Bounds on point-conic incidences over finite fields and applications

FGC-IPM Number Theory Seminar

Ali Mohammadi
Joint work with Thang Pham and Audie Warren
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Institute for Research in Fundamental Sciences (IPM)

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- Bold lowercase letters, e.g. $\boldsymbol{p}$, will be points.
- $A \ll B$ and $B \gg A$ will mean $A \leq CB$ for some absolute constant C (not always the same).

Objective

Definition (Incidences)

Let $\mathbb{F}$ be a field. Given a finite set of points $\mathcal{P}$ and a finite set of algebraic curves $\mathcal{C}$, we denote the number of incidences between $\mathcal{P}$ and $\mathcal{C}$ by

$$I(\mathcal{P}, \mathcal{C}) = |\{(\boldsymbol{p}, C) \in \mathcal{P} \times \mathcal{C} : \boldsymbol{p} \in C\}|.$$

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This talk will be mostly about upper bounds on $I(\mathcal{P}, \mathcal{C})$, where $\mathcal{P} \subset \mathbb{F}_p^2$ and $\mathcal{C}$ is a set of irreducible conics over $\mathbb{F}_p$.

Incidences: trivial bounds

Theorem (Kővári-Sós-Turán)

Suppose that the incidence graph on $\mathcal{P} \times \mathcal{C}$ (over $\mathbb{F}^2$) contains no copy of $K_{s,t}$, i.e. for any s points in $\mathcal{P}$ there are fewer than t curves in $\mathcal{C}$ incident to it, then

$$I(\mathcal{P}, \mathcal{C}) \ll t^{1/s} |\mathcal{P}| |\mathcal{C}|^{1-1/s} + s|\mathcal{C}|.$$

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Writing $(\mathcal{P}, \mathcal{C}, I)$ for the incidence graph on $\mathcal{P}$ and $\mathcal{C}$, we have:

$\mathcal{C}$	$(\mathcal{P}, \mathcal{C}, I)$ has no	$(\mathcal{C}, \mathcal{P}, I)$ has no
Lines	$K_{2,2}$	$K_{2,2}$
General irreducible conics	$K_{5,2}$	$K_{2,5}$
Circles and parabolas	$K_{3,2}$	$K_{2,3}$

Incidences: trivial bounds

- Two lines over the plane $\mathbb{F}^2$ meet in at most one point. Also for any two points, there is at most one line passing both. So

$$I(\mathcal{P},\mathcal{L}) \ll \min\{|\mathcal{L}|^{1/2}|\mathcal{P}| + |\mathcal{L}|, |\mathcal{P}|^{1/2}|\mathcal{L}| + |\mathcal{P}|\}.$$

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- In particular, if $|\mathcal{P}| = |\mathcal{L}| = N$, then $I(\mathcal{P},\mathcal{L}) \ll N^{3/2}$.
- An irreducible conic is determined uniquely by five points (with no three collinear). Also by Bézout's theorem, any two distinct conics meet in at most four distinct points. So

$$I(\mathcal{P},\mathcal{C}) \ll \min\{|\mathcal{P}||\mathcal{C}|^{4/5} + |\mathcal{C}|, |\mathcal{P}|^{1/2}|\mathcal{C}| + |\mathcal{P}|\}.$$

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- Any two parabolas or circles meet in at most two points. Also they are determined uniquely by three (non-collinear) points. So

$$I(\mathcal{P},\mathcal{C}) \ll \min\{|\mathcal{P}||\mathcal{C}|^{2/3} + |\mathcal{C}|, |\mathcal{P}|^{1/2}|\mathcal{C}| + |\mathcal{P}|\}.$$

Incidences: some well-known results

Over $\mathbb{R}^2$: (This was extended to $\mathbb{C}^2$ by Toth in 2015.)

