Selberg's Central Limit Theorem

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Outline

Background

Selberg's central limit theorem

Proof of Selberg's theorem

Remarks, Analogues and An Application

Riemann Zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{for} \quad \Re s > 1$$

$$= \prod_{p:\text{prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \qquad \text{for} \quad \Re s > 1.$$

Riemann Zeta-function

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- $\zeta(s)$ has an analytic continuation to $\mathbb{C}$ except a simple pole at $s = 1$.
- It has symmetries : one with respect to the real axis so that $\zeta(\bar{s}) = \overline{\zeta(s)}$; one with respect to the so-called critical line $\Re s = \frac{1}{2}$ which is a result of its functional equation:

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

Zeros of the Riemann Zeta-Function

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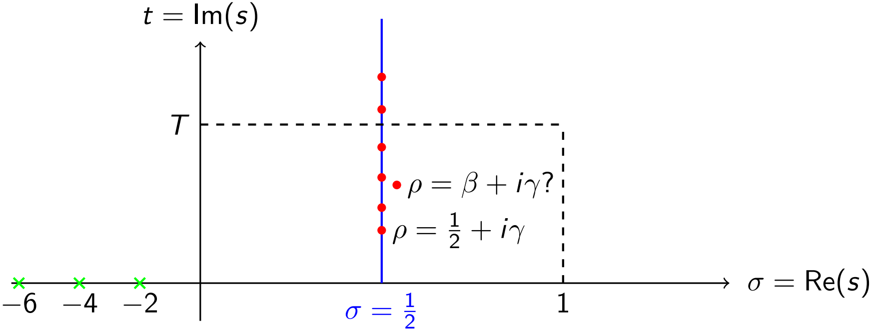
The zeros in the critical strip $\{s \in \mathbb{C} : 0 \leq \Re s \leq 1\}$ are called nontrivial zeros. A generic one is denoted by $\rho = \beta + i\gamma$, its multiplicity is denoted by $m(\rho)$.



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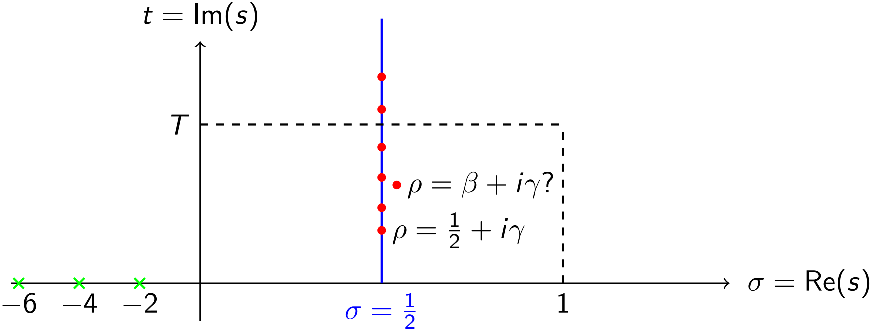

- Riemann Hypothesis (RH): $\mathrm{Re}(\rho) = \beta = \frac{1}{2}$ for all ρ.

Some Conjectures on the Zeros

Some Conjectures on the Zeros

- **Montgomery's Pair Correlation Conjecture**

$$\frac{1}{N(T)} \sum_{\frac{2\pi\alpha}{\log T} \le \gamma - \gamma' \le \frac{2\pi\beta}{\log T}} 1 \sim \int_{\alpha}^{\beta} \left(1 - \frac{\sin^2(\pi x)}{(\pi x)^2} + \delta_0\right) dx$$

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- **Zero-Spacing Hypothesis** (Assuming RH)
  Let $0 < \alpha \le 1$ be fixed.

  $$\limsup_{T \to \infty} \frac{1}{N(T)} \# \left\{ 0 < \gamma_n \le T : 0 \le \gamma_{n+1} - \gamma_n \le \frac{c}{\log T} \right\} \ll c^{\alpha}$$

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- **Simplicity Conjecture**
  All zeros are simple, i.e. $m(\rho) = 1$ for all ρ.

