Dualities in Noncommutative Geometry

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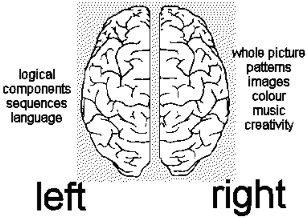
What is Noncommutative Geometry?

- In mathematics each object (or subject) can be looked at in two different ways:
  Geometric or Algebraic



NCG

- NCG, created by A. Connes, is the name of a very young and fast developing mathematical theory (1980).
- A great idea here is to find algebraic generalizations of most of the structures currently available in mathematics: measurable, topological, differential, metric, vector bundle, ...

NCG Idea

The fundamental idea, implicitly used in A. Connes' noncommutative geometry is a powerful extension of R. Decartes' analytic geometry:

- To trade geometrical spaces X of points with their Abelian algebras of (complex or real valued) functions $f : X \longrightarrow \mathbb{C}$;

Example

$$\text{Manifold } M \rightsquigarrow C^{\infty}(M).$$

- Then we find a way to translate the geometrical properties of spaces into algebraic properties of the associated algebras.

NCG Idea

To develop noncommutative geometries, we usually proceed as follows:

- 1) First we find a suitable way to codify or translate the geometric properties of a space X (topology, measure, differential structure, metric, . . .) in algebraic terms, using a commutative algebra of functions over X.
- 2) Then we try to see if this codification survives" generalizing to the case of noncommutative algebras.
- 3) Finally the generalized properties are taken as axioms defining what a dual of a noncommutative" (topological, measurable, differential, metric, . . .) space is, without referring to any underlying point space.

  Of course the process of generalization of the properties from the commutative to the noncommutative algebra case is highly non trivial and, as a result, several alternative possible axiomatizations arise in the noncommutative case, corresponding to a unique commutative limit".

Original space $X \Longleftarrow$ *Algebra*

A reverse process: To reconstruct the original geometric space X, a technique that appeared for the first time in the work of I. Gelfand on Abelian C^*-algebras. (1939).

A similar ideas previously used also in algebraic geometry in P. Cartier, A. Grothendieck's definition of schemes.

Heisenberg's Commutatation Relation- 1925

- Why NONCOMMUTATIVE?
- Heisenberg's commutation relation:

$$\textit{classical} \quad \textit{mechanics} \rightsquigarrow \textit{quantum} \quad \textit{mechanics}$$

$$pq - qp = \frac{h}{2\pi i} 1$$

$$\textit{commutative} \quad \textit{algebra} \quad \textit{functions} \rightsquigarrow \textit{noncommutative} \quad \textit{algebra} \quad \textit{opera}$$

Serre(1955)-Swan(1962) Theorem

R. G. Swan, Transactions of the American Mathematical Society, Vol. 105, No. 2 (Nov., 1962), pp.264-277.

Serre has shown that there is a one-to-one correspondence between algebraic vector bundles over an affine variety and finitely generated projective modules over its coordinate ring.

For some time, it has been assumed that a similar correspondence exists between topological vector bundles over a compact Hausdorff space X *and finitely generated projective modules over the ring of continuous real-valued functions on* X.

However, no rigorous treatment of the correspondence seems to have been given. I will give such a treatment here.

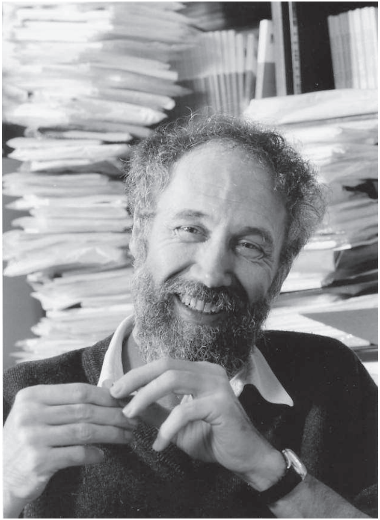
Serre(1955)-Swan(1962) Theorem

- $M =$ compact manifold.
- $C^\infty(M) =$ smooth real-valued functions on M.
- $V =$ smooth vector bundle over M.
- $\Gamma(V) =$ space of smooth sections of V.
- Γ is a module over $C^\infty(M)$

$$C^\infty(M) \times \Gamma \longrightarrow \Gamma, \quad (f \triangleright s)(m) = f(m)s(m)$$

- Swan $\rightsquigarrow$ Γ is finitely generated projective module on $C^\infty(M)$.
- One to one correspondence:
  $\{$ smooth vector bundles on $M\}$ $\leftrightsquigarrow$ $\{$ F.G.P modules on $C^\infty(M)\}$