Theorem (Szemerédi-Trotter (1983))

For finite sets of points $\mathcal{P}$ and lines $\mathcal{L}$ over $\mathbb{R}^2$, we have

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Over $\mathbb{F}_p^2$: (Various explicit forms, based on the same principal ideas, appeared later.)

Theorem (Bourgain-Katz-Tao (2004))

For any point set $\mathcal{P}$ and any line set $\mathcal{L}$ in $\mathbb{F}_p^2$ with $|\mathcal{P}| = |\mathcal{L}| = N = p^\alpha$, $0 < \alpha < 2$, we have

$$I(\mathcal{P},\mathcal{L}) \ll N^{\frac{3}{2}-\varepsilon}, \text{ where } \varepsilon = \varepsilon(\alpha) > 0.$$

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Theorem (Stevens-de Zeeuw (2017))

Given finite sets of points $\mathcal{P}$ *and lines* $\mathcal{L}$ *over* $\mathbb{F}^2$, *with* $|\mathcal{L}|^{7/8} \ll |\mathcal{P}| \ll |\mathcal{L}|^{8/7}$, *if* $char(\mathbb{F}) = p > 0$, *suppose* $|\mathcal{P}|^{13}|\mathcal{L}|^{-2} \ll p^{15}$. *Then*

$$I(\mathcal{P}, \mathcal{L}) \ll |\mathcal{P}|^{11/15}|\mathcal{L}|^{11/15}.$$

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- For $|\mathcal{P}| = |\mathcal{L}| = N \ll p^{15/11}$, the above result gives $I(\mathcal{P}, \mathcal{L}) \ll N^{3/2-1/30} = N^{4/3+2/15}$ (the bound $N^{4/3}$ would be tight).

Incidences: some well-known results

Point-line incidences for "large sets" in $\mathbb{F}_q$:

Theorem (Vinh (2011))

Given finite sets of points $\mathcal{P}$ and lines $\mathcal{L}$ over $\mathbb{F}_q^2$, we have

$$\left| I(\mathcal{P},\mathcal{L}) - \frac{|\mathcal{P}||\mathcal{L}|}{q} \right| \leq \sqrt{q|\mathcal{P}||\mathcal{L}|}.$$

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More generally:

Theorem (Vinh (2011))

Let $\mathcal{P}$ be a set of points and $\mathcal{H}$ be a set of hyperplanes in $\mathbb{F}_q^d$. The number of incidences between $\mathcal{P}$ and $\mathcal{H}$ satisfies

$$\left| I(\mathcal{P}, \mathcal{H}) - \frac{|\mathcal{P}||\mathcal{H}|}{q} \right| \leq \sqrt{q^{d-1}|\mathcal{P}||\mathcal{H}|}.$$

Non-linear incidence theorems over $\mathbb{F}_p$

Recall a Möbius transformation is the mapping

$$f(x) = \frac{ax+b}{cx+d}, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_p).$$

- Bourgain (2012) proved the first (non-quantitative) bound between points and Möbius transformations over $\mathbb{F}_p$.

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- So $I(\mathcal{P}, \mathcal{C}) = I(\phi(\mathcal{P}), \mathcal{L})$ where $|\mathcal{L}| = |\mathcal{C}|$ and $|\mathcal{P}| = |\phi(\mathcal{P})|$.
- This idea was due to Pham, Vinh and de Zeeuw (2018).

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Theorem (M., Pham and Warren (2021+))

For any set $\mathcal{C}$ of irreducible conics in $\mathbb{F}_p^2$, and any set of points $\mathcal{P} \subseteq \mathbb{F}_p^2$ with $|\mathcal{P}| \ll p^{15/13}$, we have

$$I(\mathcal{P},\mathcal{C}) \ll |\mathcal{P}|^{23/27}|\mathcal{C}|^{23/27} + |\mathcal{P}|^{13/9}|\mathcal{C}|^{12/27} + |\mathcal{C}|.$$

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Let $\mathcal{P} \subseteq \mathbb{F}_p^2$*, with* $|\mathcal{P}| \ll p^{15/13}$ *and let* $\mathcal{C}$ *be either a set of*

- *circles (in which case, suppose* $p \equiv 3 \pmod 4$*), or*
- *parabolas of the form* $y = ax^2 + bx + c$*, or*
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Then

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- **The key observation for circles etc.:** If we fix a point $\boldsymbol{q} \in \mathcal{P}$, then the incidence structure arising from $\mathcal{P}$ and the curves through $\boldsymbol{q}$ resembles that of a point-line one.

