Solving Fermat Type Equations by Modular Approach

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Diophantine Equations

Diophantine Equation: an indeterminate polynomial equation with integral coefficients for which integral solutions are sought

- $ax + by = c$: linear equation
- $y^2 = x^3 + ax + b$: elliptic curve
- $x^n + y^n + z^n = 0$: Fermat's equation
- $Ax^n + By^n + Cz^n = 0$: generalized Fermat's Equation
- $x^p + y^q + z^r = 0$: Fermat's equation with signature (p, q, r)

Fermat's Last Theorem

- **Fermat's Last Theorem:** The equation $x^n + y^n + z^n = 0$, where x, y, z and n are integers, has no non-trivial solutions ($xyz \neq 0$) for $n > 2$.

- It is sufficient to consider cases $n = 4$ and n is an odd prime.

- $n = 4$: Fermat
  $p = 3, 5, 7$: Euler, Legendre, Dirichlet, Gauss, Lamé

- p is regular prime i.e. $p \nmid h(\mathbb{Q}(\zeta_p))$, $\zeta_p = e^{2\pi i/p}$: Kummer

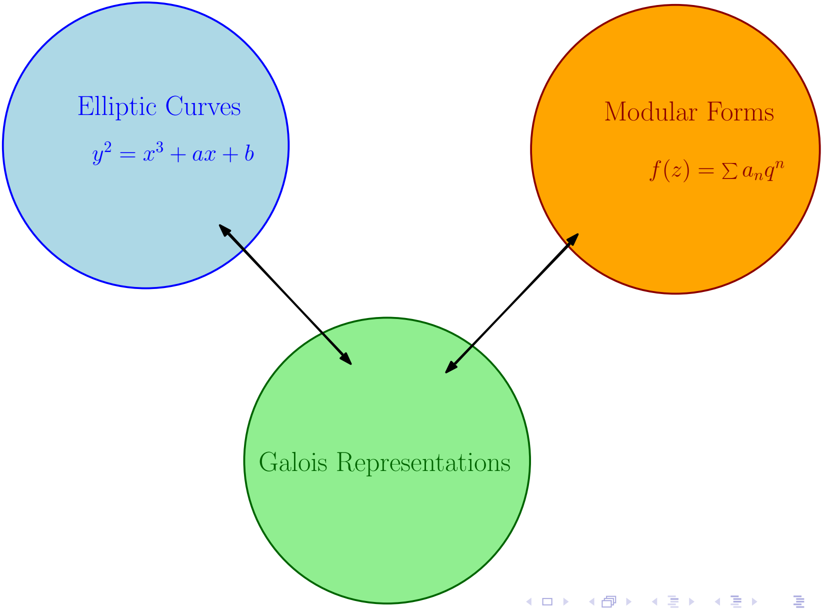
Theorem (Wiles, Taylor, 1995)

Let $p \geq 3$ be a prime. Then $x^p + y^p + z^p = 0$ has no non-trivial integer solutions.

Modular approach

- Find an elliptic curve associated to a putative solution
- Show that this elliptic curve has properties contradicting to each other
    1. Modularity theorem (Wiles, Taylor-Wiles)
    2. Irreducibility of Galois representations (Mazur)
    3. Level lowering theorem (Ribet)

Comparing Different Worlds

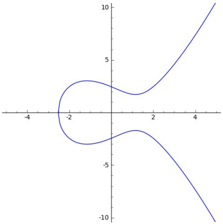


Elliptic curves

Definition

An elliptic curve E over a field K is a smooth, projective algebraic curve of genus one, on which there is a specified point $\mathcal{O}$. The point $\mathcal{O}$ is called point at infinity.

- Given by $y^2 = x^3 + ax + b$, where a, b are in K if $\mathrm{char}(K) \neq 2, 3$.
- $E(K) = \{(x, y) \in K^2 : y^2 = x^3 + Ax + B\} \cup \{\mathcal{O}\}$.
- No cusp or self-intersection, $E : y^2 = x^3 - 4x + 6$ over $\mathbb{R}$



Elliptic curves mod p

- Given E, we can reduce it modulo p.
- Then we say E has good reduction at p if the reduced curve $\tilde{E}$ is smooth.
- We say E has bad reduction at p if the reduced curve $\tilde{E}$ is singular.
- There can be only finitely many bad primes. The conductor of E, denoted by N_E, is an invariant of E which encodes the bad primes.