Riemann-von Mangoldt formula:

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iT\right) + \frac{7}{8} + O\left(\frac{1}{T}\right)$$

Riemann-von Mangoldt formula:

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{1}{\pi} \arg \zeta\Big(\frac{1}{2} + iT\Big) + \frac{7}{8} + O\Big(\frac{1}{T}\Big)$$

Define

$$S(t) = \frac{1}{\pi} \arg \zeta\Big(\frac{1}{2} + it\Big).$$

This is defined via continuous variation over the union of line segments one from 2 to $2 + it$, and the other from $2 + it$ to $\frac{1}{2} + it$ unless t is the ordinate of a nontrivial zero.

Central Limit Theorem

Central Limit Theorem

Let $X_1, X_2, \dots$ be a sequence of bounded random variables with respective means $\mathbb{E}X_1, \mathbb{E}X_2, \dots,$ and variances $\mathrm{Var}(X_1), \mathrm{Var}(X_2), \dots$.
If $\sum_{i=1}^{n} \mathrm{Var}(X_i) \to \infty$ as $n \to \infty$, then

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_1 - \mathbb{E}X_1 + \cdots + X_n - \mathbb{E}X_n}{\sqrt{\mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)}} \in [a, b] \right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx \, .$$

• The value on the left-hand side is the value over $[a, b]$ of the distribution function of the standard Gaussian random variable.

Selberg's central limit theorem

Theorem (Selberg, 1946)

For $k \in \mathbb{Z}^+$,

$$\int_0^T S(t)^{2k}\,dt = \frac{(2k)!}{k!\,(2\pi)^{2k}}\,T(\log\log T)^k + O_k\big(T(\log\log T)^{k-1/2}\big).$$

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- Moments of a random variable Z that has Gaussian distribution of mean 0 and variance $\frac{1}{2}\log\log T$ are given as

$$\mathbb{E}[Z^K] = \begin{cases} \frac{(2k)!}{k!\,2^{2k}}(\log\log T)^k & K\text{:even and } K = 2k, \\ 0 & K\text{:odd.} \end{cases}$$

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- Distribution is completely determined by moments in the case of Gaussian distribution.

Selberg's central limit theorem

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Theorem (From Tsang's thesis, 1984)

$$\frac{1}{T}\mu\left\{0 < t < T : \frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}} \in [a,b]\right\} = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\,dx + O\left(\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right)$$

$$\frac{1}{T}\mu\left\{0 < t < T : \frac{\arg\zeta(\frac{1}{2}+it)}{\sqrt{\frac{1}{2}\log\log T}} \in [a,b]\right\} = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\,dx + O\left(\frac{\log\log\log T}{\sqrt{\log\log T}}\right)$$

Selberg's central limit theorem

Theorem

Let $\mathcal{R} \subset \mathbb{C}$ be a rectangle with sides parallel to the coordinate axes.

$$\frac{1}{T}\mu\Big\{0 < t < T : \frac{\log\zeta(\frac{1}{2}+it)}{\sqrt{\log\log T}} \in \mathcal{R}\Big\} = \frac{1}{2\pi}\int_{\mathcal{R}} e^{-(x^2+y^2)/2}\,dx\,dy + O\left(\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right)$$

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- $S(t) = O\left(\frac{\log t}{\log \log t}\right)$ on RH (Littlewood, 1924)

Here, $f(t) = O(g(t))$ as $t \to \infty$ means that for some constant $C > 0$, we have $|f(t)| \leq C|g(t)|$ for all sufficiently large t.

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- $S(t) = \Omega_{\pm}\left(\frac{(\log t)^{1/3}}{(\log \log t)^{1/3}}\right)$ unconditionally (Tsang, 1986)
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- On RH,

$$S(t) = \Omega_{\pm}\left(\frac{\sqrt{\log t}\sqrt{\log\log\log t}}{\sqrt{\log\log t}}\right)$$

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| = \Omega\left(\exp\left((1+o(1))\frac{\sqrt{\log t}\sqrt{\log\log\log t}}{\sqrt{\log\log t}}\right)\right)$$

(Bondarenko and Seip, 2018)

Maximum of $\log \zeta\left(\frac{1}{2}+it\right)$

Let $\epsilon > 0$ be close to 0 and $0 < t < T$.