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- Based on these, initially we show that there cannot be too many such curves with too many points on them.
- This easily leads to a similar statement about all of $\mathcal{C}$. Specifically, we bound the number of *k-rich curves*: curves in $\mathcal{C}$ containing at least k points of $\mathcal{P}$.

The overall strategy

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- **The key observation for general conics:** If we fix two points $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathcal{P}$, then the incidence structure arising from $\mathcal{P}$ and the curves through both points resembles that of a point-Möbius transformations one.
- Based on these, initially we show that there cannot be too many such curves with too many points on them.
- This easily leads to a similar statement about all of $\mathcal{C}$. Specifically, we bound the number of *k-rich curves*: curves in $\mathcal{C}$ containing at least k points of $\mathcal{P}$.
- This is converted to a bound on $I(\mathcal{P}, \mathcal{C})$.

From k-rich curves to incidences

- We write $\mathcal{C}_k$ for the set of k-rich curves in $\mathcal{C}$ (w.r.t. $\mathcal{P}$), i.e.

$$\mathcal{C}_k = \{C \in \mathcal{C} : |C \cap \mathcal{P}| \geq k\}.$$

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- Then note that

$$|\mathcal{C}_k| \ll \frac{1}{k} \sum_{\boldsymbol{q} \in \mathcal{P}} |\mathcal{C}_{\boldsymbol{q},k}|.$$

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- Applying a bound on $|\mathcal{C}_k|$ and optimizing for Δ results in an incidence bound.

Bounding $|\mathcal{C}_{\boldsymbol{q},k}|$ (for circles)

- Fix $\boldsymbol{q} \in \mathcal{P}$ and let C be a circle through it:

$$(x, y) \in \mathbb{F}_q^2 : \quad (x - c)^2 + (y - d)^2 = r \text{ for some } c, d, r \in \mathbb{F}_q.$$

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- In higher dimensions, the same scheme reduces the point-circle problem to a hyperplane-point problem.

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Key observation: Given (the homogenized form of) an irreducible conic γ, we have

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- Then

$$|\mathcal{C}_k| \leq \binom{k}{2}^{-1} \sum_{\boldsymbol{q}_1,\boldsymbol{q}_2 \in \mathcal{P}} |\mathcal{C}_{\boldsymbol{q}_1,\boldsymbol{q}_2,k}| \ll \frac{|\mathcal{P}|^2}{k^2} \max_{\boldsymbol{q}_1,\boldsymbol{q}_2 \in \mathcal{P}} |\mathcal{C}_{\boldsymbol{q}_1,\boldsymbol{q}_2,k}|.$$

The Erdős distinct distances problem

Given $\boldsymbol{p} = (p_1, p_2)$ and $\boldsymbol{q} = (q_1, q_2)$, define their distance by

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For finite $\mathcal{P} \subset \mathbb{R}^2$, we have $|D(\mathcal{P})| \gg |\mathcal{P}|(\log|\mathcal{P}|)^{-1/2}$.

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Theorem (Murphy, Petridis, Pham, Rudnev and Stevens (2021))

Let $\mathcal{P} \subset \mathbb{F}^2$ be finite and if char$(\mathbb{F}) = p$, suppose $p \equiv 3 \pmod 4$ and $|\mathcal{P}| \leq p^{4/3}$. Then $|D_{pin}(\mathcal{P})| \gg |\mathcal{P}|^{2/3}$.

