Pólya and pre-Pólya Groups in Dihedral Number Fields

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    1. Integer valued polynomials

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    2. Pólya fields

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   3. Maximum ramification in Pólya D_4-fields

Integer Valued Polynomials in $\mathbb{Q}[X]$

Theorem (Pólya, 1919)

For a polynomial $f(X) \in \mathbb{Q}[X]$, we have $f(\mathbb{Z}) \subseteq \mathbb{Z}$ if and only if it can be written as a $\mathbb{Z}$-linear combination of the polynomials

$$\binom{X}{n} = \frac{X(X-1)(X-2)\cdots(X-n+1)}{n!} \qquad n = 0, 1, 2, \cdots .$$

▸ regular basis

Replace $\mathbb{Q}$ with K

Definition (Ring of integer valued polynomials)

$\mathrm{Int}(\mathscr{O}_K) = \{f \in K[X] \mid f(\mathscr{O}_K) \subseteq \mathscr{O}_K\}$.

$\mathfrak{J}_n(K) = \{\text{leading coefficients of } \quad f(X) \in \mathrm{Int}(\mathscr{O}_K), \deg(f) = n\} \cup \{0\}$.

Proposition

$\mathrm{Int}(\mathscr{O}_K) \simeq \bigoplus_{n=0}^{\infty} \mathfrak{J}_n(K)$ (as $\mathscr{O}_K$-module).

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$\mathrm{Int}(\mathcal{O}_K) \simeq \bigoplus_{n=0}^{\infty} \mathfrak{I}_n(K)$ (as $\mathcal{O}_K$-module).

$\mathrm{Int}(\mathscr{O}_K)$ is a free $\mathscr{O}_K$-module

Definition

If $\mathrm{Int}(\mathscr{O}_K)$ has an $\mathscr{O}_K$ basis with exactly one member from each degree, we say that $\mathrm{Int}(\mathscr{O}_K)$ has a regular basis.

$\mathrm{Int}(\mathcal{O}_K)$ is a free $\mathcal{O}_K$-module

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Example

By Pólya's result, $\{\binom{X}{n}\}_{n\geq 0}$ is a regular basis for $\mathrm{Int}(\mathbb{Z})$.

Go back

Existence of a regular basis

Theorem (Pólya, 1919)

$\mathrm{Int}(\mathscr{O}_\mathrm{K})$ has a regular basis iff all the ideals $\mathfrak{J}_\mathrm{n}(\mathrm{K})$'s are principal.

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Proposition (Bhargava, 1997)

$$n!_{\mathscr{O}_K} = \mathfrak{J}_n(K)^{-1}.$$

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Convention

If p^f is not the norm of any maximal ideal of $\mathscr{O}_K$, put $\Pi_{p^f}(K) = \mathscr{O}_K$.

Definition (Zantema, 1982)

A number field K is called a Pólya field, if the following equivalent conditions hold:

- $\mathrm{Int}(\mathscr{O}_K)$ has a regular basis;
- All the characteristic ideals $\mathfrak{J}_n(K)$'s are principal;
- All the Ostrowski ideals $\Pi_q(K)$'s are principal;
- All the Bhargava factorial ideals of K are principal.

Some Families of Pólya Fields

Example

The following number fields are Pólya fields:

1. Class number one number fields (Obviously, but not conversely!);
2. Cyclotomic fields (Zantema, 1982);
3. Abelian number fields with only one ramified prime (Zantema, 1982);
4. Hilbert class field of a number field (Leriche, 2014).
5. Genus field of an abelian number field (Leriche, 2014).

measure of the obstruction for K to be Pólya

Definition(Cahen-Chabert, 1997)

The Pólya group of K is the subgroup **Po**(K) of the class group **Cl**(K) generated by the classes of the ideals $\Pi_{p^f}(K)$:

$$\mathbf{Po}(K) = < \left[\Pi_{p^f}(K)\right] : p \in \mathbb{P}, f \in \mathbb{N} > .$$

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Remark (Ostrowski)

For a Galois extension $K/\mathbb{Q}$,

$$\mathbf{Po}(K) = < \left[\Pi_{p^f}(K)\right] : p \text{ is a ramified prime in } K, f \in \mathbb{N} > .$$

Non-Galois case

Galois action on ideal class group

Proposition(Brumer-Rosen, 1963)

For a Galois extension $K/\mathbb{Q}$, $\mathscr{P}o(K)$ and the group of strongly ambiguous ideal classes in K, coincide.

