Shimura varieties modulo p with many compact factors

Oliver Bültel

May 11th, 2021
Istanbul

Plan I

1. Shimura Varieties
   - Theorems of Riemann and Mumford
   - Integral Models: Results and Conjectures

2. Construction of $\mathfrak{M}_n^{\mathfrak{f}}$ and ${}_{K^\infty,p}\mathcal{M}$
   - Hodge cycles
   - Poly-unitary moduli problems
   - Proof of the main theorem

3. Related results and questions

Plan

1. Shimura Varieties
   - Theorems of Riemann and Mumford
   - Integral Models: Results and Conjectures
2. Construction of $\mathfrak{M}_n^{\mathfrak{f}}$ and ${}_{K^\infty,p}\mathcal{M}$
   - Hodge cycles
   - Poly-unitary moduli problems
   - Proof of the main theorem
3. Related results and questions

Siegel space I

Fix an auxiliary integer $n \geq 3$. By a level n-structure on an abelian g-fold Y we mean a group isomorphism from $(\mathbb{Z}/n\mathbb{Z})^{2g}$ to $Y[n]$ taking the Weil pairing on $Y[n]$ to a multiple of the standard alternating pairing on $(\mathbb{Z}/n\mathbb{Z})^{2g}$ (i.e. $x^t J_g x$). The following result is basic to the study of moduli spaces of polarized abelian g-folds.

Theorem (Mumford)

The functor that takes a scheme S to the set of isomorphism classes of principally polarized abelian schemes of relative dimension g with level-n structure is representable by a quasi-projective $\mathbb{Z}[\frac{1}{n}]$-scheme. This scheme is smooth of relative dimension $\frac{g(g+1)}{2}$, and traditionally denoted by $\mathcal{A}_{g,n}$.

Siegel space II

Historically the knowledge of the mere set $\mathcal{A}_{g,n}(\mathbb{C})$ preceded the above result, to describe it in the modern adelic language we need a little bit of notation:

- Let $K_\infty \subset \mathrm{GSp}(2g,\mathbb{R})$ be the centralizer of $\begin{pmatrix} aE_g & -bE_g \\ bE_g & aE_g \end{pmatrix}$ (Deligne torus).
- Let K^∞ be the kernel of $\mathrm{GSp}(2g,\hat{\mathbb{Z}}) \to \mathrm{GSp}(2g,\mathbb{Z}/n\mathbb{Z})$, where $\hat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$.

Siegel space III

K^∞ is a compact open subgroup of $\mathrm{GSp}(2g, \mathbb{A}^\infty)$, where $\mathbb{A}^\infty = \mathbb{Q} \otimes \hat{\mathbb{Z}}$ is the ring of finite adeles, and $K_\infty K^\infty$ is a subgroup of $\mathrm{GSp}(2g, \mathbb{A})$, where $\mathbb{A} = \mathbb{R} \times \mathbb{A}^\infty$ is the ring of adeles.

Theorem (Riemann)

There exists a canonical bijection between $\mathcal{A}_{g,n}(\mathbb{C})$ and the complex orbifold $\mathrm{GSp}(2g, \mathbb{Q}) \backslash \mathrm{GSp}(2g, \mathbb{A}) / K_\infty K^\infty$ (which has $\phi(n)$ connected components, each of which look like $\Gamma_g(n) \backslash \mathfrak{h}_g$).

The above double quotient may also be written as

$$\mathrm{GSp}(2g, \mathbb{Q}) \backslash (\mathfrak{h}_g^{\pm} \times \mathrm{GSp}(2g, \mathbb{A}^\infty)) / K^\infty,$$

where $\mathfrak{h}_g^{\pm} := \mathrm{GSp}(2g, \mathbb{R}) / K_\infty$ is the so-called double half-space.

Plan

Shimura data I

Deligne's notion of Shimura datum (G, X) axiomatizes the properties of $(\mathrm{GSp}(2g), \mathfrak{h}_g^{\pm})$. To any such pair he attaches a canonical number field (called the reflex field) and an adelic system of complex manifolds

$${}_{K^\infty}M(G, X) := G(\mathbb{Q})\backslash(X \times G(\mathbb{A}^\infty))/K^\infty$$

as K^∞ runs through the set of compact open subgroups of $G(\mathbb{A}^\infty)$. The projective limit acquires a right $G(\mathbb{A}^\infty)$-action and is denoted $M(G, X)$.