Example

$E : y^2 = x^3 - x + 1$, find solutions mod 3.
$E(\mathbb{F}_3) = \{(0,1),(0,2),(1,1),(1,2),(2,1),(2,2)\}$. Note that point at ∞ is always in the set of solutions, $|E(\mathbb{F}_3)| = 7$.

- We can do this modulo many other primes p.
  Say N_p is the number of solutions mod p.

p	3	5	7	11	13	17	19	29	31	37	41
N_p	7	8	12	10	19	14	22	37	35	36	51

- Number of solutions increase as p increases.

Definition

For an elliptic curve E, define $a_p = p + 1 - N_p$.

- Can we predict a_p?
- For example, for $E : y^2 = x^3 - x + 1$ we have

p	3	5	7	11	13	17	19	29	31	37	41
N_p	7	8	12	10	19	14	22	37	35	36	51
a_p	-3	-2	-4	2	-5	4	-2	-7	-3	2	-9

Modular Forms

- A modular form of weight $2k$ wrt $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function on the upper half plane satisfying :

$$f(\frac{az+b}{cz+d}) = (cz+d)^{2k} f(z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

  and a growth condition.

- **newform of level N**: a special kind of modular form for the group $\Gamma_0(N)$.

- Modular forms have power series representations i.e. they can be written as $\sum_{n=0}^{\infty} c_n q^n$ where $q = e^{2\pi i}$.

- Let E/K be an elliptic curve and $m \in \mathbb{Z}$ with $m \geq 1$. The m-torsion subgroup of E, denoted by $E[m]$, is the set of points of E of order m,

$$E[m] = \{P \in E(K) : [m]P = O\}.$$

- $E[m] = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ if $char(\overline{K}) = 0$.

- Let $G_K = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K. For an elliptic curve E/K,

$$\overline{\rho}_{E,p} : G_K \longrightarrow \mathrm{Aut}(E[p]) \cong \mathrm{GL}_2(\mathbb{F}_p)$$

denotes the mod p Galois representation of E.

Solving $x^p + y^p + z^p = 0$

Three main ingredients:

- Modularity Thm: Every elliptic curve over $\mathbb{Q}$ is associated to a rational newform of level N i.e.
  there is a newform $f(z) = \sum_{n=1}^{\infty} c_n q^n$ such that
  $c_l = a_l(E) = l + 1 - |E(\mathbb{F}_l)|$.
- Mazur's Thm: If E is an elliptic curve over $\mathbb{Q}$ with full two torsion then E doesn't have any p isogenies for $p \geq 5$ (irreducibility of Galois representations attached to the elliptic curve for each p)
- Ribet's Thm: Sometimes it is possible, due to Ribet's work, to replace f by another newform of smaller level if we have modularity of E and irreducibility of mod p Galois representations.

Basic sketch

- Suppose (a, b, c) is a solution.
  Scale a, b and c so that they become coprime integers.

- Attach the Frey elliptic curve to this solution:

$$E : y^2 = x(x - a^p)(x + b^p).$$

- Let $\bar{\rho}_{E,p}$ be its mod p Galois representation.

- $\bar{\rho}_{E,p}$ is irreducible by Mazur[1978] and modular by Wiles[1995].
- Apply Ribet's level lowering theorem[1990] and conclude that $\bar{\rho}_{E,p}$ arises from a weight 2 newform f of level 2.
- There are no such newforms at this level, so we get a contradiction.
- Hence, the equation $x^p + y^p + z^p = 0$ does not have any solutions.

Fermat equation over higher degree number fields

- K: a number field
- $\mathcal{O}_K$: its ring of integers and p be a prime.
- We refer the equation

$$a^p + b^p + c^p = 0, \quad a, b, c \in \mathcal{O}_K$$

as the *Fermat equation over K* with the exponent p.

Conjecture (Asymptotic Fermat Conjecture)

Let K be a number field such that $\zeta_3 \notin K$. There is a constant B_K depending only on K such that for any prime $p > B_K$, all solutions to the Fermat equation are trivial i.e. $abc = 0$.

- treat the Fermat equation with fixed exponent p as a curve and determine the points of low degree (i.e. points defined over number fields of low degree) on the Fermat curve

  For $p = 3, 5, 7$ and 11, Gross and Rohrlich (78) determined the solutions to $x^p + y^p + z^p = 0$ over all number fields K of degree $\leq (p-1)/2$.