Example(Hilbert, 1897)

Let K be a quadratic field and denote the number of ramifid primes in $K/\mathbb{Q}$ by s_K. Then

$$|\mathscr{P}o(K)| = \begin{cases} 2^{s_k - 2} & : K \text{ is real and } N_{K/\mathbb{Q}}(\mathscr{O}_K^\times) = \{1\} \\ 2^{s_k - 1} & : \text{Otherwise} \end{cases}$$

Ramification is very restricted

Proposition (Zantema, 1982)

Let $K/\mathbb{Q}$ be a Galois extension with Galois group G. Then the following sequence is exact:

$$\{0\} \to H^1(G, U_K) \to \bigoplus_{p \text{ prime}} \mathbb{Z}/e(p)\mathbb{Z} \to \mathscr{P}o(K) \to \{0\}$$

where $e(p)$ denotes the ramification index of p in $K/\mathbb{Q}$.

Corollary

For a Galois extension $K/\mathbb{Q}$, if $\gcd(h_K, [K:\mathbb{Q}]) = 1$ then K is Pólya; but not converesely.

relative case | Max ramification in S_3-extensions | Improving Ishida's result

Relative Pólya group

Definition (-, A. Rajaei, 2020 & Chabert, 2019)

For a finite extension K/P of number fields, relative Pólya group $\mathbf{Po}(K/P)$, is defined as follows:

$$\mathbf{Po}(K/P) = < \left[\Pi_{\mathfrak{P}^f}(K/P) = \prod_{\substack{\mathfrak{M} \in \mathrm{Max}(\mathcal{O}_K) \\ N_{K/P}(\mathfrak{M}) = \mathfrak{P}^f}} \mathfrak{M} \right] : \mathfrak{P} \text{ is a prime of } P, f \in \mathbb{N} > .$$

In particular, $\mathbf{Po}(K/\mathbb{Q}) = \mathbf{Po}(K)$ and $\mathbf{Po}(K/K) = \mathbf{Cl}(K)$.

Theorem (-, A. Rajaei, J. Number Theory, 2020)

Let K/P be a finite Galois extension of number fields with Galois group G. Then we have an exact sequence as follows:

$$\{0\} \to \mathrm{Ker}(\varepsilon_{K/P}) \to H^1(G, U_K) \to \bigoplus_{\mathfrak{P}\text{ a prime of } P} \mathbb{Z}/e_{\mathfrak{P}}\mathbb{Z} \to \frac{\mathrm{Po}(K/P)}{\varepsilon_{K/P}(\mathrm{Cl}(P))} \to \{0\},$$

where $\varepsilon_{K/P} : [\mathfrak{a}] \in \mathrm{Cl}(P) \mapsto [\mathfrak{a}\mathscr{O}_K] \in \mathrm{Cl}(K)$ denotes the transfer of ideal classes, and $e_{\mathfrak{P}}$ denotes the ramification index of $\mathfrak{P}$ in K/P. ◂ Zantema's exact sequence

▸ even dihedral fields

Corollary

For a finite Galois extension K/P, if $\gcd(h_K, [K:P]) = 1$, then $\mathrm{Po}(K/P) = \varepsilon_{K/P}(\mathrm{Cl}(P))$. ▸ even dihedral fields

Lemma (-, A. Rajaei, J. Number Theory, 2020)

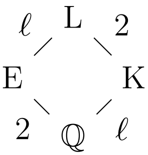
Let $P \subseteq K \subseteq L$ be a tower of finite extensions of number fields.

(i) If K/P and L/P are Galois extensions, then $\varepsilon_{L/K}(\mathbf{Po}(K/P)) \subseteq \mathbf{Po}(L/P)$.

(ii) If L/K is a Galois extension, then $\mathbf{Po}(L/P) \subseteq \mathbf{Po}(L/K)$.

even dihedral fields

Pólya odd prime dihedral extensions of $\mathbb{Q}$



$\mathrm{Gal}(L/\mathbb{Q}) \simeq D_\ell$, ℓ an odd prime

Lemma (-, A. Rajaei, J. Number Theory, 2020)

Let K be a D_ℓ-field with Galois closure L, and denote by E the unique quadratic subfield of L. For each ramified prime p in $L/\mathbb{Q}$:

1. if $p\mathcal{O}_L = (\gamma_1\gamma_2\ldots\gamma_\ell)^2$, then $p\mathcal{O}_K = \beta_1\beta_2^2\ldots\beta_{\frac{\ell+1}{2}}^2$, and

$$\Pi_p(L) \text{ is principal} \Leftrightarrow \Pi_p(E) \text{ is principal}$$

proof for maximum ramification

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2. If $p\mathscr{O}_L = \gamma^\ell$ or $p\mathscr{O}_L = (\gamma_1\gamma_2)^\ell$, then $p\mathscr{O}_K = \beta^\ell$, and