Maximum of $\log \zeta\left(\frac{1}{2} + it\right)$

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$$S(t) > \epsilon \sqrt{\frac{1}{2} \log \log T}$$

with probability of almost

$$\frac{1}{\sqrt{2\pi}} \int_{\epsilon}^{\infty} e^{-x^2/2}\, dx,$$

so almost half of the time.

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so almost half of the time. Similar result holds for

$$S(t) < -\epsilon\sqrt{\frac{1}{2}\log\log T}.$$

Thus, almost all the time

$$|S(t)| > \epsilon\sqrt{\frac{1}{2}\log\log T}.$$

Conjectures on the maximum

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Conjecture (Gonek, Farmer, Hughes, 2007)

$$\max_{0\le t\le T} S(t) = \left(\frac{1}{\pi\sqrt{2}} + o(1)\right)\sqrt{\log T}\sqrt{\log\log T}$$

and

$$\max_{0\le t\le T} \log\left|\zeta\left(\frac{1}{2}+it\right)\right| = \left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\log T}\sqrt{\log\log T}$$

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Conjecture (By ?, in ?)

$$\max_{0\le t\le T} S(t) = (C + o(1))\frac{\log T}{\log\log T}$$

and

$$\max_{0\le t\le T} \log\left|\zeta\left(\frac{1}{2}+it\right)\right| = (C + o(1))\frac{\log T}{\log\log T}$$

Remarks about the proof of Selberg's central limit theorem

Method of Moments

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Step 1. Approximate $S(t)$ by the imaginary part of a short polynomial

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$$\int_0^T \Big(\operatorname{Im} \sum_{p \le X^2} \frac{1}{p^{1/2+it}} \Big)^{2k} dt$$

$$= \frac{1}{(2i)^{2k}} \sum_{j=0}^{2k} \binom{2k}{j} \left\{ \int_0^T \Big(\sum_{p \le X^2} \frac{1}{p^{1/2+it}} \Big)^j \Big(\overline{\sum_{p \le X^2} \frac{-1}{p^{1/2+it}}} \Big)^{2k-j} \right\} dt$$

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Note. It can be shown that these moments approximate moments of $S(t)$.

Let $s = \sigma + it$. Start with the logarithmic derivative of $\zeta(s)$.

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \quad \text{for} \quad \sigma > 1.$$

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For convenience, we assume RH.

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Let $s = \sigma + it$ be other than $1, \rho, -2n$ and $X \le t^2$.

Selberg:

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n \le X^2} \frac{\Lambda_X(n)}{n^s} + \frac{1}{\log X} \sum_{\rho} \frac{X^{\rho - s} - X^{2(\rho - s)}}{(\rho - s)^2} + \frac{1}{\log X} \sum_{n=1}^{\infty} \frac{X^{-2n-s} - X^{2(-2n-s)}}{(2n+s)^2} + \frac{X^{2(1-s)} - X^{1-s}}{(1-s)^2 \log X}$$

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This follows from a Perron type formula for $c = \max\{2, 1 + \sigma\}$

$$\frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \frac{\zeta'}{\zeta}(w) \frac{x^{w-s} - x^{2(w-s)}}{(w-s)^2} \, dw \, .$$

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$$+ \frac{1}{\log X} \sum_{n=1}^{\infty} \frac{X^{-2n-s} - X^{2(-2n-s)}}{(2n+s)^2} + \frac{X^{2(1-s)} - X^{1-s}}{(1-s)^2 \log X}$$

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(Compare this with $\sum_{n \le X} \frac{\Lambda(n)}{n^s} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta'}{\zeta}(w) \frac{x^{w-s}}{w-s}\, dw\,.$)