- try to use modular approach

Theorem (Jarvis and Meekin, 2004)

The Fermat equation $x^n + y^n = z^n$ has no solutions $x, y, z \in \mathbb{Q}(\sqrt{2})$ with $xyz \neq 0$ and $n \geq 4$.

Example (Serre and Mazur)

- Consider the equation

$$x^p + y^p + L^r z^p = 0,$$

where $L = 1$ or an odd prime and $0 < r < p$, $p \neq L$, $p \geq 5$ prime.

- Assume $(x, y, z) \in \mathbb{Z}^3$ is a non-trivial solution and $gcd(x, y, Lz) = 1$.

- Let $(A, B, C) = (x^p, y^p, L^r z^p)$, $A \equiv -1 \pmod 4$ and $2|B$.

- $E : Y^2 = X(X - A)(X + B)$ (Frey curve)

- Mazur, Wiles, Ribet $\implies$ $E \sim f$ and f newform of weight 2 and level $N = 2L$

- If $L = 1$, then $N = 2$ (the FLT case). Since there are no newforms of weight 2 and level 2, there are no non-trivial solutions for $L = 1$.

- There are no newforms of weight 2 and levels $6, 10, 22$, i.e. there are non non-trivial solutions for $L = 3, 5, 11$.

- There are newforms of weight 2 and level $2L$ for $L = 7, 13, 17, \ldots$

- We need to study the relationship between E and f where f has a q-expansion

$$f = q + \sum_{n=1}^{\infty} c_n q^n$$

.

- Let $K_f = \mathbb{Q}(c_1, c_2, \dots)$. K_f is totally real and actually c_n belongs to the ring of integers $\mathcal{O}$ of K_f.

Definition

*A newform f is **irrational** if $K_f \neq \mathbb{Q}$ and **rational** if $K_f = \mathbb{Q}$.*

- It can be shown that if f is irrational,then the exponent p is bounded for non-trivial solutions to $x^p + y^p + L^r z^p = 0$, e.g. this is the case when $L = 37$.

- There is a rational f corresponding to E for the remaining $L = 7, 13, 17, 19, 23, \ldots$.
- **Eichler-Shimura Relation:** A rational weight 2, level N newform f corresponds to an isogeny class of elliptic curves E' defined over $\mathbb{Q}$ of conductor N.
- Actually, this case also can be reduced to the case in which E' is isogenous to an elliptic curve with full 2-torsion.
- **Question:** What are the odd primes L for which there is an elliptic curve E' over $\mathbb{Q}$ with full 2-torsion and conductor $2L$?

Lemma

Let L be an odd prime. Then there is an elliptic curve E' over $\mathbb{Q}$ with full 2-torsion and conductor $2L$ if and only if L is a Mersenne or a Fermat prime and $L \geq 31$.

Proof.

E' has a model

$$E' : y^2 = x(x-a)(x+b)$$

where $a, b \in \mathbb{Z}$, $ab(a+b) \neq 0$, and $\Delta_{E'} = 16a^2b^2(a+b)^2$. We can choose a, b so that the model is minimal away from 2. Hence,

$$a^2b^2(a+b)^2 = 2^u L^v$$

for some nonnegative integers u, v. Then we obtain

$$a = \pm 2^{u_1} L^{v_1}, \quad b = \pm 2^{u_2} L^{v_2}, \quad a+b = \pm 2^{u_3} L^{v_3}$$

It follows that L is a Mersenne or a Fermat prime and $L \geq 31$. □

Theorem (Serre and Mazur)

Let L be an odd prime. Suppose $L < 31$, or L is neither a Mersenne nor a Fermat prime. Then there is a constant C_L s.t.for all primes $p > C_L$ the only solutions $(x, y, z) \in \mathbb{Z}^3$ to the equation $x^p + y^p + L^r z^p = 0$ are the trivial ones satisfying $xyz = 0$.

Summary

- The equation

$$\pm 2^{u_1} L^{v_1} \pm 2^{u_2} L^{v_2} = \pm 2^{u_3} L^{v_3}$$

  is called an S-unit equation with $S = \{2, L\}$.
- It is possible to relate non-trivial solutions to Fermat type equations to solutions of certain S-unit equations.