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3. If $p\mathscr{O}_L = \gamma^{2\ell}$, then $p = \ell$, $\ell\mathscr{O}_K = \beta^\ell$ and

$$\Pi_\ell(L) \text{ is principal} \Leftrightarrow \Pi_\ell(K) \text{ and } \Pi_\ell(E) \text{ are principal}$$

proof for maximum ramification

Theorem (-, A. Rajaei, J. Number Theory, 2020)

Let K be a D_ℓ-field with Galois closure L, and denote by E the unique quadratic subfield of L. Then we have an exact sequence as follows:

$$\{0\} \longrightarrow \mathscr{P}o(E) \xrightarrow[{[\mathfrak{a}]\mapsto[\mathfrak{a}\mathscr{O}_L]}]{\epsilon_{L/E}} \mathscr{P}o(L) \xrightarrow{\mathscr{N}_{L/K}\circ\pi_\ell} \mathscr{P}o(K).$$

Moreover, $(\mathscr{P}o(L))_2 \simeq \mathscr{P}o(E)$ and $(\mathscr{P}o(L))_\ell \hookrightarrow \mathscr{P}o(K)$.

proof for maximum ramification | Improving Ishida's result

$(\mathscr{P}o(L))_2 \simeq \mathscr{P}o(E)$ and $(\mathscr{P}o(L))_\ell \hookrightarrow \mathscr{P}o(K)$

Example

Let $K = \mathbb{Q}(\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a root of

$$f(X) = X^5 - 2X^4 + 7X^3 - 11X^2 + 5X + 1.$$

Then

- K is a D_5-field;
- $\mathrm{disc}(K) = 1367^2$;
- $h_K = 4$;
- $E = \mathbb{Q}(\sqrt{-1367})$ is the unique quadratic subfield of Galois closure L of K.

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E is Pólya and $5 \nmid h_K \implies$ L is a Pólya D_5-extension of $\mathbb{Q}$.

$$\{0\} \longrightarrow \mathscr{P}o(E) \longrightarrow \mathscr{P}o(L) \longrightarrow \mathscr{P}o(K).$$

Corollary (-, A. Rajaei, J. Number Theory, 2020)

Let K be a D_ℓ-field with Galois closure L, and denote the unique quadratic subfield of L by E. If either

- $\ell \nmid h_K$;
- or L/E is unramified,

then $Po(L) \simeq Po(E)$.

Corollary (-, A. Rajaei, J. Number Theory, 2020)

With the above notations, if L is the splitting field of an irreducible polynomial

$$f(X) = X^\ell + a_2X^{\ell-2} + a_3X^{\ell-3} + \cdots + a_{\ell-1}X + a_\ell, \quad a_i \in \mathbb{Z}$$

over $\mathbb{Q}$. If $\gcd(a_2, a_3, \ldots, a_{\ell-1}, \ell.a_\ell) = 1$, then $Po(E) \simeq Po(L)$.

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Theorem (A. Leriche, J. Number Theory, 2013)

Let $M/\mathbb{Q}$ be a Pólya Galois extension of degree n, and denote the number of ramified primes in $M/\mathbb{Q}$ by s_M. Then

$$s_M \leq \sum_{p|n} n\left(\frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{v_p(n)}}\right) + v_p(n)$$

Example

For $[M:\mathbb{Q}] = 2\ell$ (ℓ an odd prime), $s_M \leq \ell + 4$. D_ℓ-extensions

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Maximum ramification in Pólya S_3-extensions of $\mathbb{Q}$

Theorem (-, A. Rajaei, Proc. Roy. Soc. Edinburgh Sect. A, 2019)

Let K be a non-Galois cubic field with Galois closure L and $E = \mathbb{Q}(\sqrt{\operatorname{disc}(K)})$. Denote by s_L the number of ramified primes in $L/\mathbb{Q}$. If L is Pólya,

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1. for L real, $s_L \leq 4$ and this is sharp. Moreover, if $\varepsilon_E \in \mathrm{Norm}_{L/E}(U_L)$ where ε_E is the fundamental unit of E, then $s_L \leq 3$.

