On Drinfeld modular forms of higher rank and quasi-periodic functions

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Modular Forms for $\mathrm{SL}_2(\mathbb{Z})$

Definition

Let $\mathbb{H}$ be the upper half plane. *A modular form of weight* $k \in \mathbb{Z}_{\geq 0}$ (for $\mathrm{SL}_2(\mathbb{Z})$) is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following properties:

(i) $f(\gamma \cdot z) := f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}), \quad z \in \mathbb{H}.$

(ii) Set $\mathfrak{q} := e^{2\pi i z}$. The $\mathfrak{q}$-expansion of f is of the form

$$f(z) = \sum_{n=0}^{\infty} a_n \mathfrak{q}^n, \quad a_n \in \mathbb{C}.$$

Furthermore, f is called *a cusp form* if $a_0 = 0$.

Examples of modular forms

Example

Let $k > 2$ and consider the $\mathbb{Z}$-lattice $\Lambda_z := z\mathbb{Z} + \mathbb{Z}$ for any $z \in \mathbb{H}$. *The Eisenstein series* $G_k(\Lambda_z)$ is defined by

$$G_k(\Lambda_z) := \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(mz+n)^k}.$$

It is a modular form of weight k when k is even and identically zero if k is odd. One can normalize $G_k(\Lambda_z)$ and denote it by $\mathcal{G}_k(\Lambda_z)$ so that its $\mathfrak{q}$-expansion is given by

$$\mathcal{G}_k(\Lambda_z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) \mathfrak{q}^n$$

where

$$\sigma_{k-1}(n) := \sum_{\substack{d \geq 1 \\ d \mid n}} d^{k-1},$$

and B_k is the k-th Bernoulli number.

False Eisenstein series

Example

Consider *the discriminant function* $\Delta : \mathbb{H} \to \mathbb{C}$ defined by

$$\Delta(z) = \frac{1}{1728}\Big(\mathcal{G}_4(\Lambda_z)^3 - \mathcal{G}_6(\Lambda_z)^2\Big).$$

It is a non-vanishing function on $\mathbb{H}$ which is also a cusp form of weight 12.

- Define *the false Eisenstein series* $G_2 : \mathbb{H} \to \mathbb{C}$ given by

$$G_2(z) := \frac{1}{2\pi i}\frac{1}{\Delta(z)}\frac{d}{dz}\Delta(z).$$

- The $\mathfrak{q}$-expansion of G_2 is

$$G_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)\mathfrak{q}^n.$$

False Eisenstein series, quasi-periods and transcendence

- G_2 is *a quasi-modular form* and indeed we have

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)\left((cz+d)G_2(z) + \frac{6c}{\pi i}\right), \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}).$$

Theorem (Deligne, Katz)

Let $z \in \mathbb{H}$ and let η_1 and η_2 be quasi-periods of the elliptic curve corresponding to the $\mathbb{Z}$-lattice Λ_z satisfying the Legendre's relation $\eta_1 - z\eta_2 = 2\pi i$. Then we have

$$G_2(z) = -\frac{3}{\pi^2}\eta_2.$$

Theorem (Chang, 2012)

Let $z \in \mathbb{H}$ be a CM point satisfying $G_2(z) \neq 0$. Then $G_2(z)$ is transcendental over $\mathbb{Q}$.

Introduction to Drinfeld A-modules and Tate algebras

$$
\begin{aligned}
p &:= \text{ a prime number} \\
\mathbb{F}_q &:= \text{finite field with } q = p^m \text{ elements for some } m \in \mathbb{Z}_{\geq 1} \\
\mathbb{Z} \sim A &:= \mathbb{F}_q[\theta] \\
\mathbb{N} \sim A_+ &:= \text{the set of monic polynomials in } A \\
\mathbb{Q} \sim K &:= \mathbb{F}_q(\theta) = \text{rational functions in the variable } \theta \text{ over } \mathbb{F}_q \\
|\cdot| &:= \text{a fixed absolute value corresponding to the place at } \infty \text{ so that} \\
&\quad |\theta| = q \\
\mathbb{R} \sim K_\infty &:= \mathbb{F}_q((1/\theta)) = \text{the completion of } K \text{ with respect to} \\
&\quad \text{the absolute value } |\cdot| \\
\mathbb{C} \sim \mathbb{C}_\infty &:= \text{completion of an algebraic closure of } K_\infty
\end{aligned}
$$

