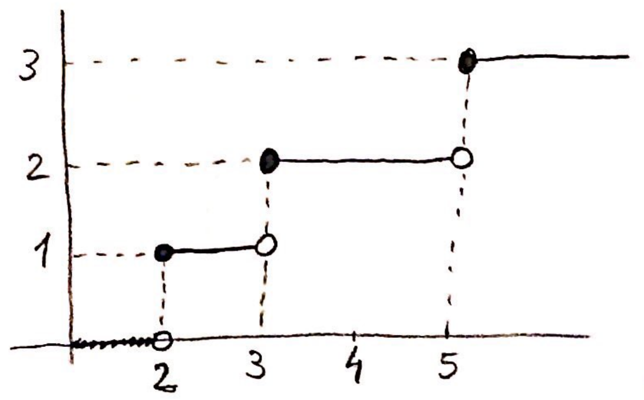
Prime Counting Functions: What is the behavior of $\pi(x)$?

$\pi(x) \Rightarrow$ number of prime numbers that are $\leq x$. $\pi(x)$ is a monotonically increasing function with jumps of height 1 at the primes.



As we go far away to the right, it becomes more and more challenging to sketch the graph of $\pi(x)$. It is not easy to see what happens to the graph of $\pi(x)$ on a macro scale.

(Euclid) $\pi(x) \to \infty$ as $x \to \infty$. How does $\pi(x)$ grow? Rate of growth?

$\pi(x)$ is $\underset{?}{\underline{\text{about}}}$ $\dfrac{x}{\log x}$ (Here $\log x = \ln x$).

(Legendre) $\pi(x)$ is $\underset{?}{\underline{\text{approximately}}}$ given by $\dfrac{x}{\log x - \underbrace{1.083\ldots}_{\text{correction term}}}$. Legendre cooked up his formula as a good fit to the tables of prime counts of his time (1830's).

(Legendre's formula) $\pi(x) - \pi(\sqrt{x}) + 1 = [x] - \sum_{p_1 \leq \sqrt{x}} \left[\frac{x}{p_1}\right] + \sum_{p_1 < p_2 \leq \sqrt{x}} \left[\frac{x}{p_1 p_2}\right]$

$- \cdots\cdots \Rightarrow$ Inclusion-Exclusion principle $\Rightarrow$ counts the number of all integers $\leq x$ that are divisible by no prime $\leq \sqrt{x}$. Any integer $\leq \sqrt{x}$ (except 1) is divisible by a prime $\leq \sqrt{x}$. Any composite (i.e. not prime) integer in $(\sqrt{x}, x]$ is divisible by a prime $\leq \sqrt{x}$. Thus the inclusion-exclusion also gives the number of primes in $(\sqrt{x}, x]$ together with 1 i.e. $\pi(x) - \pi(\sqrt{x}) + 1$.

Let P be the product of all primes $\leq \sqrt{x}$. Then $\pi(x) - \pi(\sqrt{x}) + 1 = \sum_{d \mid P} \mu(d) \left[\frac{x}{d}\right]$

where $\mu(d)$ is the Möbius function. We will revisit Legendre's findings.

$\pi(x) = \sum_{p \leq x} 1$ (p always denotes a prime number).

