Periods of 1-motives and their polynomial relations

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Elliptic curves

$\mathcal{E}/\mathbb{C}$ elliptic curve, i.e. plane curve whose points are solutions of the equation

$$y^2 = x^3 + ax + b \qquad a, b \in \mathbb{C}$$

The Weierstrass $\wp$-function

$$\wp(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

(meromorphic function on $\mathbb{C}$ having a double pole with residue zero at each point of $\Lambda \cong \mathrm{H}_1(\mathcal{E}(\mathbb{C}), \mathbb{Z})$ and no other poles)

The exponential map

$$\begin{aligned} \exp_{\mathcal{E}} : \mathbb{C} &\longrightarrow \mathcal{E}(\mathbb{C}) \subseteq \mathbb{P}^2(\mathbb{C}) \\ z &\longmapsto \exp_{\mathcal{E}}(z) = [\wp(z), \wp(z)', 1] \end{aligned}$$

In particular $\mathcal{E}(\mathbb{C}) \cong \mathbb{C}/\Lambda$.

Elliptic functions

The Weierstrass σ-function

$$\sigma(z) := z \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right) e^{z/\lambda + \frac{1}{2}(z/\lambda)^2}$$

(holomorphic function on all of $\mathbb{C}$)

The Weierstrass ζ-function

$$\zeta(z) := \frac{1}{z} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{z - \lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2}\right).$$

(meromorphic function on $\mathbb{C}$ with simple poles at each point of Λ and no other poles)

$$\frac{d}{dz}\zeta(z) = -\wp(z)$$

Differentials on an elliptic curves

The kind of a differential

A meromorphic differential 1-form is

- of the first kind (I) if it is holomorphic everywhere,
- of the second kind (II) if the residue at any pole vanishes, and
- of the third kind (III) in general.

We have $(I) \subset (II) \subset (III)$.

On the elliptic curve $\mathcal{E}$ we have the following differential 1-forms:

1. the first kind $\omega = \frac{dx}{y}$ and $\exp_{\mathcal{E}}^*(\omega) = dz$.
2. the second kind $\eta = \frac{xdx}{y}$ and $\exp_{\mathcal{E}}^*(\eta) = \wp(z)dz$
3. the third kind
$$\xi_Q = \frac{1}{2}\frac{y - y(Q)}{x - x(Q)}\frac{dx}{y}$$
for any point Q of $\mathcal{E}(\mathbb{C})$, $Q \neq 0$.

Serre's function

J.-P. Serre introduced the function

$$f_q(z) = \frac{\sigma(z+q)}{\sigma(z)\sigma(q)}\ e^{-\zeta(q)z} \qquad \text{with } q \in \mathbb{C} \setminus \Lambda$$

whose logarithmic differential is

$$\frac{f_q'(z)}{f_q(z)} dz = \frac{1}{2} \frac{\wp'(z) - \wp'(q)}{\wp(z) - \wp(q)} dz = \exp^*_{\mathcal{E}}(\xi_Q)$$

where $q \in \mathbb{C}$ is an elliptic logarithm of the point Q (that is $\exp_{\mathcal{E}}(q) = Q$).

Elliptic Integrals and periods

Let γ_1, γ_2 be a basis for $\mathrm{H}_1(\mathcal{E}(\mathbb{C}), \mathbb{Z})$. Remark that

1. the elliptic integrals of the first kind $\int_{\gamma_i} \omega = \omega_i$ $(i = 1, 2)$ are the periods of the Weierstrass $\wp$-function:

$$\wp(z + \omega_i) = \wp(z) \qquad \text{for } i = 1, 2.$$

2. the elliptic integrals of the second kind $\int_{\gamma_i} \eta = \eta_i$ $(i = 1, 2)$ are the quasi-periods of the Weierstrass ζ-function:

$$\zeta(z + \omega_i) = \zeta(z) + \eta_i \qquad \text{for } i = 1, 2.$$

3. the *exponentials* of the elliptic integrals of the third kind $\int_{\gamma_i} \xi_Q = \eta_i q - \omega_i \zeta(q)$ $(i = 1, 2)$ are the quasi-quasi periods of the function $f_q(z)$:

$$f_q(z + \omega_i) = f_q(z)\ e^{\eta_i q - \omega_i \zeta(q)} \qquad \text{for } i = 1, 2.$$