The Tate Algebra

- Consider t as an independent variable over $\mathbb{C}_\infty$. Define *the Tate algebra* $\mathbb{T}$ by the set

$$\mathbb{T} := \Big\{ \sum_{i=0}^{\infty} c_i t^i \;\mid\; c_i \in \mathbb{C}_\infty, \;\; |c_i| \to 0 \text{ as } i \to \infty \Big\}.$$

- $\mathbb{T}$ is indeed the set of holomorphic functions, in the rigid analytic sense, on the closed unit disk $\mathfrak{D} := \{c \in \mathbb{C}_\infty | \;\; |c| \leq 1\}$.
- The Gauss norm $\|\cdot\|_\infty$ on $\mathbb{T}$ is defined by setting for $g = \sum_{i\geq 0} c_i t^i \in \mathbb{T}$,

$$\|g\|_\infty := \max\{|c_i| : i \in \mathbb{Z}_{\geqslant 0}\}.$$

- $\mathbb{T}$ is complete with respect to $\|\cdot\|_\infty$.

- For any $g = \sum_{i \geq 0} c_i t^i \in \mathbb{T}$ and $j \in \mathbb{Z}$, we set *the j-fold twist of g* by

$$g^{(j)} := \sum_{i=0}^{\infty} c_i^{q^j} t^i \in \mathbb{T}.$$

- Let $(-\theta)^{1/(q-1)}$ be a fixed $(q-1)$-st root of $-\theta$. *The Anderson-Thakur element* $\omega(t)$ is defined by

$$\omega(t) := (-\theta)^{1/(q-1)} \prod_{i=0}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right)^{-1} \in \mathbb{T}^{\times}.$$

- The element $\tilde{\pi} \in \mathbb{C}_\infty^{\times}$ is defined by

$$\tilde{\pi} := -\omega^{(1)}(t)_{|t=\theta} = \theta(-\theta)^{1/(q-1)} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1}.$$

A-lattices

Definition

- An A-module $M \subset \mathbb{C}_\infty$ is *strongly discrete* if the intersection of M with any ball of finite radius is finite.
- An A-module $M \subset \mathbb{C}_\infty$ is called *an A-lattice* if it is free, finitely generated and strongly discrete.

- Two A-lattices Λ and Λ' are *isogenous* if there exists an element $c \in \mathbb{C}_\infty^\times$ such that $c\Lambda \subset \Lambda'$ where the quotient $\Lambda'/c\Lambda$ is a finite A-module.
- We call c *an isogeny between* Λ *and* Λ'.
- If $c\Lambda = \Lambda'$, we say that Λ and Λ' are *isomorphic*.
- The set of A-lattices of rank $r \in \mathbb{Z}_{\geq 1}$ forms a category whose morphisms are given by isogenies between A-lattices.

Drinfeld A-modules

- We define the power series ring $\mathbb{C}_\infty[[\tau]]$ subject to the condition

$$\tau c = c^q \tau, \quad c \in \mathbb{C}_\infty,$$

and $\mathbb{C}_\infty[\tau] \subset \mathbb{C}_\infty[[\tau]]$ to be the ring of polynomials in τ.

Definition

Let $r \in \mathbb{Z}_{\geq 1}$. *A Drinfeld A-module* ϕ *of rank* r is an $\mathbb{F}_q$-algebra homomorphism

$$\phi : A \to \mathbb{C}_\infty[\tau]$$

uniquely defined by

$$\phi_\theta := \phi(\theta) = \theta + \phi_{\theta,1}\tau + \cdots + \phi_{\theta,r}\tau^r, \quad \phi_{\theta,r} \neq 0.$$

- The Carlitz module C is a Drinfeld A-module of rank 1 defined by $C_\theta = \theta + \tau$.
- ϕ gives an A-module structure on $\mathbb{C}_\infty$ defined as

$$a \cdot z := \phi_a(z) := az + \phi_{a,1}z^q + \cdots + \phi_{a,r}z^{q^{\deg(a)r}}, \quad a \in A, \quad z \in \mathbb{C}_\infty.$$

More on Drinfeld A-modules

Definition

- The set of Drinfeld A-modules forms a category such that any morphism $u : \phi \to \psi$ between Drinfeld A-modules is given by a non-zero $u \in \mathbb{C}_\infty[\tau]$ satisfying $u\phi_\theta = \psi_\theta u$ in $\mathbb{C}_\infty[\tau]$.
- If $u \in \mathbb{C}_\infty^\times$, then we say ϕ is isomorphic to ψ.
- Define $\mathrm{End}(\phi) := \{v \in \mathbb{C}_\infty[\tau] \mid v\phi_\theta = \phi_\theta v\}$. Drinfeld showed that it is a commutative ring and free and finitely generated A-module whose rank is less than or equal to r. Moreover, $\mathrm{End}(\phi) \otimes_A K$ is a field extension of K of finite degree.
- We say that ϕ has *complex multiplication* (CM) if the dimension of $\mathrm{End}(\phi) \otimes_A K$ over K is r and furthermore we call ϕ a *CM Drinfeld A-module*.