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   1. for non-pure K, $s_L \leq 2$ and this is sharp;

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1. for L real, $s_L \leq 4$ and this is sharp. Moreover, if $\varepsilon_E \in \mathrm{Norm}_{L/E}(U_L)$ where ε_E is the fundamental unit of E, then $s_L \leq 3$.
2. for L imaginary:
    1. for non-pure K, $s_L \leq 2$ and this is sharp;
    2. for pure K, $s_L \leq 3$ and this is sharp. Moreover, if $\zeta_3 \in \mathrm{Norm}_{L/E}(U_L)$ where ζ_3 is a primitive third root of unity, then $s_L \leq 2$.

Proof.

Using ◂ Zantema's exact sequence we find

$$\#H^1(G, U_L) = \prod_{p \mid \mathrm{disc}(L)} e(p); \qquad G = \mathrm{Gal}(L/\mathbb{Q})$$

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For $G_2 = \mathrm{Gal}(L/K)$ and $G_3 = \mathrm{Gal}(L/E)$

$$\mathrm{res} : H^1(G, U_L) \to H^1(G_2, U_L),$$

and

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are injective on the 2-primary and 3-primary part of $H^1(G, U_L)$, respectively.

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$$Q(G_2, U_L) = \frac{\#\hat{H}^0(G_2, U_L)}{\#H^1(G_2, U_L)} = \begin{cases} \frac{1}{2} & : \text{L is real}, \\ 1 & : \text{L is imaginary}, \end{cases}$$

$$Q(G_3, U_L) = \frac{\#\hat{H}^0(G_3, U_L)}{\#H^1(G_3, U_L)} = \frac{1}{3}$$

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$$\hat{H}^0(G_2, U_L) = \frac{U_K}{\mathrm{Norm}_{L/K}(U_L)} \leq \frac{U_K}{U_K^2} \leq \begin{cases} \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} & : L \text{ is real} \\ \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} & : L \text{ is imaginary} \end{cases}$$

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Proof.

$$\Rightarrow \prod_{p \mid disc(L)} e(p) = \#H^1(G, U_L) \mid \begin{cases} 2^4.3^2 & : L \text{ is real} \\ 2^2.3^2 & : L \text{ is imaginary} \end{cases}$$

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1. for L real and $\varepsilon_E \in \mathrm{Norm}_{L/E}(U_L)$,

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L is real

Example

$K = \mathbb{Q}(\boldsymbol{\theta})$; $\boldsymbol{\theta}$ is a root of $f(X) = X^3 - 20X - 30$;

$\mathrm{disc}(K) = 2^2 \times 5^2 \times 7 \times 11$;

$L = K(\sqrt{77})$ and $E = \mathbb{Q}(\sqrt{77})$;

$h_K = 1$ and E is Pólya;

$\Rightarrow$ L is a real Pólya S_3-extension of $\mathbb{Q}$ with $s_L = 4$.

L is imaginary and K is pure

Example

$K = \mathbb{Q}(\sqrt[3]{10})$;

$\mathrm{disc}(K) = -3 \times 2^2 \times 5^2$;

$L = K(\sqrt{\zeta_3})$ and $E = \mathbb{Q}(\sqrt{\zeta_3})$;

$h_K = 1$ and E is Pólya;

$\Rightarrow$ L is an imaginary Pólya S_3-extension of $\mathbb{Q}$ with $s_L = 3$;

L is imaginary and K is not pure

Example

$K = \mathbb{Q}(\boldsymbol{\theta})$ where $\boldsymbol{\theta}$ is a root of $f(X) = X^3 + 5X + 5$;

$\mathrm{disc}(K) = -5^2 \times 47$;

$L = K(\sqrt{-47})$ and $E = \mathbb{Q}(\sqrt{-47})$; K is not pure;

$h_K = 1$ and E is Pólya;

$\Rightarrow$ L is an imaginary Pólya S_3-extension of $\mathbb{Q}$ with $s_L = 2$;

Maximum ramification in Pólya D_ℓ-extensions of $\mathbb{Q}$ for a prime $\ell > 3$

Theorem (-, A. Rajaei, J. Number Theory, 2020)

Let K be D_ℓ-field with Galois closure of L, for $\ell > 3$ an odd prime, and E be the unique quadratic subfield of L. Denote by s_L the number of ramified primes in $L/\mathbb{Q}$. If L is Pólya, then:

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Leriche's upper bound

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◂ Leriche's upper bound

Remark

For $\ell > 3$ and E imaginary, $\gcd(\#\mathrm{U}_{\mathrm{E}}, \#\frac{\mathrm{U}_{\mathrm{E}}}{\mathrm{U}_{\mathrm{E}}^{\ell}}) = 1 \Rightarrow \mathrm{U}_{\mathrm{E}} = \mathrm{Norm}_{\mathrm{L}/\mathrm{K}}(\mathrm{U}_{\mathrm{L}})$.