S-unit Equation

- K: a number field, $\mathcal{O}_K$: its ring of integers
  S: finite set of prime ideals of $\mathcal{O}_K$
- S-integers in K:

$$\mathcal{O}_S = \{\alpha \in K^* : v_{\mathfrak{P}}(\alpha) \geq 0 \text{ for all } \mathfrak{P} \notin S\}$$

- S-units in K:

$$\mathcal{O}_S^* = \{\alpha \in K^* : v_{\mathfrak{P}}(\alpha) = 0 \text{ for all } \mathfrak{P} \notin S\}$$

- The S-unit equation is

$$\lambda + \mu = 1, \quad \lambda, \mu \in \mathcal{O}_S^*$$

Examples of S-units

- $K = \mathbb{Q}$, $\mathcal{O}_K = \mathbb{Z}$, and $S = \{2\}$.

  $\mathcal{O}_S = \{\pm 2^r m : m \in \mathbb{Z},\ r \in \mathbb{Z}\},\ \ \mathcal{O}_S^* = \{\pm 2^r : r \in \mathbb{Z}\}$

  S-unit equation solutions : $(1/2, 1/2), (2, -1), (-1, 2)$

- $K = \mathbb{Q}$, $\mathcal{O}_K = \mathbb{Z}$, and $S = \{2, L\}$.

  $\mathcal{O}_S = \{\pm 2^r L^s m : m \in \mathbb{Z},\ r, s \in \mathbb{Z}\},$

  $\mathcal{O}_S^* = \{\pm 2^r L^s : r, s \in \mathbb{Z}\}$

- $K = \mathbb{Q}(\sqrt{5})$, $\mathcal{O}_K = \{a(\frac{1+\sqrt{5}}{2}) + b : a, b \in \mathbb{Z}\}$, and $S = \{\emptyset\}$.

  $\mathcal{O}_S^* = \{\pm(\frac{1+\sqrt{5}}{2})^r : r \in \mathbb{Z}\}$,

  S-unit equation solutions : $(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$

- $K = \mathbb{Q}(\sqrt{5})$ and $S = \{2\mathcal{O}_K\}$

  $\mathcal{O}_S^* = \{\pm 2^r(\frac{1+\sqrt{5}}{2})^s : r, s \in \mathbb{Z}\}$,

Conjecture (Asymptotic Fermat Conjecture)

Let K be a number field such that $\zeta_3 \notin K$. There is a constant B_K depending only on K such that for any prime $p > B_K$, all solutions to the Fermat equation are trivial i.e. $abc = 0$.

Theorem (Freitas and Siksek, 2015)

Let K be a totally real field. The asymptotic Fermat's last theorem holds for K satisfying some explicitly given, algorithmically testable criterion.

- In particular, they show that the criterion in the above theorem is satisfied by $K = \mathbb{Q}(\sqrt{d})$ for a subset of $d \geq 2$ having density $5/6$ among the squarefree positive integers. This density becomes 1 if "Eichler-Shimura conjecture" is assumed.
- Şengün and Siksek[2018] proved the asymptotic Fermat's Last Theorem holds for any number field K by assuming "modularity".

Assumptions

- **K: totally real number field**
  (I)An "Eichler-Shimura" Conjecture over K: Let K be a totally real field. Let $\mathfrak{f}$ be a Hilbert newform of level $\mathcal{N}$ and parallel weight 2 and with rational eigenvalues. Then there is an elliptic curve $E_{\mathfrak{f}}/K$ with conductor $\mathcal{N}$ having the same L-function as $\mathfrak{f}$.

Assumptions

- **K: a general number field**

(I) Serre's modularity Conjecture over K: This associates a totally odd, continuous, finite flat, absolutely irreducible 2 dimensional mod p representation of $\mathrm{Gal}(\overline{K}/K)$ a cuspform of parallel weight 2 whose level is equal to the prime-to-p part of the Artin conductor of the representation.

(II) An "Eichler-Shimura" Conjecture over K: This associates to a weight 2 cuspform with rational Hecke eigenvalues either an elliptic curve or a "fake elliptic curve".

We call (I) and (II) together as "modularity".

Our results

- K: a number field
  $\mathcal{O}_K$: its ring of integers and p be a prime.

- **generalized Fermat equation:**

  $$Ax^p + By^p + Cz^p = 0 \text{ where } A, B, C \text{ are odd integers}$$

  i.e. if $\mathfrak{P}$ is a prime of $\mathcal{O}_K$ lying over 2, then $\mathfrak{P} \nmid ABC$.

- Our main theorem depends on the "modularity" conjecture since the analogues of modularity theorem have not been proven yet in general.