The exponential function $\exp_\phi$

Definition

For a Drinfeld A-module ϕ, *the exponential series for* ϕ is given by

$$\exp_\phi := \sum_{i=0}^{\infty} \alpha_i \tau^i \in \mathbb{C}_\infty[[\tau]],$$

subject to the conditions $\alpha_0 = 1$ and $\exp_\phi \theta = \phi_\theta \exp_\phi$.

- *The exponential function* $\exp_\phi$ *of* ϕ is the $\mathbb{F}_q$-linear function

  $$\exp_\phi : \mathbb{C}_\infty \to \mathbb{C}_\infty,$$

  defined by $\exp_\phi(z) = \sum_{i=0}^{\infty} \alpha_i z^{q^i}$ for all $z \in \mathbb{C}_\infty$ converging everywhere in $\mathbb{C}_\infty$.
- Each element in the kernel $\mathrm{Ker}(\exp_\phi)$ of $\exp_\phi$ is called *a period of* ϕ. Moreover $\mathrm{Ker}(\exp_\phi)$ is an A-module whose A-module structure is given by

  $$a \cdot z = az, \quad a \in A, \quad z \in \mathrm{Ker}(\exp_\phi).$$

  It is an A-lattice of rank r. We call $\mathrm{Ker}(\exp_\phi)$ *the period lattice of* ϕ.

Correspondence between A-lattices and Drinfeld A-modules

- (Drinfeld) There is an equivalence of categories between the category of Drinfeld A-modules of rank r and the category of A-lattices of rank r. This will be described as follows:
- Let Λ be a an A-lattice of rank r. Then there exists a unique Drinfeld A-module ϕ up to isomorphism so that its exponential function $\exp_\phi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is given by

$$\exp_\phi(z) = z \prod_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(1 - \frac{z}{\lambda}\right) = \sum_{j=0}^{\infty} \alpha_j z^{q^j}, \quad z \in \mathbb{C}_\infty.$$

Observe that $\Lambda = \mathrm{Ker}(\exp_\phi)$. We call ϕ *the Drinfeld A-module corresponding to* Λ.

Example

The Carlitz module C given by $C_\theta = \theta + \tau$ is the Drinfeld A-module corresponding to the A-lattice of rank one generated by $\tilde{\pi} = \theta(-\theta)^{1/(q-1)} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1} \in \mathbb{C}_\infty^\times$.

ϕ-biderivations

Definition

Let ϕ be a Drinfeld A-module of rank r defined by

$$\phi_\theta = \phi(\theta) = \theta + \phi_{\theta,1}\tau + \cdots + \phi_{\theta,r}\tau^r \in \mathbb{C}_\infty[\tau].$$

A *ϕ-biderivation* η is a map $\eta : A \to \tau\mathbb{C}_\infty[\tau]$ satisfying the following properties:

(i) η is an $\mathbb{F}_q$-linear map.

(ii) $\eta_{ab} = a\eta_b + \eta_a\phi_b$ where we set $\eta_c := \eta(c)$ for any $c \in A$.

Example

(i) The $\mathbb{F}_q$-linear map $\delta_0 : A \to \tau\mathbb{C}_\infty[\tau]$ defined by $\delta_0 : a \mapsto a - \phi_a$ is a ϕ-biderivation.

(ii) For each $1 \leq j \leq r-1$, the $\mathbb{F}_q$-linear map $\delta_j : A \to \tau\mathbb{C}_\infty[\tau]$ defined by $\delta_j : \theta \mapsto \tau^j$ is also a ϕ-biderivation.

Quasi-Periodic Functions

Definition

Let η be a ϕ-biderivation. There exists a unique function $F_\eta : \mathbb{C}_\infty \to \mathbb{C}_\infty$ satisfying

$$F_\eta(\theta z) - \theta F_\eta(z) = \eta_\theta(\exp_\phi(z)), \quad z \in \mathbb{C}_\infty.$$

We call F_η *the quasi-periodic function corresponding to* η. It is an $\mathbb{F}_q$-linear function and moreover $F_\eta(z)$ converges in $\mathbb{C}_\infty$ for any value of $z \in \mathbb{C}_\infty$.