1-motives

Definition

A 1-motive $M = [u : \mathbb{Z} \to G]$ over $\overline{\mathbb{Q}}$ consists of

- an extension $0 \to \mathbb{G}_m \to G \to \mathcal{E} \to 0$ defined over $\overline{\mathbb{Q}}$
- a morphism $u : \mathbb{Z} \to G, u(1) = R \in G(\overline{\mathbb{Q}})$.

Via the isomorphism $\mathcal{E}^* \cong \underline{\mathrm{Ext}}^1(\mathcal{E}, \mathbb{G}_m)$, to have an extension G of $\mathcal{E}$ by $\mathbb{G}_m$ defined over $\overline{\mathbb{Q}}$ is equivalent to have a point Q of $\mathcal{E}^*(\overline{\mathbb{Q}})$.

Examples of 1-motives $M = [u : \mathbb{Z} \to G]$

- $M = [0 : 0 \to \mathcal{E}] = \mathcal{E}$
- $M = [0 : 0 \to \mathbb{G}_m] = \mathbb{G}_m$
- $M = [u : \mathbb{Z} \to \mathcal{E}], u(1) = P \in \mathcal{E}$
- $M = [u : \mathbb{Z} \to \mathbb{G}_m], u(1) = S \in \mathbb{G}_m$

In order to define a 1-motive $M = [u : \mathbb{Z} \to G]$, I need

- an elliptic curve $\mathcal{E}$,
- a point $Q \in \mathcal{E}^*$ which gives the extension G of $\mathcal{E}$ by $\mathbb{G}_m$, i.e. $0 \to \mathbb{G}_m \to G \to \mathcal{E} \to 0$ is exact,
- a point $P \in \mathcal{E}$,
- a lifting $R \in G$ of the point P via $G \to P$.

Differential forms and paths on 1-motives

Consider the 1-motive $M = [u : \mathbb{Z} \to G]$.

Differential forms on M

Basis for the de Rham realization $\mathrm{H}_{\mathrm{dR}}(M)$ of M is

$$\left\{ df,\ \omega = \frac{dx}{y},\ \eta = \frac{xdx}{y},\ \xi_Q = \frac{1}{2}\frac{y - y(Q)}{x - x(Q)}\frac{dx}{y} \right\}$$

where df is an exact form on G such that $f(R) - f(O) = 1$.

Paths on M

Basis for the Hodge realization $T_{\mathrm{H}}(M)$ of M is

$$\left\{ \beta_R,\ \tilde{\gamma}_1,\ \tilde{\gamma}_2,\ \delta_Q \right\}$$

where β_R is a path from O to R on G, $\tilde{\gamma}_1, \tilde{\gamma}_2$ lift the basis γ_1, γ_2, δ_Q is a closed path in $\mathbb{G}_m$ such that $\int_{\delta_Q} \xi_Q^G = 2\mathrm{i}\pi$,

Periods of 1-motives

Deligne showed that the integration of differentials forms gives a canonical isomorphism

$$\mathrm{H}_{\mathrm{dR}}(M) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \longrightarrow \mathrm{Hom}(T_{\mathrm{H}}(M), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$
$$\omega \longmapsto [\gamma \mapsto \int_\gamma \omega]$$

Periods of M

The periods of M are the coefficients of a matrix which represents this isomorphism with respect to $\overline{\mathbb{Q}}$-bases.