Related problems: Falconer-type questions

- Finite field Falconer-type questions ask how large must $\mathcal{P} \subset \mathbb{F}_q^2$ be such that $D(\mathcal{P})$ (or $D_{\text{pin}}(\mathcal{P})$) is of size q (or up to constants).

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Pinned algebraic distances: variations

Noting that $d(\boldsymbol{p}, \boldsymbol{e}) = f(\boldsymbol{p} - \boldsymbol{e})$, where f is $(x, y) \mapsto x^2 + y^2$, one may naturally wish to study the distances problem for other choices of f:

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Theorem (M., Pham and Warren (2021+))

Let $\mathcal{E} \subset \mathbb{F}_p^2$ with $|\mathcal{E}| \ll p^{15/13}$ and $p \equiv 3 \pmod 4$. Let $f(x, y)$ be one of the following polynomials:

- $x^2 + y^2$ *(usual distance function), or*
- xy *(Minkowski distance function), or*
- $y + x^2$ *(parabolic distance function).*

There exists a point $\boldsymbol{p} \in \mathcal{E}$ such that $|f(\boldsymbol{p} - \mathcal{E})| \gg |\mathcal{E}|^{\frac{8}{15}}$, where

$$f(\boldsymbol{p} - \mathcal{E}) := \{f(\boldsymbol{p} - \boldsymbol{e}) \colon \boldsymbol{e} \in \mathcal{E}\}.$$

Pinned algebraic distances: proof

- For $\boldsymbol{p} \in \mathcal{E}$, let

$$\mathcal{C}_{\boldsymbol{p}} = \left\{ \boldsymbol{x} \in \mathbb{F}_p^2 : f(\boldsymbol{p} - \boldsymbol{x}) = t \quad \text{with} \quad t \in f(\boldsymbol{p} - \mathcal{E}) \setminus \{0\} \right\}$$

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$$|\mathcal{C}| \leq \sum_{\boldsymbol{p} \in \mathcal{E}} |f(\boldsymbol{p} - \mathcal{E})| \leq |\mathcal{E}| \cdot \max_{\boldsymbol{p} \in \mathcal{E}} |f(\boldsymbol{p} - \mathcal{E})|.$$

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- For $\boldsymbol{p} \in \mathcal{E}$, let

$$\mathcal{C}_{\boldsymbol{p}} = \left\{ \boldsymbol{x} \in \mathbb{F}_p^2 : f(\boldsymbol{p} - \boldsymbol{x}) = t \quad \text{with} \quad t \in f(\boldsymbol{p} - \mathcal{E}) \setminus \{0\} \right\}$$

- Let $\mathcal{C} = \bigcup_{\boldsymbol{p} \in \mathcal{E}} \mathcal{C}_{\boldsymbol{p}}$.
- Note that $I(\mathcal{E}, \mathcal{C}) \gg |\mathcal{E}|^2$.
- Further note that

$$|\mathcal{C}| \leq \sum_{\boldsymbol{p} \in \mathcal{E}} |f(\boldsymbol{p} - \mathcal{E})| \leq |\mathcal{E}| \cdot \max_{\boldsymbol{p} \in \mathcal{E}} |f(\boldsymbol{p} - \mathcal{E})|.$$

- An upper bound on $I(\mathcal{E}, \mathcal{C})$ follows from our point-conic incidence bound, yielding the result.

Related problems: higher dimensions etc.