For $s \neq 1, \rho, -2n$,

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) + O\big(\log t\big).$$

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For $\sigma \geq \sigma_1 = \frac{1}{2} + \frac{1}{\log X}$,

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n \leq X^2} \frac{\Lambda_X(n)}{n^s}$$

$$+ c\, X^{1/2-\sigma} \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{(\sigma_1 - \frac{1}{2})^2 + (t-\gamma)^2} + O\big(X^{1/2-\sigma}\big).$$

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By comparing these two formulas at $\sigma = \sigma_1$,

$$\sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{(\sigma_1 - \frac{1}{2})^2 + (t-\gamma)^2} = O\left(\Big| \sum_{n \leq X^2} \frac{\Lambda_X(n)}{n^{\sigma_1 + it}} \Big| \right) + O(\log t).$$

The sum over the zeros is decreasing in $\sigma - \frac{1}{2}$. For $\sigma \geq \sigma_1$,

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When $\frac{1}{2} \leq \sigma < \sigma_1$, use

$$\begin{aligned}
&\operatorname{Im}\left\{\frac{\zeta'}{\zeta}(\sigma_1 + it) - \frac{\zeta'}{\zeta}(s)\right\} \\
&= \operatorname{Im} \sum_{\gamma} \left(\frac{1}{\sigma_1 + it - \rho} - \frac{1}{\sigma + it - \rho}\right) + O(\log t) \\
&= \sum_{\gamma} \frac{(t-\gamma)\{(\sigma_1 - \frac{1}{2})^2 - (\sigma - \frac{1}{2})^2\}}{((\sigma - \frac{1}{2})^2 + (t-\gamma)^2)(\sigma_1 - \frac{1}{2})^2 + (t-\gamma)^2)} + O(\log t)
\end{aligned}$$

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S(t) &= -\frac{1}{\pi}\int_{1/2}^{\infty} \operatorname{Im}\frac{\zeta'}{\zeta}(\sigma+it)\,d\sigma \\
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Substitute all the expressions we previously obtained:

$$
S(t) = \frac{1}{\pi}\operatorname{Im}\sum_{n\le X^2}\frac{\Lambda_X(n)}{n^s\log n} + O\bigg(\frac{1}{\log X}\Big|\sum_{n\le X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\Big|\bigg) + O\Big(\frac{\log t}{\log X}\Big)
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Taking a step further, we can prove that

$$S(t) \approx \frac{1}{\pi} \operatorname{Im} \sum_{p \le X^2} \frac{1}{p^{1/2+it}}$$

with a small error.

Note.1

Selberg avoids the assumption of RH by using

$$\Lambda_X(n) = \begin{cases} \Lambda(n) & n \le X, \\ \Lambda(n)\frac{\log^2(X^3/n) - 2\log^2(X^2/n)}{2\log^2 X} & X \le n \le X^2, \\ \Lambda(n)\frac{\log^2(X^3/n)}{2\log^2 X} & X^2 \le n \le X^3. \end{cases}$$

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This comes from

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n \le X^3} \frac{\Lambda_X(n)}{n^s} + \frac{1}{\log^2 X} \sum_{\rho} \frac{X^{\rho-s}(1 - X^{\rho-s})^2}{(\rho - s)^3}$$
$$+ \frac{1}{\log^2 X} \sum_{n=1}^{\infty} \frac{X^{-2n-s}(1 - X^{-2n-s})^2}{(2n+s)^3} + \frac{X^{1-s}(1 - X^{1-s})^2}{(1-s)^3 \log^2 X}$$

by considering $\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta'}{\zeta}(w) \frac{x^{w-s}(1-x^{w-s})^2}{(w-s)^3 \log^2 X}\, dw$.

Note.2

Note.2

Selberg avoids the assumption of RH by using

$$\sigma_1 = \sigma_{X,t} = \frac{1}{2} + \max_{\rho=\beta+i\gamma} \left(\beta - \frac{1}{2}, \frac{2}{\log X} \right),$$

where γ of ρ satisfies

$$|\gamma - t| \leq \frac{X^{3\left(\beta - \frac{1}{2}\right)}}{\log X}.$$

This ensures that $\sigma_1 + it$ lies sufficiently away from any zero ρ.