Shimura data II

Theorem (Baily-Borel, Deligne, Milne, Borovoi)

If (G, X) and K^∞ are as above, then ${}_{K^\infty}M(G, X) = M(G, X)/K^\infty$ is a quasi-projective variety possessing a canonical model over the reflex field E.

Fix a prime number p. The group K^∞ might allow a factorization into a hyperspecial subgroup $K_p \subset G(\mathbb{Q}_p)$ and some other compact open $K^{\infty,p} \subset G(\mathbb{A}^{\infty,p})$. In this case p is unramified in E and one expects ${}_{K^\infty}M(G, X)$ to have "good reduction" at all divisors $\operatorname{Spec} \mathcal{O}_E \ni \mathfrak{p} \mid p$, more specifically we have:

Shimura data III

Conjecture (Langlands)

Assume that G^{ad} is anisotropic and that $K^\infty = K_p K^{\infty,p}$ holds for some hyperspecial group K_p. Then ${}_{K^\infty}M(G,X)$ can be extended to a smooth projective scheme $\mathcal{M}$ over the semi-local number ring $(1+p\mathcal{O}_E)^{-1}\mathcal{O}_E$.

One says that a Shimura datum (G, X) is of Hodge type if there exists an embedding $(G, X) \hookrightarrow (GSp(2g), \mathfrak{h}_g^{\pm})$ for some g. For this class of Shimura data the conjecture is known to be true if $p \neq 2$, thanks to works of Mark Kisin and Adrian Vasiu, who proved a more specific statement, which was previously conjectured by James Milne.

main theorem

Theorem (Bültel)

Let $p \neq 2$ be a prime and let (G, X) be a Shimura datum satisfying:

- *G^{der} is a simply connected almost simple group of type E_7.*
- *$G^{ad}_{\mathbb{R}}$ possesses more than four times as many compact factors than non-compact ones.*
- *$G^{ad}_{\mathbb{Q}_p}$ is quasisplit and G splits over the unramified extension $K(\mathbb{F}_{p^f})$ of degree f over $\mathbb{Q}_p$.*

Fix an embedding $\iota : K(\mathbb{F}_{p^f}) \to \mathbb{C}$. There exists a compact open subgroup $K_p \subset G(\mathbb{Q}_p)$, such that for any compact open subgroup $K^{\infty,p} \subset G(\mathbb{A}^{\infty,p})$ there exists a smooth projective $W(\mathbb{F}_{p^f})$-scheme ${}_{K^{\infty,p}}\mathcal{M}$ of which the complexification (via ι) is isomorphic to ${}_{K_pK^{\infty,p}}M(G, X)$.

Plan

absolute Hodge cycles I

Definition

Let n be an integer. A *pure Hodge structure of weight n* is a finite dimensional $\mathbb{Q}$-vector space V which is equipped with a decomposition $\mathbb{C} \otimes V = \bigoplus_{p+q=n} V^{p,q}$ such that $V^{q,p}$ is the complex conjugate of $V^{p,q}$.
If n is even, there exists a unique Hodge structure of weight n on the $\mathbb{Q}$-vector space $(2i\pi)^{n/2}$, which is denoted by $\mathbb{Q}(n/2)$.

Let Y be a smooth projective complex variety. It is known that for every non-negative integer n, the $\mathbb{Q}$-vector space $H^n(Y, \mathbb{Q})$ possesses a natural Hodge structure of weight n. A Hodge structure is just a direct sum of pure Hodge structures (of different weights), for example $\bigoplus_{n=0}^{2\dim Y} H^n(Y, \mathbb{Q})$.

absolute Hodge cycles II

Conjecture (Hodge)

Every morphism from $H^{2p}(Y,\mathbb{Q})$ to $\mathbb{Q}(-p)$ (in the category of Hodge structures) arises from a $\mathbb{Q}$-linear combination of p-dimensional subvarieties Z by means of $Z \rightsquigarrow (\eta \mapsto \int_Z \eta)$.