Example

The quasi periodic function $F_{\delta_0} : \mathbb{C}_\infty \to \mathbb{C}_\infty$ corresponding to the ϕ-biderivation δ_0 is defined by $F_{\delta_0}(z) = z - \exp_\phi(z)$.

Anderson generating functions and quasi-periodic functions

Definition

Let ϕ be a Drinfeld A-module of rank r and set $\exp_\phi(z) = \sum_{n\geq 0} \alpha_n z^{q^n}$ for any $z \in \mathbb{C}_\infty$. We define *the Anderson generating function* $s_\phi(z;t)$ by the infinite sum

$$s_\phi(z;t) := \sum_{n=0}^{\infty} \exp_\phi\left(\frac{z}{\theta^{n+1}}\right) t^n = \sum_{n=0}^{\infty} \frac{\alpha_n z^{q^n}}{\theta^{q^n} - t} \in \mathbb{T}.$$

Proposition (Pellarin)

For each $1 \leq j \leq r-1$, let $F_{\delta_j} : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be the quasi-periodic function corresponding to the ϕ-biderivation δ_j. Then for any $\lambda \in \mathrm{Ker}(\exp_\phi)$, we have

$$F_{\delta_j}(\lambda) = s_\phi^{(j)}(\lambda;t)_{|t=\theta} = \sum_{n=0}^{\infty} \exp_\phi\left(\frac{\lambda}{\theta^{n+1}}\right)^{q^j} \theta^n.$$

Definition

For any $\lambda \in \mathrm{Ker}(\exp_\phi)$ and each $1 \leq j \leq r-1$, we call $F_{\delta_j}(\lambda)$ *a quasi-period of* ϕ.

The Drinfeld upper half plane

- From now on, we let $r \geq 2$.
- *The Drinfeld upper half plane* Ω^r is defined by

$$\Omega^r := \mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus \{K_\infty\text{-rational hyperplanes}\}.$$

- It is a connected rigid analytic space.
- Ω^r will be identified as the set of elements $\mathbf{z} := (z_1, \ldots, z_r)^{\mathrm{tr}} \in \mathbb{C}_\infty^r$ whose entries are K_∞-linearly independent elements and $z_r := 1$.
- For a given $\mathbf{z} = (z_1, \ldots, z_r)^{\mathrm{tr}} \in \Omega^r$, we also set $\tilde{\mathbf{z}} := (z_2, \ldots, z_r)^{\mathrm{tr}} \in \Omega^{r-1}$.
- For any $c \in \mathbb{C}_\infty^\times$, let $c\mathbf{z}A$ ($c\tilde{\mathbf{z}}A$ respectively) be the A-lattice generated by the entries of $c\mathbf{z}$ ($c\tilde{\mathbf{z}}$ respectively).
- Let $\phi^{\mathbf{z}}$ ($\phi^{\tilde{\pi}\tilde{\mathbf{z}}}$ respectively) be the Drinfeld A-module corresponding to $\mathbf{z}A$ ($\tilde{\pi}\tilde{\mathbf{z}}A$ respectively).
- Set $u(\mathbf{z}) := \exp_{\phi^{\tilde{\pi}\tilde{\mathbf{z}}}}(\tilde{\pi} z_1)^{-1}$.

Weak modular forms

- Let $\gamma = (a_{ij}) \in \mathrm{GL}_r(A)$. We define the action of $\mathrm{GL}_r(A)$ on Ω^r by

$$\gamma \cdot \mathbf{z} := \left(\frac{a_{11}z_1 + \cdots + a_{1r}z_r}{a_{r1}z_1 + \cdots + a_{rr}z_r}, \ldots, \frac{a_{(r-1)1}z_1 + \cdots + a_{(r-1)r}z_r}{a_{r1}z_1 + \cdots + a_{rr}z_r}, 1\right)^{\mathrm{tr}} \in \Omega^r.$$

- Set $j(\gamma, \mathbf{z}) := a_{r1}z_1 + \cdots + a_{rr}z_r \in \mathbb{C}_\infty^\times$.
- We call a holomorphic function $f : \Omega^r \to \mathbb{C}_\infty$ *a weak modular form of weight $k \in \mathbb{Z}$ and type $m \in \mathbb{Z}/(q-1)\mathbb{Z}$* (for $\mathrm{GL}_r(A)$) if it satisfies

$$f(\gamma \cdot \mathbf{z}) = j(\gamma, \mathbf{z})^k \det(\gamma)^{-m} f(\mathbf{z}), \quad \gamma \in \mathrm{GL}_r(A), \quad \mathbf{z} \in \Omega^r.$$

