The independence numbers and the chromatic numbers of random subgraphs

Andrei Raigorodskii

Moscow Institute of Physics and Technology
Moscow, Russia
Main question
Main question

**Erdős–Rényi random graph**

Let $n \in \mathbb{N}$, $p \in [0, 1]$. $G(n, p)$ is obtained by drawing independently edges on $n$ vertices, each with probability $p$. 
Main question

**Erdős–Rényi random graph**

Let $n \in \mathbb{N}$, $p \in [0, 1]$. $G(n, p)$ is obtained by drawing independently edges on $n$ vertices, each with probability $p$.

**Theorem**

Let $p$ be a constant or a function tending to zero and bounded from below by a value $\frac{c}{n}$, where $c > 1$. Let $d = \frac{1}{1-p}$. Then w.h.p.

$$\alpha(G(n, p)) \sim 2 \log_d(np), \quad \chi(G(n, p)) \sim \frac{n}{2 \log_d(np)}.$$
Main question

**Erdős–Rényi random graph**

Let \( n \in \mathbb{N}, \ p \in [0, 1] \). \( G(n,p) \) is obtained by drawing independently edges on \( n \) vertices, each with probability \( p \).

**Theorem**

Let \( p \) be a constant or a function tending to zero and bounded from below by a value \( \frac{c}{n} \), where \( c > 1 \). Let \( d = \frac{1}{1-p} \). Then w.h.p.

\[
\alpha(G(n,p)) \sim 2 \log_d(np), \quad \chi(G(n,p)) \sim \frac{n}{2 \log_d(np)}.
\]

**A general random subgraph**

Let \( n \in \mathbb{N}, \ p \in [0, 1], \ G_n = (V_n, E_n) \) — an arbitrary sequence of graphs. \( G_{n,p} \) is obtained from \( G_n \) by keeping independently edges of \( G_n \), each with probability \( p \).
Main question

**Erdős–Rényi random graph**

Let $n \in \mathbb{N}$, $p \in [0, 1]$. $G(n, p)$ is obtained by drawing independently edges on $n$ vertices, each with probability $p$.

**Theorem**

Let $p$ be a constant or a function tending to zero and bounded from below by a value $\frac{c}{n}$, where $c > 1$. Let $d = \frac{1}{1-p}$. Then w.h.p.

$$\alpha(G(n, p)) \sim 2 \log_d(np), \quad \chi(G(n, p)) \sim \frac{n}{2 \log_d(np)}.$$  

**A general random subgraph**

Let $n \in \mathbb{N}$, $p \in [0, 1]$, $G_n = (V_n, E_n)$ — an arbitrary sequence of graphs. $G_{n,p}$ is obtained from $G_n$ by keeping independently edges of $G_n$, each with probability $p$.

What can be said about $\alpha(G_{n,p})$ and $\chi(G_{n,p})$?
Let $r, s, n \in \mathbb{N}$, $s < r < n$, and let $G(n, r, s) = (V, E)$, where

$$V = \{x = (x_1, \ldots, x_n) : x_i \in \{0, 1\}, \ x_1 + \ldots + x_n = r\},$$

$$E = \{\{x, y\} : (x, y) = s\}.$$
A special case

Main definition

Let $r, s, n \in \mathbb{N}$, $s < r < n$, and let $G(n, r, s) = (V, E)$, where

$$V = \{ \mathbf{x} = (x_1, \ldots, x_n) : x_i \in \{0, 1\}, x_1 + \ldots + x_n = r \},$$

$$E = \{ \{ \mathbf{x}, \mathbf{y} \} : (\mathbf{x}, \mathbf{y}) = s \}.$$

Equivalent definition

Let $r, s, n \in \mathbb{N}$, $s < r < n$. Let $[n]$ be an $n$-element set, and let $G(n, r, s) = (V, E)$, where

$$V = \binom{[n]}{r}, \quad E = \{ A, B \in V : |A \cap B| = s \}.$$
A special case

Main definition

Let \( r, s, n \in \mathbb{N} \), \( s < r < n \), and let \( G(n, r, s) = (V, E) \), where

\[
V = \{ \mathbf{x} = (x_1, \ldots, x_n) : x_i \in \{0, 1\}, \ x_1 + \ldots + x_n = r \},
\]

\[
E = \{ \{x, y\} : (x, y) = s \}.
\]

Equivalent definition

Let \( r, s, n \in \mathbb{N} \), \( s < r < n \). Let \([n]\) be an \( n\)-element set, and let \( G(n, r, s) = (V, E) \), where

\[
V = \binom{[n]}{r}, \quad E = \{A, B \in V : |A \cap B| = s \}.
\]

Again, what can be said about \( \alpha(G_p(n, r, s)) \) and \( \chi(G_p(n, r, s)) \)?
Some motivation

Why studying $G(n, r, s)$?
Some motivation

Why studying $G(n, r, s)$?