A strong evidence for this conjecture is the following result:

Theorem (Deligne)

Every Hodge cycle on an abelian variety Y over an algebraically closed subfield of $\mathbb{C}$ is an absolute Hodge cycle: Automorphisms of the field of complex numbers preserve Hodgeness of p-cycles, when regarded as elements of

$$H^{2p}_{dR}(Y)(p) \times H^{2p}_{et}(Y,\hat{\mathbb{Z}}(p)).$$

$\mathcal{O}_L$-linear tensor products of period lattices I

Let L be a CM field of finite degree over $\mathbb{Q}$, and suppose that $\mathcal{O}_L$ acts on three principally polarized abelian varieties (Y_1, λ_1), (Y_2, λ_2) and (Y_3, λ_3), of dimension $\frac{n_1[L:\mathbb{Q}]}{2}$, $\frac{n_2[L:\mathbb{Q}]}{2}$ and $\frac{n_3[L:\mathbb{Q}]}{2}$ respectively.

The triple is called *multipliable* if for each embedding $\sigma : L \to \mathbb{C}$ at least one of the three σ-eigenspaces $\mathrm{Lie}_\sigma\, Y_1$, $\mathrm{Lie}_\sigma\, Y_2$ or $\mathrm{Lie}_\sigma\, Y_3$ vanishes. In this case we obtain yet another principally polarized abelian variety

$$(\dot{\bigotimes}_{i\in\{1,2,3\}} Y_i, \lambda)$$

again with $\mathcal{O}_L$-action by decreeing its period lattice to be:

$$H_1(\dot{\bigotimes}_{i\in\{1,2,3\}} Y_i, \mathbb{Z}) = H_1(Y_1, \mathbb{Z}) \otimes_{\mathcal{O}_L} H_1(Y_2, \mathbb{Z}) \otimes_{\mathcal{O}_L} H_1(Y_3, \mathbb{Z})(-1).$$

$\mathcal{O}_L$-linear tensor products of period lattices II

This formula can be expressed by saying that a certain element in

$$H^4(Y_1 \times Y_2 \times Y_3 \times \dot{\bigotimes}_{i\in\{1,2,3\}} Y_i, \mathbb{Z}(2))$$

is a Hodge cycle, and therefore absolutely Hodge! In the algebraic context the assignment

$$(Y_1, Y_2, Y_3) \mapsto \dot{\bigotimes}_{i\in\{1,2,3\}} Y_i$$

remains meaningful, provided that the characteristic of the ground field is odd and unramified in L.

Properties of $\mathcal{O}_L$-linear tensor products

Recall the ℓ-adic Tate module $T_\ell(Y) := \lim_n Y[\ell^n]$ for any abelian variety Y over some fixed field $k \supset \mathbb{F}_{p^f}$ and prime $\ell \neq p$.

Proposition

$$\mathrm{End}_{\mathcal{O}_L}(Y_1) \otimes_{\mathcal{O}_L} \mathrm{End}_{\mathcal{O}_L}(Y_2) \otimes_{\mathcal{O}_L} \mathrm{End}_{\mathcal{O}_L}(Y_3) \subset \mathrm{End}_{\mathcal{O}_L}(\dot{\bigotimes}_{i\in\{1,2,3\}} Y_i)$$

holds for every multipliable triple (Y_1, Y_2, Y_3)*, respecting:*

$$T_\ell(Y_1) \otimes_{\mathcal{O}_L} T_\ell(Y_2) \otimes_{\mathcal{O}_L} T_\ell(Y_3)(1) \cong T_\ell(\dot{\bigotimes}_{i\in\{1,2,3\}} Y_i)$$

$$\mathbb{D}(Y_1) \otimes_{\mathcal{O}_L} \mathbb{D}(Y_2) \otimes_{\mathcal{O}_L} \mathbb{D}(Y_3)(1) \cong \mathbb{D}(\dot{\bigotimes}_{i\in\{1,2,3\}} Y_i)$$