Theorem(Ishida, J. Number Theory, 1969)

For $K/\mathbb{Q}$ a non-pure extension of degree n, if

$$\#\{\text{primes ramify totally in } K\} > \text{rank}_{\mathbb{Z}}(\mathscr{O}_K^{\times}),$$

then $n \mid h_K$.

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Example

For a D_ℓ-field K (ℓ an odd prime), we have

$$\text{Signature of K} = \begin{cases} (\ell, 0): & \operatorname{disc}(K) > 0 \\ (1, \frac{\ell-1}{2}): & \operatorname{disc}(K) < 0 \end{cases}$$

$\Rightarrow$ for $\operatorname{disc}(K) > 0$ (resp. $\operatorname{disc}(K) < 0$) if at least ℓ (resp. $\frac{\ell+1}{2}$) primes ramify totally in K, then $\ell | h_K$.

Corollary (-, A. Rajaei, J. Number Theory, 2020)

Let K be a D_ℓ-field, where ℓ is an odd prime. Denote the number of totally ramified primes in K by t_K.

(i) If $\ell = 3$:

(i−1) for $\mathrm{disc}(K) > 0$ or K pure, if $t_K \geq 3$ then $3 \mid h_K$;

(i−2) for $\mathrm{disc}(K) < 0$ and K non-pure, if $t_K \geq 2$ then $3 \mid h_K$.

(ii) If $\ell > 3$:

(ii−1) for $\mathrm{disc}(K) > 0$, if $t_K \geq 3$ then $\ell \mid h_K$.

(ii−2) for $\mathrm{disc}(K) < 0$, if $t_K \geq 2$ then $\ell \mid h_K$.

Proof.

For instance, assume that K is a D_ℓ-field ($\ell > 3$ prime) with $\mathrm{disc}(K) > 0$ and denote its Galois closure by L.

If at least three distinct primes ramify totally in $K/\mathbb{Q}$, then

$$\ell^3 \mid \prod_{p \mid \mathrm{disc}(L)} e(p) \overset{\text{Zantema's exact sequence}}{=} \#H^1(\mathrm{Gal}(L/\mathbb{Q}), U_L).\#\mathscr{P}o(L).$$

$$\#H^1(\mathrm{Gal}(L/\mathbb{Q}), U_L) \mid 2^{\omega}.\ell^2 \ (\omega \in \mathbb{N}) \Rightarrow \ell \mid \#\mathscr{P}o(L).$$

$$\frac{\mathbb{Z}}{\ell\mathbb{Z}} \leq (\mathscr{P}o(L))_\ell \overset{\text{Theorem}}{\hookrightarrow} \mathscr{P}o(K) \overset{\text{Chabert-Halberstadt}}{=} Cl(K).$$

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non-Galois extensions of $\mathbb{Q}$

Definition (Chabert-Halberstadt, J. Number Theory, 2020)

For a number field K, the subgroups $\mathbf{Po}(K)_{nr}$ (pre-Pólya group) and $\mathbf{Po}(K)_{nr1}$ of $\mathbf{Po}(K)$ are defined as follows:

$$\mathbf{Po}(K)_{nr} =< \left[\Pi_{p^f}(K)\right] : f \in \mathbb{N}, \text{p is a non-ramified prime in } K/\mathbb{Q} >,$$

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Theorem (Chabert-Halberstadt, J. Number Theory, 2020)

For a D_n-field K with n an odd integer, we have:

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Motivation

We take K to be a D_n-field with $n \geq 4$ an even integer.

Lemma (–, Int. J. Math, 2021)

Assume that $n \geq 4$ is an even integer, K is a D_n-field with the unique subfield F of degree $\frac{n}{2}$ over $\mathbb{Q}$. Then the norm map $\mathscr{N}_{K/F} : \mathbf{Cl}(K) \to \mathbf{Cl}(F)$ is surjective. In particular, $h_F | h_K$.

Proof.

