Effective height bounds for odd-degree totally real points on some curves.

Levent Alpöge

Diophantus, may your soul be with Satan because of the difficulty of the other theorems of yours, and in particular of the present theorem. — Chortasmenos, $\sim$ 1400.

Book VI of Arithmetica:

solve $y^2 = x^6 + x^2 + 1$ nontrivially.

finds $x = \frac{1}{2}$.

'98 PhD thesis of Wetherell: Diophantus found the only nontriv. sol.ns.

'83

Theorem (Faltings).

$g > 1$

Let $K/\mathbb{Q}$ be a number field. Let C/K be a smooth projective hyperbolic curve. Then: $C(K)$ is finite.

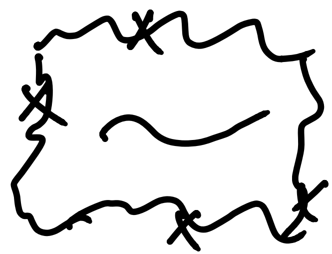
$f(x,y) = 0.$

- Famously ineffective, i.e. does *not* provide a finite-time algorithm to find $C(K)$ given K and C/K.

$$C \longrightarrow M_{\tilde{g}} \longrightarrow A_{\tilde{g}} \quad /K$$

$$\mathcal{C} \xrightarrow{\text{finite-to-one}} \mathcal{A}_{\tilde{g}} \quad /\mathcal{O}_{K,S}$$

$$C(K) = \mathcal{C}(\mathcal{O}_{K,S}) \longrightarrow \mathcal{A}_{\tilde{g}}(\mathcal{O}_{K,S})$$



$\subseteq \mathbb{P}^N.$

$x^n + y^n = 1$
$x^n + y^n = z^n.$

- Faltings' starting point was Parshin's key observation: for all hyperbolic $C/\overline{\mathbb{Q}}$, there is a $g \geq 2$ and a finite-to-one map $C \to A_g$, thanks to an amazing construction of Kodaira.

- But nothing forces us to use these maps…

- Aside: every hyperelliptic $C/\overline{\mathbb{Q}}$ admits a diagram:

Bogomolov-Tschinkel

$$\begin{tikzcd}
& \widetilde{C} \arrow[dl, "\text{ét.}"'] \arrow[dr, "\text{dom.}"] & & \\
C & & C_6 \arrow[r, "\text{univ. fam.}"] & A_2
\end{tikzcd}$$

with $C_6 : y^2 = x^6 + 1$.

Chevalley-Weil thm. $\exists$ explicit L/K s.t. all K-pts of C lift to L-pts of $\widetilde{C}$.

$\widetilde{C}(L) \longrightarrow C_6(L) \Rightarrow$ hence it suff. to deal with rat'l pts on C_6.

- Trick: $C_6/\overline{\mathbb{Q}}$ is a Shimura curve, corresponding to $[G, G] \subseteq G$ with $G := \Delta(2, 6, 6)$, an arithmetic triangle group. (Quat. alg.: $B_6/\mathbb{Q}$, indef. of disc. 6.)

- Key point of this talk: introducing modularity into Faltings' technique.

- Currently, modularity is best understood for GL_2. Thus we will map to Hilbert modular varieties instead of A_g.

- Downside: unclear for which curves one can (or can't) do this!

imagine;

$F = \mathbb{Q}(\sqrt{2})$,

$\mathcal{O} = \mathcal{O}_F = \mathbb{Z}[\sqrt{2}]$

$F/\mathbb{Q}$ tot. real, $\mathcal{O} \subseteq \mathcal{O}_F$, get $\mathcal{H}_{\mathcal{O}}$.

$\mathcal{H}_{\mathcal{O}} \subseteq \mathbb{P}^N$, $\dim. \mathcal{H}_{\mathcal{O}} = [F:\mathbb{Q}]$.

$\dim. \partial \mathcal{H}_{\mathcal{O}} = 0$.

$\subseteq \mathbb{P}^N$.

A/K, $\dim A = [F:\mathbb{Q}]$, $\mathcal{O} \hookrightarrow \mathrm{End}_K(A)$. good red. outside S.

Theorem (A.).
Let $F/\mathbb{Q}$ be a totally real field. Let $\mathfrak{o} \subseteq F$ be an order. Let $K/\mathbb{Q}$ be totally real with $[K:\mathbb{Q}]$ odd. Let S be a finite set of places of K. Then: there is an effectively computable $h_{\mathfrak{o},K,S} \in \mathbb{Z}^+$ such that, for all $[F:\mathbb{Q}]$-dimensional abelian varieties A/K with good reduction outside S and admitting $\mathfrak{o} \hookrightarrow \mathrm{End}_K(A)$,

$$h(A) \leq h_{\mathfrak{o},K,S}.$$

- Thus all $P \in \mathcal{H}_{\mathfrak{o}}(\mathfrak{o}_{K,S})$ satisfy $h(P) \leq h_{\mathfrak{o},K,S}$.
- $K = \mathbb{Q}$ and $\mathfrak{o} = \mathbb{Z}$: Murty-Pasten. Independently, $K = \mathbb{Q}$ and all $\mathfrak{o}$: von Känel.