- Basson, Breuer and Pink showed that for any weak modular form $f : \Omega^r \to \mathbb{C}_\infty$, there exists a uniquely defined holomorphic function $f_n : \Omega^{r-1} \to \mathbb{C}_\infty$ for each $n \in \mathbb{Z}$ such that the series

$$\sum_{n \in \mathbb{Z}} f_n(\tilde{\mathbf{z}}) u(\mathbf{z})^n$$

converges to $f(\mathbf{z})$ on some "neighborhood $\mathcal{N} \subset \Omega^r$ of infinity". Such an expansion is called *the u-expansion of f*. When $r = 2$, $f_n \in \mathbb{C}_\infty$.

Drinfeld modular forms

Definition

(i) A weak modular form $f : \Omega^r \to \mathbb{C}_\infty$ of weight k and type m is called *a Drinfeld modular form of weight k and type m* (for $\mathrm{GL}_r(A)$) if f has a u-expansion of the form

$$f(\mathbf{z}) = \sum_{n=0}^{\infty} f_n(\tilde{\mathbf{z}}) u(\mathbf{z})^n.$$

(ii) Furthermore we say that f is *a Drinfeld cusp form* if $f_0 : \Omega^{r-1} \to \mathbb{C}_\infty$ is identically zero.

- Drinfeld modular forms were firstly defined by David Goss in his Ph.D. thesis for the rank 2 case.
- Using the work of Häberli, Kapranov and Pink, later on Basson, Breuer and Pink were able to define Drinfeld modular forms of higher rank both algebraically and analytically.

Examples of Drinfeld modular forms of type 0

- Let $k \in \mathbb{Z}_{\geq 1}$ and $\mathbf{z} = (z_1, \ldots, z_r)^{\mathrm{tr}} \in \Omega^r$. We define *the Eisenstein series* $\mathrm{Eis}_k(\mathbf{z})$ by

$$\mathrm{Eis}_k(\mathbf{z}) := \sum_{\substack{a_1,\ldots,a_r \in A \\ (a_1,\ldots,a_r) \neq (0,\ldots,0)}} \frac{1}{(a_1 z_1 + \cdots + a_r z_r)^k}.$$

  It is a Drinfeld modular form of weight k and type 0.

- Set

$$\phi_\theta^{\mathbf{z}} = \theta + g_1(\mathbf{z})\tau + \cdots + g_{r-1}(\mathbf{z})\tau^{r-1} + \Delta_r(\mathbf{z})\tau^r.$$

  For each $1 \leq i \leq r-1$, $g_i : \Omega^r \to \mathbb{C}_\infty$ defined by $\mathbf{z} \mapsto g_i(\mathbf{z})$ is a Drinfeld modular form of weight $q^i - 1$ and type 0.

- *The discriminant function* $\Delta_r : \Omega^r \to \mathbb{C}_\infty$ *defined by* $\mathbf{z} \mapsto \Delta_r(\mathbf{z})$, is a non-vanishing holomorphic function on Ω^r which is also a Drinfeld cusp form of weight $q^r - 1$ and type 0.

The h-function of Gekeler

- Let S be a set of representatives of the quotient space $((A/\theta A)^r \setminus \{0\})/\mathbb{F}_q^\times$ given by

$$S := \theta^{-1}\{(0, \dots, 0, 1), (0, \dots, 0, 1, *), \dots, (0, 1, *, \dots, *), (1, *, \dots, *)\}.$$

- Let $\mathbf{z} = (z_1, \dots, z_r)^{\mathrm{tr}} \in \Omega^r$. For any $\mu = (\mu_1, \dots, \mu_r) \in S$, we define

$$\mathrm{Eis}_\mu(\mathbf{z}) := \sum_{a_1, \dots, a_r \in A} \frac{1}{(a_1 + \mu_1)z_1 + \cdots + (a_r + \mu_r)z_r}.$$

- We define *the h-function of Gekeler* $h_r : \Omega^r \to \mathbb{C}_\infty$ *for rank r* given by

$$h_r(\mathbf{z}) := -(-\theta)^{1/(q-1)} \prod_{\mu \in S} \mathrm{Eis}_\mu(\mathbf{z}).$$

- It is a non-vanishing Drinfeld cusp form of weight $\frac{q^r-1}{q-1}$ and type 1.