Our results

Main Theorem (K., Ozman)

K: a number field satisfying the "modularity"

$\mathcal{O}_S^*$: the set of S-units of K,
S: set of primes dividing $2ABC$

S-unit equation: $\lambda + \mu = 1, \qquad \lambda, \mu \in \mathcal{O}_S^*$

Suppose that for every solution (λ, μ) to the S-unit equation, there is some $\mathfrak{P} \in U$ that satisfies

$$max\{|v_{\mathfrak{P}}(\lambda)|, |v_{\mathfrak{P}}(\mu)|\} \leq 4v_{\mathfrak{P}}(2).$$

Then there is a constant $\mathcal{B} = \mathcal{B}(K, A, B, C)$ such that the generalized Fermat equation with exponent p and coefficients A, B, C does not have non-trivial solutions with $p > \mathcal{B}$.

Existence and Density Theorems

Density Theorem (K., Ozman)

Assuming the "modularity", the asymptotic Fermat's Last Theorem holds for 5/6 of the imaginary quadratic number fields.

Theorem 1 (K., Ozman)

$K = \mathbb{Q}(\sqrt{-d})$, and $-d \equiv 2, 3 \pmod 4$

$q \geq 29$: prime, and $q \equiv 5 \pmod 8$ and $\left(\frac{-d}{q}\right) = -1$

Assume the "modularity".

Then there exists a constant depending on K and q, namely $B_{K,q}$, such that for all $p > B_{K,q}$ the Fermat equation $x^p + y^p + q^r z^p = 0$ doesn't have any non-trivial solutions.

Theorem 2 (K., Ozman)

$K = \mathbb{Q}(\sqrt{-d})$ and $d \equiv 7 \pmod 8, d \equiv 5 \pmod 6$ and $d \not\equiv 7 \pmod{14}$

Assume the “modularity”.

Then there exists a constant depending on K, namely B_K, such that for all $p > B_K$ the Fermat equation $x^p + y^p + z^p = 0$ doesn't have any nontrivial solutions.

Comparing the two density results

Density Theorem (Freitas,Siksek)

Assuming the "Eichler-Shimura", the asymptotic Fermat's Last Theorem holds for a set of real quadratic fields of density 1.

Density Theorem (K., Ozman)

Assuming the "modularity", the asymptotic Fermat's Last Theorem holds for 5/6 of the imaginary quadratic number fields.

What is the reason for the disparity?

- Because the conclusion of the Eichler-Shimura conjecture over real quadratic fields is stronger than the conclusions of the E-S over imaginary quad. fields

Comparing the two density results

- **K:real quadratic field**

  a rational weight 2 Hilbert eigenform $\mathfrak{f}$ over K corresponds to an ell. curve E/K

- **K: imag. quad. field**

  a rational weight 2 Bianchi eigenform over K corresponds to either an ell. curve E/K or a fake ell. curve A/K (an abelian surface whose endomorphism alg. is an indefinite division quaternion algebra)
- If 2 splits or ramifies in K, then we can eliminate the fake ell. curve case

- If 2 is inert in $\mathbb{Q}(\sqrt{-d})$ i.e. $-d \equiv 5 \pmod 8$, we cannot eliminate fake ell. curves

- This is exactly 1/6 of all imag. quad.fields.

Idea behind the proof

- attach the Frey curve
- enough of modularity, irreducibility and level lowering known for totally real fields and assume "modularity" for a general number field
- get newform of weight 2 and some level $\mathcal{N}$
- there are newforms at the level $\mathcal{N}$, so no contradiction yet
- for p sufficiently large, we can get E' with full 2-torsion and good reduction outside the prime factors of $\mathcal{N}$
- parametrize such elliptic curves with the solution of S-unit equation and get a contradiction by using the valuation condition on S-units

Sketch of the proof

- Let $G_K = \mathrm{Gal}(\bar{K}/K)$ be the absolute Galois group of K. For an elliptic curve E/K,

$$\bar{\rho}_{E,p} : G_K \longrightarrow \mathrm{Aut}(E[p]) \cong \mathrm{GL}_2(\mathbb{F}_p)$$

denotes the mod p Galois representation of E.

- Let (a, b, c) be a solution of the Fermat equation.

- Attach the Frey curve

$$E : y^2 = x(x - Aa^p)(x + Bb^p).$$

- Compute the discriminant Δ_E and the j-invariant of j_E of the Frey elliptic curve.