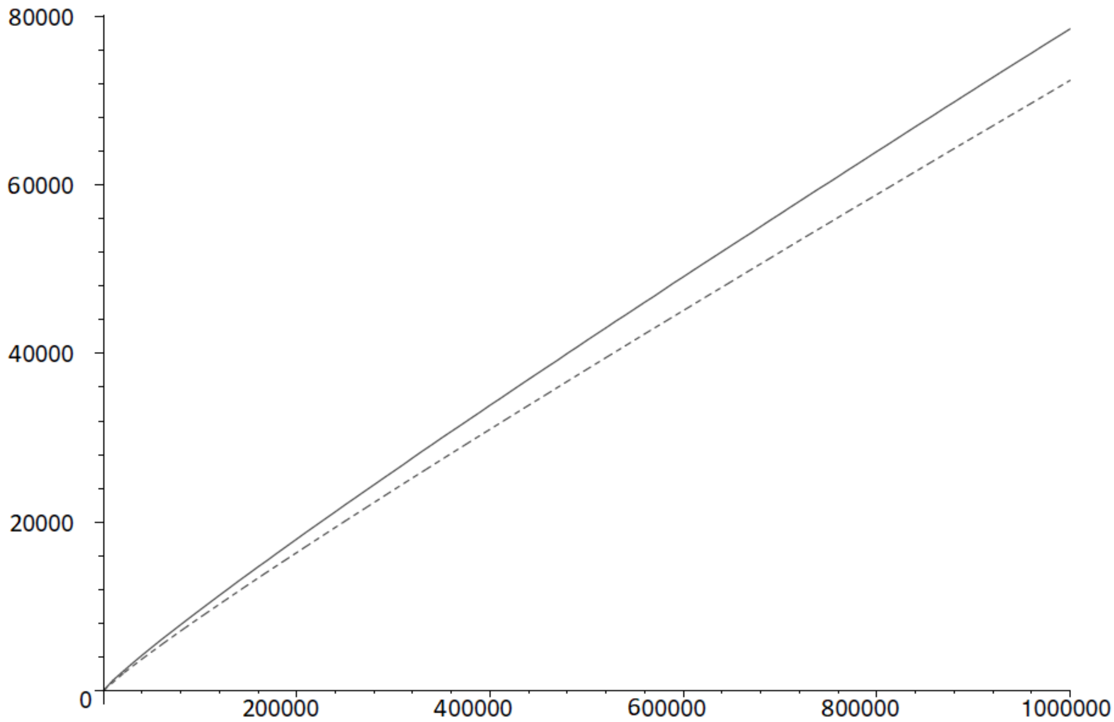


Figure 1.1 Graph of $\pi(x)$ (solid) and $x/\log x$ (dotted) for $2 \leq x \leq 10^6$.

Gauss (1800's) A better fit for $\pi(x)$ is $\operatorname{li} x = \int_2^x \frac{1}{\log t}\,dt$.

(Reasonable as $\frac{\pi(x)}{x}$ is about $\frac{1}{\log x}$ so the average value of $\frac{1}{\log t}$ over $[2, x]$ is expected to be a better fit for $\frac{\pi(x)}{x}$).

$\operatorname{li} x \Rightarrow$ logarithmic integral, $\frac{1}{\log t}$ does not have an elementary antiderivative so there is no easy closed form evaluation of $\int_2^x \frac{1}{\log t}\,dt$. Note that

$$\int_2^x \frac{1}{\log t}\,dt = \frac{x}{\log x} \underbrace{- \frac{2}{\log 2} + \int_2^x \frac{1}{(\log t)^2}\,dt}_{\text{smaller terms}}. \text{ Indeed } \int_2^x \frac{1}{(\log t)^2}\,dt$$

$\ll \frac{x}{(\log x)^2}$. What happens in the long run as $x \to \infty$? This is the asymptotic behavior. $f(x) \sim g(x)$, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

$\sim$ is an equivalence relation between functions.

Chebyshev (1850's) $(0.92\ldots)\frac{x}{\log x} < \pi(x) < (1.105\ldots)\frac{x}{\log x}$ for large x.

Chebyshev suggested a linearization of the prime counting function by introducing $\vartheta(x) = \sum_{p \le x} \log p$, $\psi(x) = \sum_{p^m \le x} \log p = \sum_{n \le x} \Lambda(n)$, where $\Lambda(n)$ is the von-Mangoldt function. He settled Bertrand's postulate that there is always a prime number in $(x, 2x]$ when $x \geq 1$. This is equivalent to showing that $\pi(2x) - \pi(x) > 0$ when $x \geq 1$. He also showed that $\liminf \frac{\pi(x)}{\operatorname{li} x} \le 1 \le \limsup \frac{\pi(x)}{\operatorname{li} x}$. His findings strongly suggest the asymptotic relation $\pi(x) \sim \operatorname{li} x \sim \frac{x}{\log x}$.

Riemann (1860's) "On the number of primes less than a given magnitude"

Classical problem: Find an efficient formula for the n^{th} prime number.