Matrix of periods of M is

$$\begin{pmatrix} \int_{\beta_R} \xi_Q & \int_{\beta_R} \omega & \int_{\beta_R} \eta & \int_{\beta_R} df \\ \int_{\tilde{\gamma}_1} \xi_Q & \int_{\gamma_1} \omega & \int_{\gamma_1} \eta & 0 \\ \int_{\tilde{\gamma}_2} \xi_Q & \int_{\gamma_2} \omega & \int_{\gamma_2} \eta & 0 \\ \int_{\delta_Q} \xi_Q & 0 & 0 & 0 \end{pmatrix}$$

Explicitly ...

Matrix of periods of M becomes

$$\begin{pmatrix} \log f_q(p) - l & p & \zeta(p) & 1 \\ \eta_1 q - \omega_1 \zeta(q) & \omega_1 & \eta_1 & 0 \\ \eta_2 q - \omega_2 \zeta(q) & \omega_2 & \eta_2 & 0 \\ 2\mathrm{i}\pi & 0 & 0 & 0 \end{pmatrix}$$

where $\exp_{\mathcal{E}}(p) = P$ is the projection of R via $G \to \mathcal{E}$ and $\exp_G(l, p) = R$.

Conjecture of Periods

According to Grothendieck, any polynomial relation with rational coefficients between the periods of an abelian variety A should have *a geometrical origin*,

that is *the existence of algebraic cycles on A and on the products of A with itself*, should affect the transcendence degree of the field generated by the periods of A.

Grothendieck's conjecture of periods

Let M be a motive defined over $\overline{\mathbb{Q}}$, then

$$\mathrm{transc.deg.}_{\mathbb{Q}}\, \overline{\mathbb{Q}}(\mathrm{periods}(M)) = \dim \mathrm{MT}(M)$$

where $\mathrm{MT}(M) \subseteq \mathrm{GL}(4, \mathbb{Q})$ is the Mumford-Tate group of M.

known case

This conjecture is an hard conjecture which is still wide open.

Only one case has been proved:

Chudnovsky Theorem

$\mathcal{E}$ is an elliptic curve defined over $\overline{\mathbb{Q}}$ with complex multiplication, i.e. $\mathrm{End}(\mathcal{E}) \supset \mathbb{Z}$
$P = 0, Q = 0, R = 0$,
The Conjecture of Periods applied to this 1-motive $M = [0 : 0 \to \mathcal{E}]$ is the Chudnovsky Theorem:

$$\mathrm{tran.deg}_{\mathbb{Q}}\, \overline{\mathbb{Q}}(\omega_1, \omega_2, \eta_1, \eta_2) = 2 = \dim \mathrm{MT}(\mathcal{E}).$$

About the $\mathrm{MT}(M)$

We have the exact sequence

$$0 \longrightarrow \mathrm{UR}(M) \longrightarrow \mathrm{MT}(M) \longrightarrow \mathrm{MT}(\mathcal{E}) \longrightarrow 0$$

where $\mathrm{UR}(M)$ is the unipotent radical of $\mathrm{MT}(M)$ and the Mumford-Tate group $\mathrm{MT}(\mathcal{E}) \subseteq \mathrm{GL}(2,\mathbb{Q})$ of $\mathcal{E}$ is its largest reductive quotient.

In particular $\dim \mathrm{MT}(M) = \dim \mathrm{MT}(\mathcal{E}) + \dim \mathrm{UR}(M)$

We have

$$\dim \mathrm{MT}(\mathcal{E}) = \begin{cases} 2 & \text{if } \mathcal{E} \text{ CM (i.e. } \mathrm{End}(\mathcal{E}) \supset \mathbb{Z}) \\ 4 & \text{if } \mathcal{E} \text{ not CM (i.e. } \mathrm{End}(\mathcal{E}) = \mathbb{Z}) \end{cases}$$