Moments of the polynomial

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$$\int_0^T \Big(\operatorname{Im} \sum_{p \le X^2} \frac{1}{p^{1/2+it}} \Big)^{2k} \, dt$$

$$= \frac{1}{(2i)^{2k}} \sum_{j=0}^{2k} \binom{2k}{j} \int_0^T \Big(\sum_{p \le X^2} \frac{1}{p^{1/2+it}} \Big)^{j} \Big(\overline{\sum_{p \le X^2} \frac{-1}{p^{1/2+it}}} \Big)^{2k-j} \, dt \, .$$

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Here the jth term includes

$$\sum_{p_i, q_i \le X^2} \frac{1}{\sqrt{p_1 \dots p_j q_1 \dots q_{2k-j}}} \int_0^T \Big(\frac{q_1 \dots q_{2k-j}}{p_1 \dots p_j} \Big)^{it} dt\,.$$

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The primes need to 'pair up' for the maximum contribution. This happens at the middle term $j = k$ and when $p_i = q_i$ with the p_i all distinct.

This diagonal term gives

$$\sum_{\substack{p_i \le X^2, \\ p_i \text{distinct}}} \frac{1}{p_1 \dots p_k} = \Big(\sum_{p \le X^2} \frac{1}{p} \Big)^k + O_k \bigg(\Big(\sum_{p \le X^2} \frac{1}{p} \Big)^{k-1} \bigg)$$

$$\approx (\log \log X)^k \approx (\log \log T)^k.$$

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The coefficients

$$\frac{(2k)!}{2^{2k} k!}$$

result from counting the primes $p_1, \dots, p_j, q_1, \dots, q_{2k-j}$ that pair up. The $(2k)$th moment of a standard Gaussian random variable is

$$\frac{(2k)!}{2^k k!}.$$

Remarks, Analogues and An Application

How sharp is Selberg's CLT?

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Theorem

$$\frac{1}{T}\mu\left\{0<t<T : \frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}} \in [a,b]\right\} = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\,dx + O\left(\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right)$$

$$\frac{1}{T}\mu\left\{0<t<T : \frac{\arg\zeta(\frac{1}{2}+it)}{\sqrt{\frac{1}{2}\log\log T}} \in [a,b]\right\} = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\,dx + O\left(\frac{\log\log\log T}{\sqrt{\log\log T}}\right)$$

Let $(\theta_p)_p$ denote a collection of i.i.d. random variables that are uniformly distributed over the unit interval.

$$\mathcal{P}(\underline{\theta}) = \sum_{p \leq X^2} \frac{e^{2\pi i \theta_p}}{\sqrt{p}}$$

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Note that as a sum of random variables, $\mathrm{Im}\mathcal{P}(\underline{\theta})$ satisfies

$$\mathrm{Var}(\mathrm{Im}\mathcal{P}(\underline{\theta})) = \frac{1}{2} \sum_{p \le X^2} \frac{1}{p} \approx \frac{1}{2} \log \log X.$$

This tends to ∞ as $X \to \infty$.

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Theorem (Central limit theorem)

$$\mathbb{P}\Big(\frac{\mathrm{Im}(\mathcal{P}(\underline{\theta}))}{\sqrt{\frac{1}{2}\log\log X}} \in [a, b] \Big) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx + O\Big(\frac{1}{\sqrt{\log\log X}} \Big)$$

Central Limit Theorem

Let $X_1, X_2, \ldots$ be a sequence of bounded random variables with respective means $\mathbb{E}X_1, \mathbb{E}X_2, \ldots,$ and variances $\mathrm{Var}(X_1), \mathrm{Var}(X_2), \ldots$.
If $\sum_{i=1}^{n} \mathrm{Var}(X_i) \to \infty$ as $n \to \infty$, then