Zantema proved K/F is not unramified. Hence $K \cap H(F) = F$. Surjectiveness of the norm map $\mathscr{N}_{K/F} : \mathbf{Cl}(K) \to \mathbf{Cl}(F)$ is a well known result in Class Field Theory. □

▸ proof of main theorem ▸ Corollary

Theorem (–, Int. J. Math, 2021)

Let K be a D_n-field, for $n \geq 4$ an even integer, and denote the unique subfield of K of degree $\frac{n}{2}$ (over $\mathbb{Q}$) by F. Then $Po(K)_{nr1} = Po(K)_{nr} = \varepsilon_{K/F}(Cl(F))$. ▸ Corollary ▸ Max ramification

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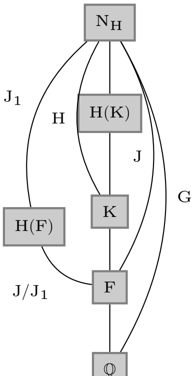
Let K be a D_n-field, for $n \geq 4$ an even integer, and denote the unique subfield of K of degree $\frac{n}{2}$ (over $\mathbb{Q}$) by F. Then $\mathrm{Po}(K)_{nr1} = \mathrm{Po}(K)_{nr} = \varepsilon_{K/F}(\mathrm{Cl}(F))$. Corollary Max ramification

Proof. Let N_H be the Galois closure of H(K) (over $\mathbb{Q}$) and:

- $G = \mathrm{Gal}(N_H/\mathbb{Q})$;
- $H = \mathrm{Gal}(N_H/K)$;
- $H_1 = \mathrm{Gal}(N_H/H(K))$;
- $J = \mathrm{Gal}(N_H/F)$;
- $J_1 = \mathrm{Gal}(N_H/H(F))$.

$$\Rightarrow H/H_1 \simeq \mathrm{Cl}(K),$$
$$J/J_1 \simeq \mathrm{Cl}(F).$$



$G = Gal(N_H/\mathbb{Q}),\ H = Gal(N_H/K),\ H_1 = Gal(N_H/H(K)),\ J = Gal(N_H/F),\ J_1 = Gal(N_H/H(F))$

Let $\Omega = \{Hs : s \in G\}$; $\Rightarrow \#\Omega = [K : \mathbb{Q}] = n$.

G acts transitively on Ω, as $(Hs)^g \mapsto Hsg^{-1}$, $\forall g, s \in G$.

By $S_\Omega \simeq S_n$, $g \longleftrightarrow \pi_1 \dots \pi_t$, where π_i's are disjoint cycles of orders f_i, and $f_i = 1$ for fixed points.

Let $Hs_i \in \Omega$ belongs to the orbit of π_i, i.e. Hs_i is permuted by π_i.

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For a non-ramified prime p in $K/\mathbb{Q}$, a prime $\mathfrak{P}$ of N_H above p and $g = (\mathfrak{P}, N_H/\mathbb{Q})$, the Frobenius element of $\mathfrak{P}$ in $N_H/\mathbb{Q}$, we have

$$\left(\frac{\Pi_{p^f}(K)}{H(K)/K}\right) = \prod_{\{i | f_i = f\}} s_i g^{f_i} s_i^{-1} \bmod H_1,$$

where $\left(\frac{\Pi_{p^f}(K)}{H(K)/K}\right)$ is the Artin symbol of $\Pi_{p^f}(K)$.

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Let $Hs_i \in \Omega$ belongs to the orbit of π_i, i.e. Hs_i is permuted by π_i.

$$Cl(K) \simeq Gal(H(K)/K) \simeq H/H_1$$

$$Po(K)_{nr} = \left\langle \left[\Pi_{p^f}(K)\right] \right\rangle \overset{\simeq}{\longleftrightarrow} \left\langle \left(\frac{\Pi_{p^f}(K)}{H(K)/K} \right) \right\rangle \overset{\simeq}{\longleftrightarrow} \mathscr{A} = \left\langle \prod_{\{i|f_i=f\}} s_i g^{f_i} s_i^{-1} \bmod H_1 \right\rangle.$$

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Let $\Omega = \{Hs : s \in G\}$; $\Rightarrow \#\Omega = [K : \mathbb{Q}] = n$.

G acts transitively on Ω, as $(Hs)^g \mapsto Hsg^{-1}$, $\forall g, s \in G$.

By $S_\Omega \simeq S_n$, $g \longleftrightarrow \pi_1 \ldots \pi_t$, where π_i's are disjoint cycles of orders f_i, and $f_i = 1$ for fixed points.

Let $Hs_i \in \Omega$ belongs to the orbit of π_i, i.e. Hs_i is permuted by π_i.

$\mathrm{Cl}(K) \simeq H/H_1$

$$\mathrm{Po}(K)_{nr} \overset{\simeq}{\longleftrightarrow} \mathscr{A} = \left\langle \prod_{\{i|f_i=f\}} s_i g^{f_i} s_i^{-1} \bmod H_1 \ : g \text{ has the orbit structure } 1,1,2,\ldots,2 \right\rangle$$

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g has the orbit structure $1, 1, 2, \ldots, 2 \iff p\mathscr{O}_K = \underbrace{(\beta_1.\beta_2)}_{\Pi_{p1}(K)}\underbrace{(\beta_3 \ldots \beta_t)}_{\Pi_{p2}(K)}$.