Dimension of $\mathrm{UR}(M)$

We have

$$\dim \mathrm{UR}(M) = 2 \dim B + \dim T$$

1. $B \subseteq \mathcal{E} \times \mathcal{E}^*$ is the smallest abelian subgroup containing the point (P, Q) (modulo isogenies). If $k = \mathrm{End}(\mathcal{E}) \otimes \mathbb{Q}$

$$\dim B = \dim_k kp + kq/(k\omega_1 + k\omega_1)$$

2. $T \subseteq \mathbb{G}_m$ is generated by the image $[B, B]$ of the Lie bracket $[\ ,\ ] : B \times B \to \mathbb{G}_m$ constructed using the Weil pairing ($\mathcal{E} \otimes \mathcal{E}^* \to \mathbb{G}_m$) and l (recall that $\exp_G(l, p) = R$)

Remark : $$[B, B] = 0 \Leftrightarrow \begin{cases} \text{if } Q = \phi(P) \text{ with } \phi \in k \text{ antisymmetric, or} \\ \text{if } P \text{ and/or } Q \text{ are torsion.} \end{cases}$$

Examples of dimension

	dim $UR(\mathrm{MT}(M))$	dim $\mathrm{MT}(M)$ $\mathcal{E}$ CM	dim $\mathrm{MT}(M)$ $\mathcal{E}$ not CM
P,Q k-L.I.	5=4+1	7	9
P,Q k-L.D. with Φ sym.	3= 2+1	5	7
P,Q k-L.D. with Φ antisym.	3= 2+1 2=2+0	5 4	7 6
Q torsion (P and R not torsion)	3= 2+1	5	7
R torsion ($\Rightarrow$ P torsion)	2= 2+0	4	6
P,Q torsion (R not torsion)	1=0+1	3	5
Q, R torsion ($\Rightarrow$ P torsion)	0	2	4

polynomials VS endomophisms

Let $\mathcal{I}$ be the ideal generated by all polynomial relations between periods of M. By Grothendieck's conjecture

Numbers of periods of M - Rank of $\mathcal{I} = \dim \mathrm{MT}(M)$

that is a decrease in the dimension of $\mathrm{MT}(M)$ is equivalent to an increase of the rank of the ideal $\mathcal{I}$.

Relation between polynomials and endomophisms

an increase in the numbers of endomorphisms $\Leftrightarrow$

an decrease in the dimension of $\mathrm{MT}(M)$ $\Leftrightarrow$

an decrease in the transcendental degree of $\overline{\mathbb{Q}}(\mathrm{periods}(M))$ $\Leftrightarrow$

an increase in the numbers of polynomial relations between periods of M.

Elliptic case

Periods of $\mathcal{E}$ are $\omega_1, \omega_2, \eta_1, \eta_2$

" Numbers of periods of $\mathcal{E}$ - Rank of $\mathcal{I} = \dim \mathrm{MT}(\mathcal{E})$", that is
" 4 - $\dim \mathrm{MT}(\mathcal{E}) =$ Rank of $\mathcal{I}$"

Not CM case

- $\dim \mathrm{MT}(\mathcal{E}) = 4$
- Rank of $\mathcal{I} = 0$. No polynomials relations between periods
- a transcendence base of the field generated by periods is $\omega_1, \omega_2, \eta_1, \eta_2$

CM case - Chudnovsky Theorem

- $\dim \mathrm{MT}(\mathcal{E}) = 2$
- Rank of $\mathcal{I} = 2$. Generators are $\omega_2 - \tau\omega_1 = 0, \quad \bar{\tau}\eta_1 - \eta_2 - \frac{\kappa}{\tau}\omega_1 = 0$
- a transcendence base of the field generated by periods is ω_1, η_1

But I have Legendre relation !!

Legendre relation : $\omega_1\eta_2 - \omega_2\eta_1 = 2\mathrm{i}\pi$

Legendre relation is not a polynomial relation between the 1-periods $\omega_1, \omega_2, \eta_1, \eta_2$ of $\mathcal{E}$, since $2\mathrm{i}\pi$ is a **2-period** of $\mathcal{E}$!