- "Masser-Wüstholz/Bost" argument:

given B/K, want all K-isog. factors A/K of B/K.

Poincaré $\Rightarrow$ $\exists\, C/K$ s.t. $B \sim_K A \times C$.

Raynaud/M.-W. $\Rightarrow$ $h(A \times C) \ll_B 1$.

$$h(A\times C) = h(A) + h(C) \geq h(A) - 10^{10} \cdot \dim B$$

Bost $\rightarrow$ (for the $\geq$ step)

$\Rightarrow$ $h(A) \ll_B 1$.

- Example: let's find all $E/\mathbb{Q}$ of conductor N.

mod. $\Rightarrow$
$$\begin{tikzcd} X_0(N) \arrow[d] \\ E \end{tikzcd}$$
, so done by previous slide.

$$\begin{tikzcd} L(s,E) \arrow[r, equal, "\text{mod.}"] & L(s,f) \arrow[d, equal, "\text{Shimura}"] \\ & L(s,A_f) \end{tikzcd}$$

- Proof of the theorem, assuming modularity:

Jac. $C_{sh.,s}$, so done.

$$\downarrow$$

$$A$$

mod.

$$L(s,A) \overset{?}{=} L(s,f)$$

$$\| \Leftarrow \text{Hida}$$

$$L(s,A_f)$$

- So: done if $\exists$ explicit finite $\mathcal{F}$ of odd-deg. tot. real L/K s.t. every such ab. var. A/K is modular over some $L \in \mathcal{F}$.

- Every such A/K is *potentially* modular (Taylor, we use Snowden).

- Roughly: let λ s.t. $\overline{\rho} := \overline{\rho}_{A,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathrm{GL}_2(\mathfrak{o}/\lambda)$ has large image. Then there is $L_{\overline{\rho}}/K$ tot. real of odd deg. s.t. $A/L_{\overline{\rho}}$ modular.

- $L_{\overline{\rho}}/K$ determined following Taylor's proof. Key step: producing a tot. real point on a certain variety using Moret-Bailly/Rumely (— can just brute force!!).

- ($\sim$ Lem. of Dimitrov:) once $\mathrm{Nm}\,\lambda \gg_{\mathfrak{o},K,S} 1$ (explicit), $\overline{\rho}$ has large image.

- Hermite-Minkowski: the possible $\overline{\rho}$ are explicitly determined ($\overline{\rho}$ gives a num. field of bdd. deg. that is unram. outside S and $\mathrm{Nm}\,\lambda$).

- Thus by doing the brute force (or cleverer) for each such $\overline{\rho}$ we produce $\mathcal{F}$, QED.

Deines–Fuselier–Long–Swisher–Tu

- Example: let $K/\mathbb{Q}$ tot. real of odd deg., $a \in K^\times$, $C_a : x^6 + 4y^3 = a^2$.

$$x^6 + 4y^3 = 1.$$

- Hypergeom. fam. assoc. to $\Delta(3,6,6)$ (arithmetic!): let $f : C_a \to \mathbb{P}^1$ via $f(P) := \frac{x(P)^6}{a^2}$, and let A_P be the 2-dim.'l quot. of Jac. of (desing. of) $y^6 = x^4(1-x)^3(1-f(P)\cdot x)$.

- Because the eq. is $y^6 = \cdots$, we see $\mathbb{Z}[\zeta_3] \hookrightarrow \mathrm{End}_{K(\zeta_3)}(A_P)$. So A_P is $\mathrm{GL}_2(\mathbb{Z}[\zeta_3])$-type over $K(\zeta_3)$. Not enough...

$$\alpha^6 + 4\beta^3 = 1.$$

$$y^6 = x^4 (1-x)^3 (1 - t \cdot x)$$

$\dim . A_{\mathbb{D}} = 2$,

$$\rho : \mathrm{Gal.}(\overline{\mathbb{Q}}/K(\zeta_3)) \longrightarrow GSp_4(\mathbb{Q}_\ell)$$

$$\rho \cong \rho' \oplus \rho'',$$

$\dim . \rho' = \dim . \rho'' = 2.$

$F_1(t) = \eta(t) \cdot \eta\left(\frac{1-t}{4}\right)^2 \cdot F_2(t)$, η order 6 char.

- Arith. of $\Delta(3,6,6)$ means $B_6 \hookrightarrow \mathrm{End}^0_{\overline{\mathbb{Q}}}(A_P)$, so there are other endo.s… In fact (compare Frob. traces of the 2-dim'l rep.s over $K(\zeta_3)$) A_P/K is GL_2-type over K.

- Easy to compute an explicit $\mathcal{F}$ s.t. each A_P is $\mathrm{GL}_2(F)$-type over K for some $F \in \mathcal{F}$, done.

Thanks!