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$$\Rightarrow \mathrm{Po}(K)_{nr} \simeq \mathrm{Ver}\,(i(H/H_1))\,,$$

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Using the following commutative diagram

$$\begin{array}{ccc} \mathrm{Cl}(F) & \xrightarrow{\varepsilon_{K/F}} & \mathrm{Cl}(K) \\ \downarrow & & \downarrow \\ J/J_1 & \xrightarrow{\mathrm{Ver}} & H/H_1 \end{array}$$

we have $\mathrm{Po}(K)_{nr} \simeq \mathrm{Ver}(J/J_1) \simeq \varepsilon_{K/F}(\mathrm{Cl}(F))$, where the vertical arrows are isomorphisms induced by the Artin map.

Corollary (–, Int. J. Math, 2021)

Let K be a D_n-field, for $n \geq 4$ an even integer, and denote the unique subfield of K of degree $\frac{n}{2}$ (over $\mathbb{Q}$) by F. If h_K is odd, then

$$\mathrm{Po}(K)_{nr1} = \mathrm{Po}(K)_{nr} = \mathrm{Po}(K) = \mathrm{Po}(K/F) \simeq \mathrm{Cl}(F).$$

Proof.

$$\varepsilon_{K/F}(\mathrm{Cl}(F)) \overset{\text{Theorem}}{=} \mathrm{Po}(K)_{nr} \subseteq \mathrm{Po}(K) \overset{\text{Lemma}}{\subseteq} \mathrm{Po}(K/F) \overset{\text{Corollary}}{=} \varepsilon_{K/F}(\mathrm{Cl}(F)).$$

by Lemma [Lemma] h(F) is odd. Hence $\mathcal{N}_{K/F} \circ \varepsilon_{K/F} : \bar{\mathfrak{a}} \in \mathrm{Cl}(F) \mapsto \bar{\mathfrak{a}}^2 \in \mathrm{Cl}(F)$ is injective, and so is $\varepsilon_{K/F}$. □

Example

Proposition (C. J. Parry, 1975)

For a prime $p \equiv 3 \pmod 8$, the pure quartic fields $\mathbb{Q}(\sqrt[4]{2p})$ and $\mathbb{Q}(\sqrt[4]{2p^2})$ have odd class numbers.

$\Rightarrow$ for $p \equiv 3 \pmod 8$ prime:

$$\mathrm{Po}(\mathbb{Q}(\sqrt[4]{2p}))_{nr1} = \mathrm{Po}(\mathbb{Q}(\sqrt[4]{2p}))_{nr} = \mathrm{Po}(\mathbb{Q}(\sqrt[4]{2p})) = \mathrm{Po}(\mathbb{Q}(\sqrt[4]{2p})/\mathbb{Q}(\sqrt{2p})) \simeq \mathrm{Cl}(\mathbb{Q}(\sqrt{2p}));$$

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Maximum ramification in Pólya D_4-fields

Proposition (Zantema, 1982)

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Example (Chabert-Halberstadt, 2020)

Let $K = \mathbb{Q}(\alpha)$, where α is a root of

$$f(X) = X^4 - 17X^2 + 31.$$

We have $\mathrm{disc}(K) = 2^4.3^2.5^2.11^2.31$, $h_K = 2$ and K is a Pólya D_4-field.

Lemma (–, Int. J. Math, 2021)

Let K be a Pólya D_4-field with the unique quadratic subfield F. Then

$$\mathrm{Po}(K/F) = \langle [\gamma] \in \mathbf{Cl}(K) : (\gamma \cap \mathbb{Z})\,\mathscr{O}_F = \beta_1\beta_2, \text{ and } \beta_i\text{'s ramify in } K \rangle.$$

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Proof.

By ◂ Theorem, $\mathrm{Po}(K)_{nr} = \varepsilon_{K/F}(\mathbf{Cl}(F))$ is trivial. Since K is Pólya, $\Pi_{\mathfrak{p}^f}(K/F)$ might not be principal only in the case that $(\mathfrak{p} \cap \mathbb{Z})\,\mathscr{O}_F = \beta_1\beta_2$, and β_i's ramify in K. □

Notation

For M/N a finite extension of number fields, we denote the number of ramified and totally ramified primes of N in M by $r_{M/N}$, $t_{M/N}$, respectively.