0-,1-,2-periods

0-periods of $\mathcal{E} = 1$
1-periods of $\mathcal{E} = \omega_1, \omega_2, \eta_1, \eta_2$
2-periods of $\mathcal{E} = 2\mathrm{i}\pi$

Legendre relation **expresses the 2-period** $2\mathrm{i}\pi$ **as the value of a degree 2 polynomial**

$$X_1 Y_2 - X_2 Y_1$$

evaluated in 1-periods !

When $2i\pi$ is a 1-period ?

Periods of $\mathbb{G}_m \times \mathcal{E}$ are $\omega_1, \omega_2, \eta_1, \eta_2, 2i\pi$

" Numbers of periods of $\mathbb{G}_m \times \mathcal{E}$ - Rank of $\mathcal{I} = \dim \mathrm{MT}(\mathbb{G}_m \times \mathcal{E})$", that is
" 5 - $\dim \mathrm{MT}(\mathbb{G}_m \times \mathcal{E})$ = Rank of $\mathcal{I}$"

Observe that $\dim \mathrm{MT}(\mathbb{G}_m \times \mathcal{E}) = \dim \mathrm{MT}(\mathcal{E})$

Not CM case

- $\dim \mathrm{MT}(\mathbb{G}_m \times \mathcal{E}) = 4$
- Rank of $\mathcal{I} = 1$. Generator is $\omega_1\eta_2 - \omega_2\eta_1 - 2i\pi = 0$
- a transcendence base of the field generated by periods is $\omega_1, \omega_2, \eta_1, \eta_2$

CM case

- $\dim \mathrm{MT}(\mathcal{E}) = 2$
- Rank of $\mathcal{I} = 3$: $\omega_2 - \tau\omega_1 = 0, \quad \bar{\tau}\eta_1 - \eta_2 - \frac{\kappa}{\tau}\omega_1 = 0, \quad \omega_1\eta_2 - \omega_2\eta_1 - 2i\pi = 0$
- a transcendence base of the field generated by periods is ω_1, η_1

Case with R (and so P) and Q torsion

$$\begin{pmatrix} \log f_q(p) - \ell & p & \zeta(p) & 1 \\ \eta_1 q - \omega_1 \zeta(q) & \omega_1 & \eta_1 & 0 \\ \eta_2 q - \omega_2 \zeta(q) & \omega_2 & \eta_2 & 0 \\ 2i\pi & 0 & 0 & 0 \end{pmatrix}$$

" Numbers of periods of M - Rank of $\mathcal{J} = \dim \mathrm{MT}(M)$", that is
" 10 - $\dim \mathrm{MT}(M) =$ Rank of $\mathcal{J}$"

Observe that $\dim \mathrm{MT}(M) = \dim \mathrm{MT}(\mathcal{E})$

CM case

- $\dim \mathrm{MT}(\mathcal{E}) = 2$
- Rank of $\mathcal{J} = 8$
- a transcendence base of the field generated by periods is ω_1, η_1

Generators of $\mathcal{J}$

$$\begin{cases} \omega_2 - \tau\omega_1 = 0, \\ \bar{\tau}\eta_1 - \eta_2 - \frac{\kappa}{\tau}\omega_1 = 0, \\ \omega_1\eta_2 - \omega_2\eta_1 - 2\mathrm{i}\pi = 0, \\ p - \alpha_1\omega_1 - \alpha_2\omega_2 = 0, \\ \zeta(p) - \alpha_1\eta_1 - \alpha_2\eta_2 \in \overline{\mathbb{Q}}, \\ -\omega_2(\eta_1 q - \omega_1\zeta(q)) + \omega_1(\eta_2 q - \omega_2\zeta(q)) - 2i\pi(\beta_1\omega_1 + \beta_2\omega_2) = 0, \\ -\eta_2(\eta_1 q - \omega_1\zeta(q)) + \eta_1(\eta_2 q - \omega_2\zeta(q)) - 2i\pi(\beta_1\eta_1 + \beta_2\eta_2) \in \overline{\mathbb{Q}}, \\ \log f_q(p) - \ell - 2i\pi\gamma = 0. \end{cases}$$

Thanks!