Coding theory ("Johnson’s graphs"): 


Some motivation

Why studying $G(n, r, s)$?

- Coding theory ("Johnson’s graphs"): the independence number $\alpha(G(n, r, s))$ stands for the maximum size of a code with one forbidden distance;
Some motivation

Why studying $G(n, r, s)$?

- Coding theory (‘‘Johnson’s graphs’’): the independence number $\alpha(G(n, r, s))$ stands for the maximum size of a code with one forbidden distance; the clique number $\omega(G(4k, 2k, k))$ is responsible for the existence of an Hadamard matrix; etc.
Some motivation

Why studying $G(n, r, s)$?

- **Coding theory** ("Johnson’s graphs"): the independence number $\alpha(G(n, r, s))$ stands for the maximum size of a code with one forbidden distance; the clique number $\omega(G(4k, 2k, k))$ is responsible for the existence of an Hadamard matrix; etc.

- **Combinatorial geometry**: $G(n, r, s)$ is a “distance” graph, i.e., its edges are of the same length $\sqrt{2(r - s)}$. The chromatic number $\chi(G(n, r, s))$ provides important bounds in the Nelson–Hadwiger problems of space coloring as well as in the Borsuk problem of partitioning sets in spaces into parts of smaller diameter.
Some motivation

Why studying $G(n, r, s)$?

- Coding theory ("Johnson’s graphs"): the independence number $\alpha(G(n, r, s))$ stands for the maximum size of a code with one forbidden distance; the clique number $\omega(G(4k, 2k, k))$ is responsible for the existence of an Hadamard matrix; etc.

- **Combinatorial geometry**: $G(n, r, s)$ is a "distance" graph, i.e., its edges are of the same length $\sqrt{2(r - s)}$. The chromatic number $\chi(G(n, r, s))$ provides important bounds in the Nelson–Hadwiger problems of space coloring as well as in the Borsuk problem of partitioning sets in spaces into parts of smaller diameter.

- $G(n, r, 0)$ is the classical Kneser graph; $G(n, 1, 0)$ is just a complete graph.
Some motivation

Why studying $G(n, r, s)$?

- Coding theory ("Johnson’s graphs"): the independence number $\alpha(G(n, r, s))$ stands for the maximum size of a code with one forbidden distance; the clique number $\omega(G(4k, 2k, k))$ is responsible for the existence of an Hadamard matrix; etc.

- **Combinatorial geometry**: $G(n, r, s)$ is a "distance" graph, i.e., its edges are of the same length $\sqrt{2(r - s)}$. The chromatic number $\chi(G(n, r, s))$ provides important bounds in the Nelson–Hadwiger problems of space coloring as well as in the Borsuk problem of partitioning sets in spaces into parts of smaller diameter.

- $G(n, r, 0)$ is the classical Kneser graph; $G(n, 1, 0)$ is just a complete graph.

- Constructive bounds for Ramsey numbers.
Random subgraphs of $G(n, r, s)$: independence numbers

**Theorem (Frankl, Füredi, 1985)**

Let $r, s$ be fixed as $n \to \infty$.

- If $r \leq 2s + 1$, then $\alpha(G(n, r, s)) = \Theta(n^s)$. 
Random subgraphs of $G(n, r, s)$: independence numbers

**Theorem (Frankl, Füredi, 1985)**

Let $r, s$ be fixed as $n \to \infty$.

- If $r \leq 2s + 1$, then $\alpha(G(n, r, s)) = \Theta(n^s)$.
- If $r > 2s + 1$, then $\alpha(G(n, r, s)) = \Theta(n^{r-s-1})$. 
Random subgraphs of $G(n, r, s)$: independence numbers

Theorem (Frankl, Füredi, 1985)

Let $r, s$ be fixed as $n \to \infty$.

- If $r \leq 2s + 1$, then $\alpha(G(n, r, s)) = \Theta(n^s)$.
- If $r > 2s + 1$, then $\alpha(G(n, r, s)) = \Theta(n^{r-s-1})$.

Theorem (Bogoliubskiy, Gusev, Pyaderkin, A.M., 2013–2016)

Let $r, s$ be fixed as $n \to \infty$.

- If $r \leq 2s + 1$, then w.h.p. $\alpha(G_{1/2}(n, r, s)) = \Theta(\alpha(G(n, r, s)) \log n)$.  


Random subgraphs of $G(n, r, s)$: independence numbers

**Theorem (Frankl, Füredi, 1985)**

Let $r, s$ be fixed as $n \to \infty$.
- If $r \leq 2s + 1$, then $\alpha(G(n, r, s)) = \Theta(n^s)$.
- If $r > 2s + 1$, then $\alpha(G(n, r, s)) = \Theta(n^{r-s-1})$.