Plan

Poly-unitary Shimura data I

- a hyperspecial subgroup $U_p \subset G_0(K(\mathbb{F}_{p^f}))$, where G_0 is such that $G = \mathrm{Res}_{L^+/\mathbb{Q}} G_0$
- a totally imaginary quadratic extension L of L^+ which is unramified at p
- a triple of homomorphisms $\rho_i : G_0 \to \mathrm{GU}(V_i/L, \Psi_i)$ rendering each Ψ_i a skew-Hermitian polarization on V_i and such that G_0 is the intersection of the stabilizers of

$$\mathrm{End}_{G_0}(V_1 \otimes_L V_2 \otimes_L V_3),$$

  and $\mathrm{End}_{G_0}(V_i)$ for every $i \in \{1, 2, 3\}$.
- U_p-invariant, self-dual $\mathcal{O}_L$-lattices $\mathfrak{V}_i \subset V_i$
- an ideal $\mathfrak{f} \subset (1 + p\mathcal{O}_{L^+})^{-1}\mathcal{O}_{L^+}$ and a non-negative integer n which is coprime to p

Poly-unitary Shimura data II

Let the $W(\mathbb{F}_{p^f})$-scheme $\mathfrak{M}_n^{\mathfrak{f}}$ be defined by the moduli problem that assigns to given $S/W(\mathbb{F}_{p^f})$ the following set of data:

- a multipliable triple of principally polarized abelian S-schemes $(Y_i, \lambda_i)_{1\leq i\leq 3}$ with $\mathcal{O}_L$-operation
- Rosati-invariant and $\mathcal{O}_L$-linear homomorphisms

$$\iota_{\{1,2,3\}} : \mathcal{O}_L + \mathfrak{f}\,\mathrm{End}_{G_0}(\mathfrak{V}_1 \otimes_{\mathcal{O}_L} \mathfrak{V}_2 \otimes_{\mathcal{O}_L} \mathfrak{V}_3) \to \mathrm{End}_{\mathcal{O}_L}(\dot{\bigotimes}_{i\in\{1,2,3\}} Y_i)$$

  and $\iota_i : \mathcal{O}_L + \mathfrak{f}\,\mathrm{End}_{G_0}(\mathfrak{V}_i) \to \mathrm{End}_{\mathcal{O}_L}(Y_i)$ for every $i \in \{1,2,3\}$
- level n-structures $\eta_i : \mathfrak{V}_i/n\mathfrak{V}_i \overset{\cong}{\to} Y_i[n]$, compatible with $\iota_{\{1,2,3\}}$ and ι_i for every $i \in \{1,2,3\}$
  N.B.: $\mathrm{End}_{G_0}(\mathfrak{V}_1 \otimes_{\mathcal{O}_L} \mathfrak{V}_2 \otimes_{\mathcal{O}_L} \mathfrak{V}_3)$ (resp. $\mathrm{End}_{G_0}(\mathfrak{V}_i)$) acts on $(\dot{\bigotimes}_{i\in\{1,2,3\}} Y_i)[n]$ (resp. $Y_i[n]$).

Plan

A commutative diagram I

Fix a reductive group scheme $\mathcal{G}$ over $\mathbb{Z}_p$ and let μ be a cocharacter of $\mathcal{G}_{W(\mathbb{F}_{p^f})}$ all of whose weights in $\operatorname{Lie}\mathcal{G}$ are contained in the set $\{-1, 0, 1\}$. To this data one associates a canonical fpqc stack $\mathcal{B}(\mathcal{G}, \mu)$ over $W(\mathbb{F}_{p^f})$. Its points over an algebraically closed field $k \supset \mathbb{F}_{p^f}$ are given by the quotient $\mathcal{G}(W(k))\mu(p)\mathcal{G}(W(k))/\sim$ where $b_2 \sim b_1$ if and only if $b_2 = g^{-1} b_1 {}^F g$ for some $g \in \mathcal{G}(W(k))$ (Bültel, Hedayatzadeh, Lau, Pappas). The integral models of a Shimura variety ${}_{K^\infty}M(G, X)$ are expected to allow a morphism to $\mathcal{B}(\mathcal{G}, \mu_X)$, where μ_X arises from the Shimura datum (G, X) and $\mathcal{G}$ is the reductive $\mathbb{Z}_p$-model of G whose $\mathbb{Z}_p$-points $\mathcal{G}(\mathbb{Z}_p)$ yield the hyperspecial factor K_p of the level $K^\infty = K_p K^{\infty,p}$.

