w.r.t. a set of lines $L$ with $|L| = |P|$). That is $|C_{q,k}| = |Q_k|$ for some point set $Q_k$. The number of $k$-rich points $Q_k$ (w.r.t. $L$) can be bounded simply using existing point-line incidence bounds based on the observation $k|Q_k| \leq I(Q_k, L)$. In higher dimensions, the same scheme reduces the point-circle problem to a hyperplane-point problem.
We have transformed our point-circle incidence relation to a line-point one. I.e. \( C_{a,b} \mapsto (a, b) \) and \( (\alpha_i, \beta_i) \mapsto l_{\alpha_i, \beta_i} \).

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Key observation: Given (the homogenized form of) an irreducible conic $\gamma$, we have

$\gamma$ is a Möbius transformation $\iff \{[0 : 1 : 0], [1 : 0 : 0]\} \subseteq \gamma$. 
Strategy for general conics

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- For $q_1, q_2 \in P$, let $C_{q_1,q_2,k}$ be the set of $k$-rich conics in $C$ incident to $q_1$ and $q_2$. 

Let $\pi$ be a projective transformation sending $q_1 \rightarrow [0 : 1 : 0], q_2 \rightarrow [1 : 0 : 0]$. 

$\pi(C_{q_1,q_2,k})$ corresponds to a set of $k$-rich Möbius transformations, whose size can be bounded by a result of Warren and Wheeler.

Then $|C_k| \leq (k^2)^{-1} \sum_{q_1, q_2 \in P} |C_{q_1,q_2,k}| \ll |P|^2 k^{2\max q_1, q_2 \in P} |C_{q_1,q_2,k}|$. 

Ali Mohammadi  
Point-conic incidences in $\mathbb{F}_q$
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- Then
  
  $$|C_k| \leq \binom{k}{2}^{-1} \sum_{q_1,q_2 \in \mathcal{P}} |C_{q_1,q_2,k}| \ll \frac{|\mathcal{P}|^2}{k^2} \max_{q_1,q_2 \in \mathcal{P}} |C_{q_1,q_2,k}|.$$
Given $p = (p_1, p_2)$ and $q = (q_1, q_2)$, define their distance by

$$d(p, q) = (p_1 - q_1)^2 + (p_2 - q_2)^2.$$
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The number of distances formed by \( \mathcal{P} \) is

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D(\mathcal{P}) = |\{d(p, q) : p, q \in \mathcal{P}\}|.
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The Erdős distinct distances problem

Given $p = (p_1, p_2)$ and $q = (q_1, q_2)$, define their distance by

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Conjecture (Erdős (1946))

For finite $P \subset \mathbb{R}^2$, we have $|D(P)| \gg |P|(\log |P|)^{-1/2}$.
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Ali Mohammadi

Point-conic incidences in $\mathbb{F}_q$
The pinned distinct distances problem

The number of distances of $\mathcal{P}$, pinned at $q \in \mathcal{P}$ is defined by

$$D(\mathcal{P}; q) = |\{d(p, q) : p \in \mathcal{P}\}|.$$
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Theorem (Katz and Tardos (2004))

For finite $\mathcal{P} \subset \mathbb{R}^2$, we have $|D_{\text{pin}}(\mathcal{P})| \gg |\mathcal{P}|^{0.8641}$. 

Theorem (Murphy, Petridis, Pham, Rudnev and Stevens (2021))

Let $\mathcal{P} \subset \mathbb{F}_q^2$ be finite and if $\text{char}(\mathbb{F}_q) = p$, suppose $p \equiv 3 \pmod{4}$ and $|\mathcal{P}| \leq p^{4/3}$. Then $|D_{\text{pin}}(\mathcal{P})| \gg |\mathcal{P}|^{2/3}$. 

Ali Mohammadi
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Related problems: Falconer-type questions

Finite field Falconer-type questions ask how large must \( P \subset \mathbb{F}_q^2 \) be such that \( D(P) \) (or \( D_{\text{pin}}(P) \)) is of size \( q \) (or up to constants).

Chapman, Erdogan, Hart, Iosevich and Koh (2012) showed if \( |P| \gg q^{4/3} \) then \( D(P) \gg q \).

Bennett, Hart, Iosevich, Pakinathan and Rudnev (2017) proved the same result for pinned distances problem.

Murphy and Petridis (2019) gave examples showing that for general \( \mathbb{F}_q \), the exponent \( 4/3 \) is sharp.

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Noting that \( d(p, e) = f(p - e) \), where \( f \) is \((x, y) \mapsto x^2 + y^2\), one may naturally wish to study the distances problem for other choices of \( f \):
Pinned algebraic distances: variations

Noting that $d(p, e) = f(p - e)$, where $f$ is $(x, y) \mapsto x^2 + y^2$, one may naturally wish to study the distances problem for other choices of $f$:

**Theorem (M., Pham and Warren (2021+))**

Let $\mathcal{E} \subset \mathbb{F}_p^2$ with $|\mathcal{E}| \ll p^{15/13}$ and $p \equiv 3 \pmod{4}$. Let $f(x, y)$ be one of the following polynomials:

- $x^2 + y^2$ *(usual distance function)*, or
- $xy$ *(Minkowski distance function)*, or
- $y + x^2$ *(parabolic distance function)*.