Lemma (–, Int. J. Math, 2021)

Let K be a Pólya D_4-field with the unique quadratic subfield F. Then

$$r_{K/\mathbb{Q}} = r_{K/F} + r_{F/\mathbb{Q}} - \left(t_{K/\mathbb{Q}} + u\right),$$

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Proof.

There exist $r_{K/F} - u$ primes of $\mathbb{Q}$ which are ramified in K/F (note that here we count the number of totally ramified primes in $K/\mathbb{Q}$). On the other hand, for a totally ramified prime p in $K/\mathbb{Q}p$ also ramifies in $F/\mathbb{Q}$. □

Theorem (–, Int. J. Math, 2021)

Let K be a Pólya D_4-field with the unique quadratic subfield F. Denote by $t_{K/\mathbb{Q}}$ and $p_{K/\mathbb{Q}}$ the number of totally and partially ramified primes in $K/\mathbb{Q}$.

(i) For F imaginary:

$$2t_{K/\mathbb{Q}} + p_{K/\mathbb{Q}} \leq \begin{cases} 2: & F \neq \mathbb{Q}(\sqrt{-1}) \text{ and } -1 \in N_{K/F}(U_K) \\ 3: & F \neq \mathbb{Q}(\sqrt{-1}) \text{ and } -1 \notin N_{K/F}(U_K) \\ 2: & F = \mathbb{Q}(\sqrt{-1}) \text{ and } \sqrt{-1} \in N_{K/F}(U_K) \\ 3: & F = \mathbb{Q}(\sqrt{-1}) \text{ and } \sqrt{-1} \notin N_{K/F}(U_K) \end{cases}$$

(ii) For F real, denote the fundamental unit of F by ε, and denote the number of infinite places of F ramified in K by s. Then:

$$2t_{K/\mathbb{Q}} + p_{K/\mathbb{Q}} \leq \begin{cases} 5: & N_{F/\mathbb{Q}}(\varepsilon) = +1, s = 0 \\ 4: & N_{F/\mathbb{Q}}(\varepsilon) = -1, s = 0 \\ 3: & N_{F/\mathbb{Q}}(\varepsilon) = +1, s = 2 \\ 2: & N_{F/\mathbb{Q}}(\varepsilon) = -1, s = 2 \end{cases}$$

Proof.

Since $\varepsilon_{K/F}(\mathrm{Cl}(F))$ is trivial, by Theorem for $G = \mathrm{Gal}(K/F)$ we have

$$\#\mathrm{Po}(K/F) = \frac{h_F \cdot \prod_{\mathfrak{P} \mid \mathrm{disc}(K/F)} e_{\mathfrak{P}}}{\#H^1(G, U_K)}.$$

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By Lemma , $\mathbf{Po}(K/F)$ is an elementary abelian 2-group of rank at most u, where

$$u := \#\{p \in \mathbb{Z} \text{ prime} : p\mathscr{O}_F = \beta_1\beta_2,\ \beta_i' \text{s ramify in } K/F\}$$

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$$\implies \frac{h_F.2^{r_{K/F}}}{\#H^1(G, U_K)} \mid 2^u.$$

For F imaginary, $2^{r_{F/\mathbb{Q}}-1} | h_F \Longrightarrow \frac{2^{r_{K/F}+r_{F/\mathbb{Q}}-1}}{\#H^1(G,U_K)} \,|\, 2^u$.
Using Herbrand quotient we find

$$\#H^1(G,U_K) = 2.\left(U_F : N_{K/F}(U_K)\right) \mid 2^2.$$

Hence

$$r_{K/F} + r_{F/\mathbb{Q}} - u \leq \begin{cases} 2: & F \neq \mathbb{Q}(\sqrt{-1}) \text{ and } -1 \in N_{K/F}(U_K) \\ 3: & F \neq \mathbb{Q}(\sqrt{-1}) \text{ and } -1 \notin N_{K/F}(U_K) \\ 2: & F = \mathbb{Q}(\sqrt{-1}) \text{ and } \sqrt{-1} \in N_{K/F}(U_K) \\ 3: & F = \mathbb{Q}(\sqrt{-1}) \text{ and } \sqrt{-1} \notin N_{K/F}(U_K) \end{cases}$$

$$\Longrightarrow 2t_{K/\mathbb{Q}} + p_{K/\mathbb{Q}} = r_{K/\mathbb{Q}} + t_{K/\mathbb{Q}} \overset{\text{Lemma}}{=} r_{K/F} + r_{F/\mathbb{Q}} - u.$$

Thanks for your attention