- **Irreducibility of Galois representations**: If p is large enough, then $\bar{\rho}_{E,p}$ is irreducible.

- **Modularity:** modularity conjecture from Langlands programme

- **Level lowering:**
  - There is a non-trivial weight 2 new complex eigenform $\mathfrak{f}$ which has an associated elliptic curve $E_{\mathfrak{f}}/K$ of conductor $\mathfrak{N}'$ dividing $\mathfrak{N}$ with $\bar{\rho}_{E,p} \sim \bar{\rho}_{E_{\mathfrak{f}},p}$.

  - There is an elliptic curve E'/K where E' has full 2-torsion with $\bar{\rho}_{E,p} \sim \bar{\rho}_{E,p}$.

Elliptic Curves with full 2-torsion and solutions to the S-unit equation

- $\mathfrak{S}_S$: the set of all elliptic curves over K with full 2-torsion and potentially good outside S
- $E_1 \sim E_2$ on $\mathfrak{S}_S$: E_1 and E_2 are isomorphic $\bar{K}$
- $\Delta_S = \{(\lambda, \mu) : \lambda + \mu = 1,\ \lambda, \mu \in \mathcal{O}_S^*\}$
- $\mathfrak{S}_3$, the symmetric group on 3 letters, acts on Δ_S.
- There is a bijection between $\mathfrak{S}_3 \setminus \Delta_S$ and $\mathfrak{S}_S / \sim$
- The orbit of (λ, μ) is sent to the class of the Legendre elliptic curve $y^2 = x(x-1)(x-\lambda)$.

Proof of the Main Theorem

- Let K be a number field satisfying Conjectures and S, U, V be the sets we defined before with $U \neq \emptyset$.
- Let (a, b, c) be a non-trivial solution to the Fermat equation.
- For the solution above, attach the Frey curve
- Apply level lowering and obtain an elliptic curve E'/K having full 2-torsion and potentially good reduction away form S with j-invariant j' satisfying $v_{\mathfrak{P}}(j') < 0$ for all $\mathfrak{P} \in U$.
- We can express j' in terms of λ and μ and by using the condition

$$max\{|v_{\mathfrak{P}}(\lambda)|, |v_{\mathfrak{P}}(\mu)|\} \leq 4v_{\mathfrak{P}}(2)$$

we deduce that $v_{\mathfrak{P}}(j') > 0$, contradiction.

Signature $(p, p, 2)$

- We consider the equation $x^p + y^p = z^2$ over number fields K.
- The strategy is the same, we apply the modular approach.
- The Frey curve attached to $x^p + y^p = z^2$ is not "symmetric".

$$E : y^2 = x(x - Aa^p)(x + Bb^p) \text{ for the Fermat equation.}$$

$$E = E_{a,b,c} : Y^2 = X^3 + 4cX^2 + 4a^pX.$$

for the equation $x^p + y^p = z^2$.

- Solving the S-unit equation over K is not enough but we need to solve them over some extensions L of K.

Our Result

Theorem (Isik, K., Ozman)

K:a totally real number field with narrow class number $h_K^+ = 1$,

$L = K(\sqrt{a})$ for each $a \in K(S_K, 2)$,

S_K-unit equation: $\lambda + \mu = 1, \qquad \lambda, \mu \in \mathcal{O}_{S_K}^*$,

Suppose that for every solution (λ, μ) to the S_K-unit equation, there is some $\mathfrak{P} \in T_K$ that satisfies

$$\max\{|v_{\mathfrak{P}}(\lambda)|, |v_{\mathfrak{P}}(\mu)|\} \leq 4v_{\mathfrak{P}}(2)$$

.

Suppose also that for each L, for every solution (λ, μ) to the S_L-unit equation, there is some $\mathfrak{P}' \in T_L$ that satisfies

$$\max\{|v_{\mathfrak{P}'}(\lambda)|, |v_{\mathfrak{P}'}(\mu)|\} \leq 4v_{\mathfrak{P}'}(2)$$

Theorem

Then there is a constant B_K -depending only on K- such that for $p > B_K$, the equation $x^p + y^p = z^2$ has no solution $(a, b, c) \in W_K$.

W_K: the set of $(a, b, c) \in \mathcal{O}_K$ such that $a^p + b^p = c^2$ with $\mathfrak{P} | b$ for every $\mathfrak{P} \in T_K$

In this case we say that asymptotic Fermat's Last Theorem holds for W_K.

Thank you!