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_1 - \mathbb{E}X_1 + \cdots + X_n - \mathbb{E}X_n}{\sqrt{\mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)}} \in [a, b] \right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx\,.$$

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- We thus have no central limit theorem for

$$\sum_{p \le X^2} \frac{e^{2\pi i \theta_p}}{p^\sigma} \quad \text{when} \quad \sigma > \frac{1}{2}.$$

Bohr & Jessen

For $\frac{1}{2} < \sigma \leq 1$ fixed and $\mathcal{R}$ a rectangle in $\mathbb{C}$ with sides parallel to the coordinate axes, as $T \to \infty$

$$\frac{1}{T}\mu\Big\{T < t \leq 2T : \log\zeta(\sigma + it) \in \mathcal{R}\Big\} \to \mathbb{F}_\sigma(z \in \mathcal{R})$$

for a probability distribution function $\mathbb{F}_\sigma$.

Other central limit theorems

Example.

Example. Let χ be a primitive Dirichlet character modulo q and

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad \text{for } \operatorname{Re}(s) > 1.$$

We have

$$\begin{aligned} &\frac{1}{T}\mu\left\{0 < t < T : \frac{\log |L(\frac{1}{2}+it,\chi)|}{\sqrt{\frac{1}{2}\log\log T}} \in [a,b]\right\} \\ &\qquad = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx + O\left(\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right). \end{aligned}$$

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Selberg. Then 2020, Hsu and Wong: A linear combination of type

$$a_1 \log\left|L\left(\frac{1}{2}+it,\chi_1\right)\right| + \cdots + a_n \log\left|L\left(\frac{1}{2}+it,\chi_n\right)\right|$$

has an approximate Gaussian distribution with mean 0 and variance $\frac{1}{2}(a_1^2 + \cdots + a_n^2)\log\log T$.

A q-analogue of Selberg's theorem

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Selberg: For $|t| \leq q^{\frac{1}{4}-\epsilon}$,

$$\sideset{}{'}\sum_{\chi \pmod{q}} S(t,\chi)^{2k} = \frac{(2k)!}{(2\pi)^{2k} k!} \phi(q)(\log\log q)^k + O_k\left(\phi(q)(\log\log q)^{k-1}\right).$$

A CLT over arithmetic progressions

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Let $\alpha > 0$ and $\beta \in \mathbb{R}$.
Consider the arithmetic progression $\{\alpha n + \beta\}$ as $n \in \mathbb{Z}$.

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Li & Radziwiłł, 2012

On the Riemann hypothesis (RH), the asymptotic behavior of $\arg\zeta(\frac{1}{2} + i(\alpha n + \beta))$ over $n \in (T, 2T]$ can be described by a central limit theorem as $T \to \infty$.

A CLT over shifted Gram points

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Define a real-valued function $Z(t)$ from

$$\zeta(\tfrac{1}{2} + it) = Z(t)e^{-i\theta(t)}.$$

Gram points are of the form $\frac{1}{2} + it$ for t satisfying $\theta(t) \equiv 0 \pmod{\pi}$.

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$\frac{1}{2} + ig_n$ is a shifted Gram point if for $-\pi < \phi \leq \pi$,

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Lester, 2013

On the assumption of a zero-spacing hypothesis,

$$\lim_{T\to\infty} \frac{1}{N_g(T)} \#\left\{ T < g_n \leq 2T : \frac{\log|\zeta(\frac{1}{2} + ig_n)|}{\sqrt{\frac{1}{2}\log\log T}} \in [a, b] \right\}$$
$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx$$

A CLT over nontrivial zeros

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C., 2020

Assume RH. Let $z = u + iv$ be nonzero with $0 < u \ll \frac{1}{\log T}$ and $|v| \ll \frac{1}{\log T}$. Then

$$\frac{1}{N(T)} \# \left\{ 0 < \gamma \le T : \frac{\arg \zeta(\rho + z)}{\sqrt{\frac{1}{2} \log \log T}} \in [a, b] \right\}$$
$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx + O\left(\frac{\log \log \log T}{\sqrt{\log \log T}} \right).$$