- Let $\boldsymbol{p} = (p_1, \ldots, p_d)$ and $\boldsymbol{q} = (q_1, \ldots, q_d)$, be points in $\mathbb{F}_q^d$. Then we write

$$d(\boldsymbol{p}, \boldsymbol{q}) = (p_1 - q_1)^2 + \cdots + (p_d - q_d)^2.$$

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- We define the number of distances between point sets $\mathcal{E}$ and $\mathcal{F}$, in $\mathbb{F}_q^d$, by

$$D(\mathcal{E}, \mathcal{F}) = |\{d(\boldsymbol{e}, \boldsymbol{f}) : \boldsymbol{e} \in \mathcal{E} \quad \text{and} \quad \boldsymbol{f} \in \mathcal{F}\}|.$$

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Theorem (Shparlinski (2006))

For $\mathcal{E}, \mathcal{F} \subset \mathbb{F}_q^d$, we have

$$D(\mathcal{E}, \mathcal{F}) \gg \min\left\{q, \frac{|\mathcal{E}||\mathcal{F}|}{q^d}\right\}.$$

In particular, if $|\mathcal{E}||\mathcal{F}| \geq q^{d+1}$ then $D(\mathcal{E}, \mathcal{F}) \gg q$.

Related problems: higher dimensions etc.

Theorem (Koh and Sun (2014))

For $\mathcal{E}, \mathcal{F} \subset \mathbb{F}_q^d$, if $d \geq 3$ is odd, then

$$D(\mathcal{E},\mathcal{F}) \geq \begin{cases} \min\left\{\frac{q}{2}, \frac{|\mathcal{E}||\mathcal{F}|}{8q^{d-1}}\right\} & \text{if } 1 \leq |\mathcal{E}| < q^{\frac{d-1}{2}} \\ \min\left\{\frac{q}{2}, \frac{|F|}{8q^{\frac{d-1}{2}}}\right\} & \text{if } q^{\frac{d-1}{2}} \leq |E| < q^{\frac{d+1}{2}} \\ \min\left\{\frac{q}{2}, \frac{|\mathcal{E}||\mathcal{F}|}{2q^{d}}\right\} & \text{if } q^{\frac{d+1}{2}} \leq |\mathcal{E}| \leq q^{d} \end{cases}.$$

Related problems: higher dimensions etc.

Theorem (Koh and Sun (2014))

For $\mathcal{E}, \mathcal{F} \subset \mathbb{F}_q^d$, if $d \geq 2$ is even, under the assumption $|\mathcal{E}||\mathcal{F}| \geq 16q^d$, one has

$$D(\mathcal{E},\mathcal{F}) \geq \begin{cases} \frac{q}{144} & \text{for } 1 \leq |\mathcal{E}| < q^{\frac{d-1}{2}} \\ \frac{1}{144}\min\left\{q, \frac{|\mathcal{F}|}{2q^{\frac{d-1}{2}}}\right\} & \text{for } q^{\frac{d-1}{2}} \leq |\mathcal{E}| < q^{\frac{d+1}{2}} \\ \frac{1}{144}\min\left\{q, \frac{2|\mathcal{E}||\mathcal{F}|}{q^d}\right\} & \text{for } q^{\frac{d+1}{2}} \leq |\mathcal{E}| \leq q^d \end{cases}.$$

Related problems: higher dimensions etc.

Theorem (M., Pham and Warren (2021+))

Let $\mathcal{E}, \mathcal{F}$ be sets in $\mathbb{F}_q^d$. Assume that $|\mathcal{E}| \sim |\mathcal{F}| \leq q^{\frac{d+1}{2}}$, then we have

$$D(\mathcal{E}, \mathcal{F}) \gg \min\left\{q, \frac{|\mathcal{E}|^{1/2}|\mathcal{F}|^{1/2}}{q^{\frac{d-1}{2}}}\right\}.$$

Related problems: higher dimensions etc.

Theorem (M., Pham and Warren (2021+))

Let $\mathcal{E}, \mathcal{F}$ be sets in $\mathbb{F}_q^d$. Assume that $|\mathcal{E}| \sim |\mathcal{F}| \leq q^{\frac{d+1}{2}}$, then we have

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That is, the condition $|\mathcal{E}||\mathcal{F}| \gg q^d$ is removed from Koh-Sun's result for even d in the range $q^{\frac{d-1}{2}} \leq |\mathcal{E}| \leq q^{\frac{d+1}{2}}$.

Thank you for your attention!