Alain Connes



A Dictionary In NCG

commutative, noncommutative
measure space, Von Neumann algebra
locally compact space, C^*- algebra
vector bundle, finitely projective module
vector field,derivation
integral,trace
de Rham complex , Hochschild homology
de Rham cohomology, cyclic homolgy
closed de Rham current , cyclic cocycle
spin Riemannian manifold, spectral triple
group, Lie algebra, Hopf algebra, quantum group
symmetry, action of Hopf algebra
principal bundle, Hopf Galois extension

References

- A. Connes, Noncommutative Geometry, Academic Press (1994).
- M. Khalkhali, Basic noncommutative geometry, European Mathematical Society (EMS), (2009)

Gelfand-Naimark Theorem-1943

Theorem

a)

$$\{Locally\ compact\ Huausdorf\ spaces\} \cong \{Commutative\quad C^*\quad algebras\}^{op}$$

or

$$\{Compact\ Huausdorf\ spaces\} \cong \{Commutative\ unital\quad C^*\quad algebras\}^{op}$$

b) Every C^*-algebra is isomorphic to a *-subalgebra of the algebra of bounded operators on a Hilbert space.

$$C \cong C' \subseteq L(H)$$

C^* algebras

- Involutive algebra
  $* : A \longrightarrow A, \quad a \longmapsto a^*,$

$$(a+b)^* = a^* + b^*, \quad (\lambda a)^* = \overline{\lambda} a^*, \quad (ab)^* = b^* a^*, \quad (a^*)^* = a.$$

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- Banach algebra: Complete normed algebra.

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- Normed algebra:Algebra + Normed vector space, $\| ab \| \leq \| a \| \| b \|$.
- Banach algebra: Complete normed algebra.
- C^*-algebra : Involutive Banach algebra,
  $$\| a^* a \| = \| a \|^2 .$$

C^* algebra

- Duality in C^* algebra structure! C^*-norm has algebraic structure

$$\| a \|^2 = sup\{|\lambda| : a^*a - \lambda 1 \quad \text{is not invertible}\}$$

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Example

$\mathbb{C}$=Complex numbers

$$(a+bi)^* = a - bi, \quad \| a+bi \| = \sqrt{a^2+b^2}$$

Example

Example

$M_n(\mathbb{C})$

$$A^* = \overline{A^t}, \quad \| A \| = sup_{\|x\|=1} \| Ax \| .$$

Note: If $A =$ finite dimensional C^*-algebra,

$$A \cong M_{n_1} \oplus \cdots \oplus M_{n_k}$$

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Spectrum

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- $A =$ Banach $\Longrightarrow \chi$ continuous and $\| \chi \| = 1$
- $A = C^* -$ algebra $\Longrightarrow \chi = C^* -$ morphism
- $\widehat{A} =$ Spectrum of Banach algebra A: All characters.
- Weak * topology (pointwise convergence Topology) $\curvearrowright \widehat{A}$
- $\widehat{A}$ is locally compact Hausdorff space.
- $\widehat{A}$ is compact $\Longleftrightarrow$ A is unital (Two G-N theorems are equivalent).

Duality: Characters and Maximal Ideals

- Duality: If A is unital:

$$\widehat{A} \leftrightarrows \text{Maximal ideals of A}$$

$$\chi \longrightarrow Kernel(\chi)$$

$$\chi : A \to A/\mathcal{I} \cong \mathbb{C} \longleftarrow \mathcal{I}$$

- Gelfand-Mazur:

$$\mathcal{I} = \text{maximal ideal} \Longrightarrow A/\mathcal{I} \cong \mathbb{C}$$

Gelfand-Naimark Theorem

$$\text{locally compact Hausdorff space} \leftrightarrow \text{commutative} \quad C^* - \text{algebra}$$

$$\widehat{A} \longleftarrow A$$

Gelfand- Naimark Theorem

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Gelfand- Naimark Theorem

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- $C_0(X) =$ algebra of complex-valued continuous functions vanishing at infinity.

Gelfand- Naimark Theorem

- $$\text{locally compact Hausdorff space} \rightsquigarrow \text{commutative} \quad C^* - \text{algebra}$$
- X = locally compact Hausdorff space.
- $C_0(X)$ = algebra of complex-valued continuous functions vanishing at infinity.
- $C_0(X)$ is commutative C^* algebra.

$$\| f \|_\infty = sup\{|f(x)|; x \in X\}$$

$$f \longmapsto f^*, \quad f^*(x) = \overline{f(x)}$$

- $C_0(X)$ is unital $\Longleftrightarrow$ X is compact.