A close look at rank 2 case and a motivation for main results

- Identify Ω^2 with $\mathbb{C}_\infty \setminus K_\infty$. Note the analogy between Ω^2 and $\mathbb{H} = \mathbb{C} \setminus \mathbb{R}$.
- Let $z \in \mathbb{C}_\infty \setminus K_\infty$ and ϕ^z be the Drinfeld A-module of rank 2 corresponding to the A-lattice $zA + A$.

Definition

A Drinfeld modular form of weight $k \in \mathbb{Z}_{\geq 0}$ *and type* $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ is a holomorphic function $f : \mathbb{C}_\infty \setminus K_\infty \to \mathbb{C}_\infty$ satisfying the following properties:

(i) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k \det(\gamma)^{-m} f(z), \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(A), \quad z \in \mathbb{C}_\infty \setminus K_\infty.$

(ii) Recall that $u := u(z) = \exp_C(\tilde{\pi} z)^{-1}$. The u-expansion of f is of the form

$$f(z) = \sum_{n=0}^{\infty} c_n u^n, \quad c_n \in \mathbb{C}_\infty.$$

False Eisenstein series and Gekeler's work

- Consider the discriminant function $\Delta_2 : \mathbb{C}_\infty \setminus K_\infty \to \mathbb{C}_\infty$ and define *the Gekeler's false Eisenstein series* $E_2 : \mathbb{C}_\infty \setminus K_\infty \to \mathbb{C}_\infty$ given by

$$E_2(z) := \frac{1}{\tilde{\pi}} \frac{1}{\Delta_2(z)} \frac{d}{dz} \Delta_2(z).$$

- Define $u_a(z) := u(az) = u^{q^{\deg(a)}}(1 + \text{higher degree terms in u})$ for any $a \in A_+$. Then E_2 has the u-expansion given by

$$E_2(z) = \sum_{a \in A_+} a u_a(z).$$

- For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(A)$, we have the functional equation

$$E_2\left(\frac{az+b}{cz+d}\right) = (cz+d)\det(\gamma)^{-1}\left((cz+d)E_2(z) - \frac{c}{\tilde{\pi}}\right).$$

Theorem (Gekeler, 1989)

Consider the quasi-period $\eta_2^z := F_{\delta_1}(1)$ where F_{δ_1} is the quasi-periodic function corresponding to the ϕ^z-biderivation $\delta_1 : A \to \tau\mathbb{C}_\infty[\tau]$ mapping $\theta \mapsto \tau$. We have

$$E_2(z) = \tilde{\pi}^{-1+q} h_2(z)\eta_2^z.$$

The function E_r

- We go back to the rank $r \geq 2$ case and let $\mathbf{z} = (z_1, \ldots, z_r) \in \Omega^r$ and $\tilde{\mathbf{z}} = (z_2, \ldots, z_r) \in \Omega^{r-1}$ be as before.
- Consider the discriminant function $\Delta_r : \Omega^r \to \mathbb{C}_\infty$ and define $E_r : \Omega^r \to \mathbb{C}_\infty$ given by

$$E_r(\mathbf{z}) := \frac{1}{\tilde{\pi}} \frac{1}{\Delta_r(\mathbf{z})} \frac{\partial}{\partial z_1} \Delta_r(\mathbf{z}).$$

- For each $a \in A$, set $u_a(\mathbf{z}) := \exp_{\phi^{\tilde{\pi}\tilde{\mathbf{z}}}}(a\tilde{\pi} z_1)^{-1}$. One can expand $u_a(\mathbf{z})$ as an infinite series in $u(\mathbf{z}) = \exp_{\phi^{\tilde{\pi}\tilde{\mathbf{z}}}}(\tilde{\pi} z_1)^{-1}$ whose coefficients are weak modular forms on Ω^{r-1}.
- Using Basson's product formula for Δ_r, we obtain

$$E_r(\mathbf{z}) = \sum_{a \in A_+} a u_a(\mathbf{z}).$$

The first main result

Theorem (Chen-G., 2021)

Let $\mathbf{z} = (z_1, \ldots, z_r)^{\mathrm{tr}}$ *be an element in* Ω^r. *We have*

$$E_r(\mathbf{z}) = \tilde{\pi}^{-1+q+\cdots+q^{r-1}} h_r(\mathbf{z}) \det \begin{pmatrix} F_{\delta_1}(z_2) & \ldots & F_{\delta_{r-1}}(z_2) \\ \vdots & & \vdots \\ F_{\delta_1}(z_r) & \ldots & F_{\delta_{r-1}}(z_r) \end{pmatrix}.$$