There exists a point $p \in \mathcal{E}$ such that $|f(p - \mathcal{E})| \gg |\mathcal{E}|^{8/15}$, where

$$f(p - \mathcal{E}) := \{f(p - e) : e \in \mathcal{E}\}.$$

Ali Mohammadi

Point-conic incidences in $\mathbb{F}_q$
For $p \in \mathcal{E}$, let

$$C_p = \{ x \in \mathbb{F}_p^2 : f(p - x) = t \text{ with } t \in f(p - \mathcal{E}) \setminus \{0\} \}$$
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Pinned algebraic distances: proof

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Further note that

$$|C| \leq \sum_{p \in \mathcal{E}} |f(p - \mathcal{E})| \leq |\mathcal{E}| \cdot \max_{p \in \mathcal{E}} |f(p - \mathcal{E})|.$$
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An upper bound on $I(\mathcal{E}, C)$ follows from our point-conic incidence bound, yielding the result.
Let $p = (p_1, \ldots, p_d)$ and $q = (q_1, \ldots, q_d)$, be points in $F_q^d$. Then we write

$$d(p, q) = (p_1 - q_1)^2 + \cdots + (p_d - q_d)^2.$$
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We define the number of distances between point sets \( \mathcal{E} \) and \( \mathcal{F} \), in \( \mathbb{F}_q^d \), by

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Related problems: higher dimensions etc.

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  \]

**Theorem (Shparlinski (2006))**

For \( \mathcal{E}, \mathcal{F} \subset \mathbb{F}_q^d \), we have

\[
D(\mathcal{E}, \mathcal{F}) \gg \min \left\{ q, \frac{|\mathcal{E}||\mathcal{F}|}{q^d} \right\}.
\]

In particular, if \( |\mathcal{E}||\mathcal{F}| \geq q^{d+1} \) then \( D(\mathcal{E}, \mathcal{F}) \gg q \).
Theorem (Koh and Sun (2014))

For $\mathcal{E}, \mathcal{F} \subset \mathbb{F}_q^d$, if $d \geq 3$ is odd, then

$$D(\mathcal{E}, \mathcal{F}) \geq \begin{cases} \min \left\{ \frac{q}{2}, \frac{|\mathcal{E}||\mathcal{F}|}{8q^{d-1}} \right\} & \text{if } 1 \leq |\mathcal{E}| < q^{\frac{d-1}{2}} \\ \min \left\{ \frac{q}{2}, \frac{|\mathcal{F}|}{8q^{d-1}} \right\} & \text{if } q^{\frac{d-1}{2}} \leq |\mathcal{E}| < q^{\frac{d+1}{2}} \\ \min \left\{ \frac{q}{2}, \frac{|\mathcal{E}||\mathcal{F}|}{2q^d} \right\} & \text{if } q^{\frac{d+1}{2}} \leq |\mathcal{E}| \leq q^d \end{cases}$$
Theorem (Koh and Sun (2014))

For $\mathcal{E}, \mathcal{F} \subset \mathbb{F}_q^d$, if $d \geq 2$ is even, under the assumption $|\mathcal{E}||\mathcal{F}| \geq 16q^d$, one has

$$D(\mathcal{E}, \mathcal{F}) \geq \begin{cases} \frac{q}{144} & \text{for } 1 \leq |\mathcal{E}| < q^{d-1} \\ \frac{1}{144} \min \left\{ q, \frac{|\mathcal{F}|}{2q^{d-1}} \right\} & \text{for } q^{\frac{d-1}{2}} \leq |\mathcal{E}| < q^{d+1} \\ \frac{1}{144} \min \left\{ q, \frac{2|\mathcal{E}||\mathcal{F}|}{q^d} \right\} & \text{for } q^{\frac{d+1}{2}} \leq |\mathcal{E}| \leq q^d \end{cases}.$$
Theorem (M., Pham and Warren (2021+))

Let $\mathcal{E}, \mathcal{F}$ be sets in $\mathbb{F}_q^d$. Assume that $|\mathcal{E}| \sim |\mathcal{F}| \leq q^{\frac{d+1}{2}}$, then we have

$$D(\mathcal{E}, \mathcal{F}) \gg \min \left\{ q, \frac{|\mathcal{E}|^{1/2} |\mathcal{F}|^{1/2}}{q^{\frac{d-1}{2}}} \right\}.$$
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That is, the condition $|\mathcal{E}| |\mathcal{F}| \gg q^d$ is removed from Koh-Sun’s result for even $d$ in the range $q^{\frac{d-1}{2}} \leq |\mathcal{E}| \leq q^{\frac{d+1}{2}}$. 

Thank you for your attention!