Table 1.1 Values of $\pi(x)$, li(x), $x/\log x$ *for* $x = 10^k$, $1 \leq k \leq 22$.

x	$\pi(x)$	$\mathrm{li}(x)$	$x/\log x$
10	4	5.12	4.34
10^2	25	29.08	21.71
10^3	168	176.56	144.76
10^4	1229	1245.09	1085.74
10^5	9592	9628.76	8685.89
10^6	78498	78626.50	72382.41
10^7	664579	664917.36	620420.69
10^8	5761455	5762208.33	5428681.02
10^9	50847534	50849233.90	48254942.43
10^{10}	455052511	455055613.54	434294481.90
10^{11}	4118054813	4118066399.58	3948131653.67
10^{12}	37607912018	37607950279.76	36191206825.27
10^{13}	346065536839	346065458090.05	334072678387.12
10^{14}	3204941750802	3204942065690.91	3102103442166.08
10^{15}	29844570422669	29844571475286.54	28952965460216.79
10^{16}	279238341033925	279238344248555.75	271434051189532.39
10^{17}	2623557157654233	2623557165610820.07	2554673422960304.87
10^{18}	24739954287740860	24739954309690413.98	24127471216847323.76
10^{19}	234057667276344607	234057667376222382.22	228576043106974646.13
10^{20}	2220819602560918840	2220819602783663483.55	2171472409516259138.26
10^{21}	21127269486018731928	21127269486616126182.33	20680689614440563221.48
10^{22}	201467286689315906290	201467286691248261498.15	197406582683296285295.97

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On the Number of Primes Less Than a Given Magnitude

by BERNHARD RIEMANN†

I believe I can best express my gratitude for the honor which the Academy has bestowed on me in naming me as one of its correspondents by immediately availing myself of the privilege this entails to communicate an investigation of the frequency of prime numbers, a subject which because of the interest shown in it by Gauss and Dirichlet over many years seems not wholly unworthy of such a communication.

In this investigation I take as my starting point the observation of Euler that the product

$$\Pi \frac{1}{1-\frac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

where p ranges over all prime numbers and n over all whole numbers. The function of a complex variable s which these two expressions define when they converge I denote by $\zeta(s)$. They converge only when the real part of s is greater than 1; however, it is easy to find an expression of the function which always is valid. By applying the equation

$$\int_0^\infty e^{-nx}x^{s-1}\,dx = \frac{\Pi(s-1)}{n^s},$$

one finds first

$$\Pi(s-1)\zeta(s) = \int_0^\infty \frac{x^{s-1}\,dx}{e^x-1}.$$

If one considers the integral

$$\int \frac{(-x)^{s-1}\,dx}{e^x-1}$$

$P_1 = 2$, $P_2 = 3$, $P_3 = 5$, $P_4 = 7$, $P_5 = 11, \ldots, P_n, \ldots$ With an efficient formula we can easily find P_{10^6}. We plug in $n = 10^6$ into our formula and compute. This problem is almost interchangable with the problem of finding an efficient formula for $\pi(x)$, and this is exactly what Riemann considered. Note that $n = \pi(P_n) \underset{\substack{\downarrow \\ \text{we expect}}}{=} li(P_n + e_n)$, where e_n is a relatively small error, we would like to control. Then $P_n = li^{-1} n - e_n$.

Through a study of $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $Re(s) > 1$, in particular using a transform of $\log \zeta(s)$ and then inverting it, Riemann obtained an expansion for $\pi(x)$ i.e. $Li(x) + \sum_{n=2}^{\infty} \frac{\mu(n)}{n} Li(x^{\frac{1}{n}})$ is empirically a very good approximation to $\pi(x)$. Here $Li(x) = \int_0^x \frac{1}{\log t} dt = \lim_{\varepsilon \to 0^+} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^x \right) \frac{1}{\log t} dt$ and $li\, x = Li(x) - Li(2)$. For a fixed x, the series on n terminates when $x^{\frac{1}{n}} < 2$.

$$\pi(x) \approx Li(x) - \frac{1}{2} Li(x^{\frac{1}{2}}) - \frac{1}{3} Li(x^{\frac{1}{3}}) - \frac{1}{5} Li(x^{\frac{1}{5}}) + \frac{1}{6} Li(x^{\frac{1}{6}}) - \frac{1}{7} Li(x^{\frac{1}{7}}) + \ldots$$

Since $Li(x) \sim \frac{x}{\log x}$, Riemann's empirical formula suggests that $\pi(x) = Li(x) - \frac{1}{2} Li(x^{\frac{1}{2}}) + O(x^{\frac{1}{3}})$. Incidentally, this formula also gives that $\pi(x) < Li(x)$ in the long run, a prediction made by Gauss earlier.