The sketch of the proof of Theorem

- For each $\mathbf{z} = (z_1, \ldots, z_r)^{\mathrm{tr}} \in \Omega^r$ and $k \in \mathbb{Z}_{\geq 1}$, we consider *the Eisenstein series* $\mathcal{E}_k(\mathbf{z}, t) \in \mathrm{Mat}_{r\times 1}(\mathbb{T})$ *of weight* k given by

$$\mathcal{E}_{\mathbf{z}}(k,t) := \left({\sum}'_{a_1,\ldots,a_r\in A} \frac{a_1(t)}{(a_1z_1+\cdots+a_rz_r)^k}, \ldots, {\sum}'_{a_1,\ldots,a_r\in A} \frac{a_r(t)}{(a_1z_1+\cdots+a_rz_r)^k} \right)^{\mathrm{tr}}.$$

- Let $\phi^{\mathbf{z}}$ be the Drinfeld A-module of rank r corresponding to the A-lattice $\mathbf{z}A$. Define

$$\mathcal{F}(\mathbf{z},t) := \begin{pmatrix} s_{\phi^{\mathbf{z}}}(z_1;t) & \ldots & s_{\phi^{\mathbf{z}}}^{(r-1)}(z_1;t) \\ \vdots & & \vdots \\ s_{\phi^{\mathbf{z}}}(z_r;t) & \ldots & s_{\phi^{\mathbf{z}}}^{(r-1)}(z_r;t) \end{pmatrix} \in \mathrm{Mat}_r(\mathbb{T}).$$

- For each $|t_0| \leq 1$ we obtain

$$\mathcal{E}_{\mathbf{z}}(1,t_0)^{\mathrm{tr}} = \frac{\tilde{\pi}^{\frac{q^r-1}{q-1}} h_r(\mathbf{z})}{\omega^{(1)}(t_0)} (C_{11}(t_0), \ldots, C_{r1}(t_0)) \qquad (4.1)$$

where, for $1 \leq j \leq r$, $C_{j1}(t_0)$ is the $(j,1)$-cofactor of $\mathcal{F}(\mathbf{z},t)$ evaluated at $t = t_0$.

- When $r=2$, the identity (4.1) is obtained by Pellarin.
- We introduce the function $\mathbf{E}_r : \Omega^r \times \{\xi \in \mathbb{C}_\infty | \; |\xi| \leq q\} \to \mathbb{C}_\infty$ given by

$$\begin{aligned}\mathbf{E}_r(\mathbf{z},\xi) &:= -\frac{\tilde{\pi}^{q+\cdots+q^{r-1}}h_r(\mathbf{z})}{\omega^{(1)}(\xi)}C_{11}(\xi) \\ &= -\frac{\tilde{\pi}^{q+\cdots+q^{r-1}}h_r(\mathbf{z})}{\omega^{(1)}(\xi)}\det\begin{pmatrix} s_{\phi^{\mathbf{z}}}^{(1)}(z_2;t)_{|t=\xi} & \dots & s_{\phi^{\mathbf{z}}}^{(r-1)}(z_2;t)_{|t=\xi} \\ \vdots & & \vdots \\ s_{\phi^{\mathbf{z}}}^{(1)}(z_r;t)_{|t=\xi} & \dots & s_{\phi^{\mathbf{z}}}^{(r-1)}(z_r;t)_{|t=\xi}\end{pmatrix}.\end{aligned}$$

- We have the following u-expansion

$$\mathbf{E}_r(\mathbf{z},\xi) = \sum_{a\in A_+} a_{|\theta=\xi}u_a(\mathbf{z}). \tag{4.2}$$

- Since $F_{\delta_j}(z_k) = s_{\phi^{\mathbf{z}}}^{(j)}(z_k;t)_{|t=\theta}$ for $1 \leq k \leq r$ and $\tilde{\pi} = -\omega^{(1)}(\theta)$, (4.2) implies

$$\mathbf{E}_r(\mathbf{z},\theta) = \sum_{a\in A_+} au_a(\mathbf{z}) = E_r(\mathbf{z}) = \tilde{\pi}^{-1+q+\cdots+q^{r-1}}h_r(\mathbf{z})\det\begin{pmatrix} F_{\delta_1}(z_2) & \dots & F_{\delta_{r-1}}(z_2) \\ \vdots & & \vdots \\ F_{\delta_1}(z_r) & \dots & F_{\delta_{r-1}}(z_r)\end{pmatrix}.$$