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Assume RH and Montgomery's Pair Correlation Conjecture. Let $z = u + iv$ be nonzero with $0 < u \ll \frac{1}{\log T}$ and $|v| \ll \frac{1}{\log T}$. Then

$$\frac{1}{N(T)}\#\left\{0 < \gamma \leq T : \frac{\log|\zeta(\rho+z)| - M(\rho,z)}{\sqrt{\frac{1}{2}\log\log T}} \in [a,b]\right\} = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\,dx + O\left(\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right),$$

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$$M(\rho,z) = m(\rho+iv)\left(\log\left(\frac{eu\log T}{4}\right) - \frac{u\log T}{4}\right).$$

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This is uniform in u. Can take $v = 0$ and let $u \to 0$.

Corollary (C., 2020)

Assume that all the zeros ρ are simple. Under RH and Montgomery's Pair Correlation Conjecture,

$$\frac{1}{N(T)}\#\left\{0<\gamma\le T : \frac{\log\left(|\zeta'(\rho)|/\log T\right)}{\sqrt{\frac{1}{2}\log\log T}} \in [a,b]\right\}$$
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- About $\arg\zeta'(\rho)$. (Stopple, 2020)

Stopple, Notes on the Phase Statistics of the Riemann Zeros, [arXiv: 2007.08008]

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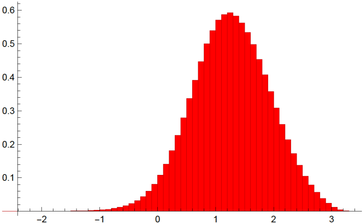

FIGURE 2. $\log|\zeta'(\rho)|$.

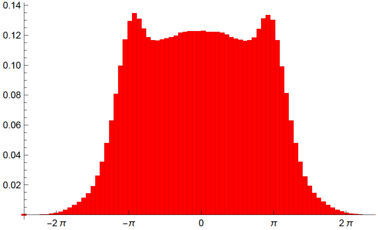

FIGURE 3. $\arg(\zeta'(\rho))$.

Both histograms use the γ_n with $5 \cdot 10^6 \leq n \leq 10^7$.

Theorem (C., 2021)

Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$ *and*

$$\mathcal{L}(\rho) = a_1 \log|L(\rho, \chi_1)| + \cdots + a_n \log|L(\rho, \chi_n)|$$

Here $\chi_1, \ldots, \chi_n$ *are distinct primitive Dirichlet characters with conductors bounded by* T*. Assume the generalized RH and that* $L(\rho, \chi_j)$ *is never* 0 *for each* j*. Further, suppose that for each* $1 \le j \le n$*, Hypothesis* $\mathscr{H}_{\alpha,\chi_j}$ *is true for some* $\alpha \in (0,1]$*. For* $A < B$*, we have*

$$\frac{1}{N(T)} \#\left\{0 < \gamma \le T : \frac{\mathcal{L}(\rho)}{\sqrt{\left(\frac{1}{2}\sum_{j=1}^{n} {a_j}^2\right) \log\log T}} \in [A, B]\right\} = \frac{1}{\sqrt{2\pi}} \int_A^B e^{-x^2/2}\, dx + O\left(\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right).$$

Zeros of linear combinations of Dirichlet L-functions

ZEROS OF LINEAR COMBINATIONS OF DIRICHLET L-FUNCTIONS

Consider the function

$$F(s) = c_1 L(s,\chi_1) + \cdots + c_n L(s,\chi_n).$$

for primitive L-functions $L(s,\chi_1),\ldots,L(s,\chi_n)$ with the same conductor. Selberg's theorem also holds for these Dirichlet L-functions.

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- **Bombieri & Hejhal, 1995** Suppose that the L_j satisfy the GRH and a zero-spacing hypothesis. Then 100% of the zeros of F are simple and are on the critical line.