Hadamard - Vallée Poussin (1896) independently proved the long sought PNT

$\pi(x) \sim li\, x \sim \frac{x}{\log x}$. This does not tell much about the error in approximation, we may deduce only $|\pi(x) - li\, x| = o(li\, x)$

x	Riemann's error	Gauss's error
1,000,000	30	130
2,000,000	−9	122
3,000,000	0	155
4,000,000	33	206
5,000,000	−64	125
6,000,000	24	228
7,000,000	−38	179
8,000,000	−6	223
9,000,000	−53	187
10,000,000	88	339

Landau- Vallée Poussin (1900) PNT with classical error term

$$\pi(x) = \operatorname{li} x + O\left(x e^{-c\sqrt{\log x}}\right)$$ for some constant $c > 0$, as $x \to \infty$

Note that $e^{c\sqrt{\log x}}$ grows faster than $(\log x)^k$ for any $k > 0$. We may write $\pi(x) = \dfrac{x}{\log x} + \dfrac{x}{(\log x)^2} + O\left(\dfrac{x}{(\log x)^3}\right)$ as $x \to \infty$.

For Legendre's approximation, $\pi(x) = \dfrac{x}{\log x - A(x)}$ i.e. define $A(x) = \log x - \dfrac{x}{\pi(x)}$ when $x \geqslant 2$. Then $\dfrac{x}{\log x - A(x)} = \dfrac{x}{\log x} + \dfrac{x}{(\log x)^2}$ $+ O\left(\dfrac{x}{(\log x)^3}\right)$, so that $A(x) = \dfrac{1 + O\left(\frac{1}{\log x}\right)}{1 + \frac{1}{\log x} + O\left(\frac{1}{(\log x)^2}\right)}$. It follows that

$\lim\limits_{x \to \infty} A(x) = 1$. Therefore, Legendre's approximation is not the best in the long run. Among all asymptotic equivalents $\pi(x) \sim \dfrac{x}{\log x - A}$, the best is the case when $A = 1$. Next, we compare the number of primes in $[0, x]$ with the number of primes in $(x, 2x]$

i.e. compare $\pi(x)$ with $\pi(2x) - \pi(x)$. We can not distinguish between them with PNT as $\pi(x) = (1 + o(1))\dfrac{x}{\log x}$ and $\pi(2x) - \pi(x) =$

$$(1+o(1))\frac{2x}{\underbrace{\log 2x}_{\text{insignificant}}} - (1+o(1))\frac{x}{\log x} = (2+o(1))\frac{x}{\log x} - (1+o(1))\frac{x}{\log x} = (1+o(1))\frac{x}{\log x}$$

Thus we get $\pi(2x) - \pi(x) \sim \pi(x)$. We have $\pi(2x) = \dfrac{2x}{\log 2x} + \dfrac{2x}{(\log 2x)^2}$ $+ O\left(\dfrac{x}{(\log x)^3}\right)$ and $\pi(x) = \dfrac{x}{\log x} + \dfrac{x}{(\log x)^2} + O\left(\dfrac{x}{(\log x)^3}\right)$ and combining

$$\pi(2x)-2\pi(x)=\frac{2x}{\log 2x}-\frac{2x}{\log x}+\frac{2x}{(\log 2x)^2}-\frac{2x}{(\log x)^2}+O\left(\frac{x}{(\log x)^3}\right).$$

$$\frac{2x}{\log 2x}=\frac{2x}{\log x+\log 2}=\frac{2x}{\log x\left(1+\frac{\log 2}{\log x}\right)}=\frac{2x}{\log x}\left(1-\frac{\log 2}{\log x}+O\left(\frac{1}{(\log x)^2}\right)\right)$$

$=\frac{2x}{\log x}-\frac{(2\log 2)\,x}{(\log x)^2}+O\left(\frac{x}{(\log x)^3}\right)$ i.e. $\frac{2x}{\log 2x}-\frac{2x}{\log x}=-\frac{(2\log 2)\,x}{(\log x)^2}$ $+O\left(\frac{x}{(\log x)^3}\right)$. Similarly, $\frac{2x}{(\log 2x)^2}=\frac{2x}{(\log x)^2\left(1+\frac{\log 2}{\log x}\right)^2}=\frac{2x}{(\log x)^2}+O\left(\frac{x}{(\log x)^3}\right)$

and $\frac{2x}{(\log 2x)^2}-\frac{2x}{(\log x)^2}=O\left(\frac{x}{(\log x)^3}\right)$. Thus $\pi(2x)-2\pi(x)=-\frac{(2\log 2)\,x}{(\log x)^2}$ $+O\left(\frac{x}{(\log x)^3}\right)$. i.e. $\lim_{x\to\infty}\big(\pi(2x)-\pi(x)\big)-\pi(x)=-\infty$. There are fewer primes in the interval $(x,2x]$. This was first remarked by Landau.