A commutative diagram II

Let $\mathfrak{B}^{\mathfrak{f}}$ be the *p*-divisible group version of the moduli problem $\mathfrak{M}_n^{\mathfrak{f}}$. Spurred by the aforementioned expectation, one constructs a commutative diagram

$$\begin{array}{ccc} {}_{K^\infty,p}\overline{\mathcal{M}} & \longrightarrow & \overline{\mathcal{B}}(\mathcal{G},\mu_X) \\ \downarrow & & \downarrow {\scriptstyle \mathrm{Flex}^{\mathbf{T}}} \\ \mathfrak{M}_n^{\mathfrak{f}} & \longrightarrow & \mathfrak{B}^{\mathfrak{f}} \end{array}$$

with horizontal arrows being formally étale and ${}_{K^\infty,p}\overline{\mathcal{M}} \to \mathfrak{M}_n^{\mathfrak{f}}$ being radicial and universally closed.

A commutative diagram III

Theorem

The $\mathbb{F}_{p^f}$-variety ${}_{K^\infty,p}\overline{\mathcal{M}}$ is smooth and it possesses a lift ${}_{K^\infty,p}\mathcal{M}$, of which the generic fiber is a finite union of quotients of X by congruence subgroups.

The lift is exhibited by requiring that the map to $\mathcal{B}(\mathcal{G}, \mu_X)$ extends to ${}_{K^\infty,p}\mathcal{M}$. The characterization of the universal covering space follows works of Varshavsky (e.g. in: J. of Differential Geom. 49(1998), p.75-113). The congruence subgroups are obtained from applying a theorem of Prasad and Rapinchuk in: Inst. Hautes Études Sci. Publ. Math. 84(1996), p.91-187.

Definition of $\mathrm{Flex}^{\mathbf{T}}$ I

Recall that an F-crystal over an algebraically closed field k of characteristic p is a pair (M, F), where M is a finitely generated torsion free $W(k)$-module and $F : M \to M$ is a Frobenius-linear injective map.

Lemma

Consider some $\mathbb{Z}/r\mathbb{Z}$-grading $\bigoplus_{i=0}^{r-1} M_i$ on some F-crystal (M, F), satisfying $F(M_i) = M_{i+1}$ for $i \in \{1, \ldots, r-1\}$ and $p^w M_1 \subset F(M_0) \subset M_1$ for some $w \leq r$. Then $M_i' := F^i M_0 + p^{\max\{w-i,0\}} M_i$ for $i \in \{0, \ldots, r-1\}$ defines a $\mathbb{Z}/r\mathbb{Z}$-graded Dieudonné module over k with $p^{w-1} M \subset M' \subset M$, i.e. $pM' \subset F(M') \subset M' = \bigoplus_{i=0}^{r-1} M_i'$.

Definition of $\mathrm{Flex}^{\mathbf{T}}$ II

Notice that giving a $\mathbb{Z}/r\mathbb{Z}$-gradation on an F-crystal over k is equivalent to giving a action of $W(\mathbb{F}_{p^r})$ thereon, this is due to $W(\mathbb{F}_{p^r}) \otimes_{\mathbb{Z}_p} W(k) \cong W(k)^r$.

Corollary

Suppose that three $\mathbb{Z}/r\mathbb{Z}$-graded F-crystals K, L and M satisfy the assumptions of the previous lemma, but in addition assume that $3w \leq r$ holds. Moreover, let M' be the Dieudonné lattice as in the previous lemma, but in addition consider $L_i' := F^i L_0 + p^{\max\{2w-i,0\}} L_i$ and $K_i' := F^i K_0 + p^{\max\{3w-i,0\}} K_i$ for $i \in \{0, \dots, r-1\}$. Then all of $K' \subset K$, $L' \subset L$, $M' \subset M$ and $K' \otimes_{W(\mathbb{F}_{p^r})} L' \otimes_{W(\mathbb{F}_{p^r})} M' \subset K \otimes_{W(\mathbb{F}_{p^r})} L \otimes_{W(\mathbb{F}_{p^r})} M$ are $\mathbb{Z}/r\mathbb{Z}$-graded Dieudonné lattices.