- Characters of $C_0(X)$?

Gelfand-Naimark Theorem-Gelfand Transform

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- $x \in X$,

$$\chi = \chi_x : C_0(X) \longrightarrow \mathbb{C}, \quad \chi_x(f) = f(x)$$

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Theorem

$A =$ commutative C^* algebra with spectrum $\widehat{A}$,

$$\Gamma : A \longrightarrow C_0(\widehat{A}), \quad a \longmapsto \widehat{a},$$

where $\widehat{a}(\chi) = \chi(a)$ is an isomorphism of C^* algebras.

Gelfand-Naimark

second part

$$C \cong C' \subseteq L(H)$$

State of C^*-Algebras-Gelfand-Naimark-Segal (GNS) construction.

- State: Linear functional $\varphi : A \longrightarrow \mathbb{C}$,

$$\varphi(aa*) \geq 0, \quad \varphi(1) = 1$$

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Example

$A = M_n(\mathbb{C})$,

$$\{all \quad states \quad \varphi\} \leftrightsquigarrow \{positive \quad matrices \quad p, \quad Tr(p) = 1\}$$

$$Hint : \varphi(a) = Tr(ap)$$

States and Probability Measures

For a locally compact Hausdorff space X there is a 1-1 correspondence between states on $C_0(X)$ and Borel probability measures on X.

To a probability measure μ is associated states is defined by

$$\varphi(f) = \int_X f d\mu$$

A Taste of the proof of Gelfand-Naimark-Segal (GNS) construction

- $$\textit{State} \rightsquigarrow \textit{Hilbert} \quad \textit{space}$$

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- $\langle a, b \rangle := \varphi(b^* a)$ inner product

A Taste of the proof of Gelfand-Naimark-Segal (GNS) construction

- $$State \rightsquigarrow Hilbert \quad space$$
- $\varphi : A \longrightarrow \mathbb{C}$ state.
- $\langle a, b \rangle := \varphi(b^* a)$ inner product
- $N := \{a \in A; \varphi(a^* a) = 0\}$.
- N closed left ideal of A, $\rightsquigarrow A/N$.
- $\langle a + N, b + N \rangle := \langle a, b \rangle$ inner product
- $H_\varphi :=$ Hilbert space completion of A/N.
- GNS Representation
  C^*-algebra Representation
  $$\pi_\varphi : A \rightsquigarrow L(H_\varphi)$$

Gelfand-Naimark $\rightsquigarrow$ A Dictionary

- **Space** $\leftrightarrows$ **Algebra**
- compact $\leftrightarrows$ unital
- closed subspace $\leftrightarrows$ closed ideal
- 1-point compactification $\leftrightarrows$ unitization
- Borel measure $\leftrightarrows$ positive functional
- probability measure $\leftrightarrows$ state

Great Identification in NCG

Gelfand-Naimark Thm $\Longrightarrow$

category of C^*-algebras as category of noncommutative topological spaces

How about category of noncommutative Riemannian manifolds?

The complete answer is NOT known, but (at least in the case of compact finite dimensional orientable Riemannian spin manifolds), the notion of Connes' spectral triples and Connes(2008, a complete proof) -Rennie-Varilly (2006) reconstruction theorem provide and adequate starting point, specifying the objects of our noncommutative category.

Motivated by the example of the Atiyah-Singer Dirac operator of a compact spin manifold

Connes Spectral Triple

A (compact) spectral triple is given by $(\mathcal{A}, \mathcal{H}, \mathcal{D})$

- $\mathcal{A}$: is a $*$-algebra.
- $\mathcal{H}$: Hilbert space.
- $\mathcal{D}$: A self adjoint operator on $\mathcal{H}$.
- a representation $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ on $\mathcal{H}$
- $(D - \lambda)^{-1}$ is a compact operator for every irrational λ.
- $[D, \pi(a)] \in \mathcal{B}(\mathcal{H})$ for every $a \in A$.

Example

Example

- $A = C^{\infty}(S^1)$,
- $H = L^2(S^1)$,
- $D = \frac{1}{i}\frac{d}{dx}$.

Connes Reconstruction Theorem

Theorem

Given a compact oriented spin manifold M

$$(C^\infty(M), L^2(M,S), \partial\!\!\!/)$$

is a spectral triple.

Theorem

(Conjectured by Connes 1996)
Given a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *where* $\mathcal{A}$ *is a "commutative"* $*$*-algebra then* $\mathcal{A} = C^\infty(M)$ *where M is a compact oriented spin manifold with a unique Riemannian structure.*

END

Thanks