Littlewood (1915) $\pi(x)-Li(x)$ changes sign infinitely often as $x\to\infty$.

The formula $\pi(x)=Li(x)-\frac{1}{2}Li(x^{\frac{1}{2}})+O(x^{\frac{1}{3}})$ is not true in the long run.

Back to Legendre's formula: $\pi(x)-\pi(\sqrt{x})+1=\sum_{d|P}\mu(d)\left[\frac{x}{d}\right]=$

$$\sum_{d|P}\mu(d)\left(\frac{x}{d}-\left\{\frac{x}{d}\right\}\right)=\underbrace{x\prod_{p\le\sqrt{x}}\left(1-\frac{1}{p}\right)}_{\text{Main term}}\underbrace{-\sum_{d|P}\mu(d)\left\{\frac{x}{d}\right\}}_{\text{error}}.$$

$\left|-\sum_{d|P}\mu(d)\left\{\frac{x}{d}\right\}\right|\le 2^{\omega(P)}=2^{\pi(\sqrt{x})}$, $2^{\pi(\sqrt{x})}\gtrsim 2^{\frac{c\sqrt{x}}{\log\sqrt{x}}}\Rightarrow$ much bigger than x so useless. What is the magnitude of the error? We know $\pi(x)-\pi(\sqrt{x})+1=(1+o(1))\frac{x}{\log x}$. Mertens showed that (1870's)

$\prod_{p\le\sqrt{x}}\left(1-\frac{1}{p}\right)=\frac{e^{-\gamma}+o(1)}{\log\sqrt{x}}=\frac{2e^{-\gamma}+o(1)}{\log x}$, where γ is the Euler-Mascheroni constant, defined by $\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log n\right)=0.5772\ldots$

$2e^{-\gamma} = 1.12\ldots$ i.e. $\sum_{d|P} \mu(d)\left\{\frac{X}{d}\right\} = (\underbrace{2e^{-\gamma}-1}_{>0} + o(1))\frac{X}{\log X}$

Error term is genuinely big! A result of Schoenfeld (1976)

RH is equivalent to $|\pi(x) - Li(x)| < \frac{1}{8\pi}\sqrt{x}\log x$ for all $x \geqslant 2657$.

Best result: $\pi(x) = Li(x) + O\left(x e^{-\frac{c(\log x)^{\frac{3}{5}}}{(\log\log x)^{\frac{1}{5}}}}\right)$ for some $c>0$.

(Vinogradov–Korobov, 1960's). Primes in arithmetic progressions

$$\pi(x, q, a) = \frac{Li(x)}{\phi(q)} + O\left(x e^{-c\sqrt{\log x}}\right).$$

Some evidence supporting RH: Analogs of prime numbers are irreducible polynomials. We have PNT for polynomials:

Let a_n be the number of monic irreducible polynomials of degree n in the polynomial ring $\mathbb{F}[x]$, where $\mathbb{F}$ is a finite field with q elements (so q is a prime power). Then $a_n = \frac{q^n}{n} + O\left(\frac{q^{\frac{n}{2}}}{n}\right)$.

We know from the work of Gauss that $a_n = \frac{1}{n}\sum_{d|n} \mu(d) q^{\frac{n}{d}} = \underbrace{\frac{q^n}{n}}_{d=1}$ $\underbrace{- \frac{q^{\frac{n}{2}}}{n}}_{d=2 \text{ if } n \text{ is even}} \pm \underbrace{\left(\text{terms with } \frac{q^m}{n}, \text{where } m \leq \frac{n}{3}\right)}_{\text{number of such terms is } \leq 2^{\omega(n)} \leq P_1 \cdots P_{\omega(n)} \leq n}$

(distinct prime divisors of n: $P_1 \cdots P_{\omega(n)}$)

Contribution of such terms with $m \leq \frac{n}{3}$ is at most $n \cdot \frac{q^{\frac{n}{3}}}{n} = q^{\frac{n}{3}} = O\left(\frac{q^{\frac{n}{2}}}{n}\right)$. Result follows. If we let $X = q^n$, then $n = \log_q X$ i.e.

$$a_n = \frac{X}{\log_q X} + O\left(\frac{X^{\frac{1}{2}}}{\log_q X}\right) \Rightarrow \text{Error term is at the quality of RH.}$$