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- **Bombieri & Hejhal, 1995** Suppose that the L_j satisfy the GRH and a zero-spacing hypothesis. Then 100% of the zeros of F are simple and are on the critical line.
- **Selberg, 1998** Unconditionally, a positive proportion of the zeros of $F(s)$ lie on the critical line.

$$F(s) = c_1 L(s, \chi_1) + \cdots + c_n L(s, \chi_n)$$

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(Montgomery)

The value distribution of $F\left(\frac{1}{2} + it\right)$ can be studied by using the central limit theorems for the $\frac{\log\left|L\left(\frac{1}{2}+it,\chi_j\right)\right|}{\sqrt{\frac{1}{2}\log\log T}}$ because:

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- $\left\{\frac{\log\left|L\left(\frac{1}{2}+it,\chi_j\right)\right|}{\sqrt{\frac{1}{2}\log\log T}}\right\}_{j=1}^{n}$ have independent Gaussian distributions.

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- $\left\{\frac{\log\left|L\left(\frac{1}{2}+it,\chi_j\right)\right|}{\sqrt{\frac{1}{2}\log\log T}}\right\}_{j=1}^{n}$ have independent Gaussian distributions.
- $\log\left|L\left(\frac{1}{2}+it,\chi_j\right)\right| - \log\left|L\left(\frac{1}{2}+it,\chi_k\right)\right|$ has a Gaussian distribution with mean 0 and variance $\log\log T$. Therefore, it is large most of the time.

Thank you!

Beurling-Selberg Functions

$$\operatorname{sgn}(x) = F_\Delta(x) + O\left(\frac{\sin^2(\pi\Delta x)}{(\pi\Delta x)^2}\right),$$

for

$$F_\Delta(x) = \operatorname{Im}\int_0^\Delta \left(\frac{2}{\pi\Delta}u + 2\left(1 - \frac{u}{\Delta}\right)\left(\frac{\pi}{\Delta}u\right)\right) e^{2\pi i x u}\frac{du}{u}.$$

- Using this, one can write an approximation for

$$\mathbb{1}_{[a,b]}(x) = \frac{\operatorname{sgn}(x-a)}{2} - \frac{\operatorname{sgn}(x-b)}{2} + \frac{\delta_a(x)}{2} + \frac{\delta_b(x)}{2}.$$

- Let $x = \operatorname{Im}\sum_{p \le X^2} \frac{1}{p^{1/2+it}}$.

Majorants and Minorants

There are nice Beurling-Selberg functions $F_-(x)$ and $F_+(x)$ such that

$$F_-(x) \leq \mathbb{1}_{[-\Delta,\Delta]}(x) \leq F_+(x).$$

2005, Goldston & Gonek: On RH,

$$|S(t)| \leq \left(\frac{1}{2} + o(1)\right) \frac{\log t}{\log\log t}.$$

Succeeding work:

- 2009, Chandee & Soundararajan: On RH,

$$\log\left|\zeta\left(\frac{1}{2} + it\right)\right| \leq \left(\frac{\log 2}{2} + o(1)\right) \frac{\log t}{\log\log t}.$$

- 2013, Chandee, Carneiro & Milinovich: On RH,

$$|S(t)| \leq \left(\frac{1}{4} + o(1)\right) \frac{\log t}{\log\log t}.$$

Example. Chandee & Soundararajan: On RH,

$$\log \left|\zeta\left(\frac{1}{2}+it\right)\right| \leq \left(\frac{\log 2}{2}+o(1)\right)\frac{\log t}{\log\log t}.$$

- $\log\left|\zeta\left(\frac{1}{2}+it\right)\right| = \log t + O(1) - \frac{1}{2}\sum_{\gamma}\log\frac{4+(t-\gamma)^2}{(t-\gamma)^2}$
- Let $f(x)=\log\frac{4+x^2}{x^2}$. There is an entire function $g_\Delta(x)$ with $\hat{g}_\Delta(x)$ having compact support and
$$f(x) \geq g_\Delta(x).$$
- Compute $\sum_{\gamma} g_\Delta(t-\gamma)$ by using an explicit formula that is applicable to nice functions.