The functional equation of E_r

- For $2 \leq j \leq r-1$, let us consider the function $E_r^{[j]} : \Omega^r \to \mathbb{C}_\infty$ defined by

$$E_r^{[j]}(\mathbf{z}) = -\frac{\tilde{\pi}^{q+\cdots+q^{r-1}} h_r(\mathbf{z})}{\omega^{(1)}(\theta)} C_{j1}(\theta)$$

where $C_{j1}(\theta)$ is the $(j,1)$-cofactor of $\mathcal{F}(\mathbf{z},t)$ evaluated at $t=\theta$.

Theorem (Chen-G., 2021)

For any $\gamma \in \mathrm{GL}_r(A)$ and $\mathbf{z} = (z_1, \ldots, z_r)^{\mathrm{tr}} \in \Omega^r$, we have

$$\begin{aligned} &E_r(\gamma \cdot \mathbf{z}) \\ &= \det(\gamma)^{-1} j(\gamma, \mathbf{z}) \Big(E_r(\mathbf{z})(c_{11}^{\gamma} - z_1 c_{1r}^{\gamma}) + E_r^{[2]}(\mathbf{z})(c_{12}^{\gamma} - z_2 c_{1r}^{\gamma}) \\ &\qquad + \cdots + E_r^{[r-1]}(\mathbf{z})(c_{1(r-1)}^{\gamma} - z_{r-1} c_{1r}^{\gamma}) + \tilde{\pi}^{-1} c_{1r}^{\gamma} \Big) \end{aligned}$$

where for each $1 \leq j \leq r$, c_{1j}^{γ} is the $(1,j)$-cofactor of γ.

Recovering the functional equation of $E_2(\mathbf{z})$

- Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(A)$. Then we obtain $c_{11}^{\gamma} = d$ and $c_{12}^{\gamma} = -c$. The functional equation implies that

$$E_2(\gamma \cdot \mathbf{z}) = \det(\gamma)^{-1} j(\gamma, \mathbf{z}) \big(E_2(\mathbf{z})(cz_1 + d) - \tilde{\pi}^{-1} c \big)$$

which was firstly discovered by Gekeler.

The relation between E_r and $E_r^{[j]}$

- For each $2 \leq j \leq r-1$, set

$$\gamma_j^{-1} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ 1 & & \ddots & \\ & & & \ddots \; 1 \end{pmatrix} \in \mathrm{GL}_r(A)$$

  where the only non-zero terms in the first column appear in the first and the j-th entry.

- We have $c_{12}^{\gamma_j} = \cdots = c_{1(j-1)}^{\gamma_j} = c_{1(j+1)}^{\gamma_j} = c_{1r}^{\gamma_j} = 0$ and $c_{11}^{\gamma_j} = c_{1j}^{\gamma_j} = 1$. The functional equation implies that

$$E_r(\gamma_j \cdot \mathbf{z}) = \det(\gamma_j)^{-1} j(\gamma_j, \mathbf{z})\big(E_r(\mathbf{z}) + E_r^{[j]}(\mathbf{z})\big) = E_r(\mathbf{z}) + E_r^{[j]}(\mathbf{z}).$$

Transcendence of special values of E_r

- Recall that a Drinfeld A-module ϕ has *complex multiplication* (CM) if the dimension of $\mathrm{End}(\phi) \otimes_A K$ over K is r and furthermore we call ϕ a *CM Drinfeld A-module*.

Definition

We say that an element $\mathbf{z} \in \Omega^r$ is *a CM point* if the Drinfeld A-module $\phi^{\mathbf{z}}$ corresponding to the A-lattice $\mathbf{z}A$ is a CM Drinfeld A-module.

Theorem (Chen-G.,2021)

Let $\mathbf{z} \in \Omega^r$ be a CM point and $\overline{K}$ be the algebraic closure of K in $\mathbb{C}_\infty$. If $E_r(\mathbf{z}) \neq 0$, then it is transcendental over $\overline{K}$.

- When $r = 2$, Chang proved the theorem for a more general class of holomorphic functions, namely a certain subset of *Drinfeld quasi-modular forms*, including E_2.
- The proof uses our first result giving the relation between E_r and quasi-periods as well as Chang and Papanikolas' transcendence theory.

THANK YOU !