Definition of $\mathrm{Flex}^{\mathbf{T}}$ III

From now on:

- Assume that $G(\mathbb{R})$ has only one single non-compact factor, in particular G_0 is simple.
- Assume that p is inert in L^+ but splits in L, so that $\mathbb{Z}_p \otimes \mathcal{O}_{L^+} \cong W(\mathbb{F}_{p^r})$ and $\mathbb{Z}_p \otimes \mathcal{O}_L \cong W(\mathbb{F}_{p^r}) \oplus W(\mathbb{F}_{p^r})$. Observe that this induces a splitting $\rho_i \cong \varrho_i \oplus \check{\varrho}_i$ for some representation ϱ_i of $G_{0,W(\mathbb{F}_{p^r})}$
- Assume that the μ-weights of ϱ_i are contained in the set $\{0, \ldots, w\}$

Definition of Flex$^{\mathbf{T}}$ IV

Sketch

1. Take some $b \in \mathcal{G}(W(k))\mu(p)\mathcal{G}(W(k))/\sim$
2. Using ϱ_1, ϱ_2 and ϱ_3 yields $\mathbb{Z}/r\mathbb{Z}$-graded F-crystals K, L and M over k, moreover $\operatorname{End}_{G_0}(\mathfrak{V}_1 \otimes_{\mathcal{O}_L} \mathfrak{V}_2 \otimes_{\mathcal{O}_L} \mathfrak{V}_3)$ (resp. $\operatorname{End}_{G_0}(\mathfrak{V}_i)$) acts on $K \otimes_{W(\mathbb{F}_{p^r})} L \otimes_{W(\mathbb{F}_{p^r})} M$ (resp. K, L and M).
3. Applying the corollary yields $\mathbb{Z}/r\mathbb{Z}$-graded Dieudonné lattices K', L' and M', moreover $p^{3w-1}\operatorname{End}_{G_0}(\mathfrak{V}_1 \otimes_{\mathcal{O}_L} \mathfrak{V}_2 \otimes_{\mathcal{O}_L} \mathfrak{V}_3)$ (resp. $p^w \operatorname{End}_{G_0}(\mathfrak{V}_i)$) acts on $K' \otimes_{W(\mathbb{F}_{p^r})} L' \otimes_{W(\mathbb{F}_{p^r})} M'$ (resp. on K', L' and M').

Take $\mathfrak{f} = p^{3w-1}$. The triple K', L' and M' together with these four actions gives rise to: $\text{Flex}^{\mathbf{T}}(b) \in \mathfrak{B}^{\mathfrak{f}}(k)$

p-adic Hodge theory

1. Can one use p-adic Hodge theory to describe ${}_{K_pK^\infty,p}M(G,X)$ in terms of ${}_{K^\infty,p}\overline{\mathcal{M}}$ as K_p varies?
2. Is the level structure K_p occuring in theorem 5 hyperspecial?
3. Are the adelic representations coming from the local systems on ${}_{K_pK^\infty,p}M(G,X)$ crystalline? (They are deRham according to a theorem of Ruochuan Liu and Xinwen Zhu, Inv. Math. 207)

Rational Tate cycles

1. Can one use Deligne's theory of absolute Hodge cycles in positive characteristic ("Rational Tate cycles" in the sense of J. Milne)?
2. Could this give a less roundabout way to prove that $\mathrm{End}_{\mathcal{O}_L}(Y_1) \otimes_{\mathcal{O}_L} \mathrm{End}_{\mathcal{O}_L}(Y_2) \otimes_{\mathcal{O}_L} \mathrm{End}_{\mathcal{O}_L}(Y_3)$ is contained in $\mathrm{End}_{\mathcal{O}_L}(\bigotimes_{i\in\{1,2,3\}} Y_i)$?