**Theorem (Bogoliubskiy, Gusev, Pyaderkin, A.M., 2013–2016)**

Let $r, s$ be fixed as $n \to \infty$.
- If $r \leq 2s + 1$, then w.h.p. $\alpha(G_{1/2}(n, r, s)) = \Theta(\alpha(G(n, r, s)) \log n)$.
- If $r > 2s + 1$, then w.h.p. $\alpha(G_{1/2}(n, r, s)) \sim \alpha(G(n, r, s))$. 
Random subgraphs of $G(n, r, s)$: independence numbers for $r > 2s + 1$

Let $r \geq 2, s = 0$. Then $G(n, r, s)$ is Kneser's graph.
Random subgraphs of $G(n, r, s)$: independence numbers for $r > 2s + 1$

Let $r \geq 2$, $s = 0$. Then $G(n, r, s)$ is Kneser’s graph.

Bollobás, Narayanan, A.M., 2016

Fix a real number $\varepsilon > 0$ and let $r = r(n)$ be a natural number such that $2 \leq r(n) = o(n^{1/3})$. Let $p_c(n, r) = ((r + 1) \log n - r \log r)/(n^{r-1})$. As $n \to \infty$, 

$$
\mathbb{P} \left( \alpha(G_p(n, r, 0)) = \alpha(G(n, r, 0)) = \frac{n - 1}{r - 1} \right) \to \begin{cases} 
1 & \text{if } p \geq (1 + \varepsilon)p_c(n, r) \\
0 & \text{if } p \leq (1 - \varepsilon)p_c(n, r). 
\end{cases}
$$
Random subgraphs of $G(n, r, s)$: independence numbers for $r > 2s + 1$

Let $r \geq 2$, $s = 0$. Then $G(n, r, s)$ is Kneser’s graph.

Bollobás, Narayanan, A.M., 2016

Fix a real number $\varepsilon > 0$ and let $r = r(n)$ be a natural number such that $2 \leq r(n) = o(n^{1/3})$. Let $p_c(n, r) = ((r + 1) \log n - r \log r)/(n - 1)$. As $n \to \infty$,

$$
P \left( \alpha(G_p(n, r, 0)) = \alpha(G(n, r, 0)) = \binom{n-1}{r-1} \right) \to \begin{cases} 
1 & \text{if } p \geq (1 + \varepsilon)p_c(n, r) \\
0 & \text{if } p \leq (1 - \varepsilon)p_c(n, r).
\end{cases}
$$

Successively improved by Das, Tran, Balogh, and others.
Random subgraphs of $G(n, r, s)$: independence numbers for $r > 2s + 1$

Let $r \geq 2$, $s = 0$. Then $G(n, r, s)$ is Kneser’s graph.

**Bollobás, Narayanan, A.M., 2016**

Fix a real number $\varepsilon > 0$ and let $r = r(n)$ be a natural number such that $2 \leq r(n) = o(n^{1/3})$. Let $p_c(n, r) = ((r + 1) \log n - r \log r)/(n - 1)$. As $n \to \infty$,

$$
\mathbb{P}\left(\alpha(G_p(n, r, 0)) = \alpha(G(n, r, 0)) = \binom{n - 1}{r - 1}\right) \to \begin{cases} 
1 & \text{if } p \geq (1 + \varepsilon)p_c(n, r) \\
0 & \text{if } p \leq (1 - \varepsilon)p_c(n, r).
\end{cases}
$$

Successively improved by Das, Tran, Balogh, and others.
Let $r \geq 4$, $s = 1$. 
Random subgraphs of $G(n, r, s)$: independence numbers for $r > 2s + 1$

Let $r \geq 2$, $s = 0$. Then $G(n, r, s)$ is Kneser’s graph.

Bollobás, Narayanan, A.M., 2016

Fix a real number $\varepsilon > 0$ and let $r = r(n)$ be a natural number such that $2 \leq r(n) = o(n^{1/3})$. Let $p_c(n, r) = ((r + 1) \log n - r \log r)/(n^{-1})$. As $n \to \infty$,

$$
\mathbb{P} \left( \alpha(G_p(n, r, 0)) = \alpha(G(n, r, 0)) = \binom{n-1}{r-1} \right) \to \begin{cases} 
1 & \text{if } p \geq (1 + \varepsilon)p_c(n, r) \\
0 & \text{if } p \leq (1 - \varepsilon)p_c(n, r).
\end{cases}
$$

Successively improved by Das, Tran, Balogh, and others. Let $r \geq 4$, $s = 1$.

Pyaderkin, A.M., 2017

W.h.p. $\alpha(G_{1/2}(n, r, s)) = \alpha(G(n, r, s))$. 
Random subgraphs of $G(n, r, s)$: independence numbers for $r > 2s + 1$

Let $r \geq 2$, $s = 0$. Then $G(n, r, s)$ is Kneser’s graph.

Bollobás, Narayanan, A.M., 2016

Fix a real number $\varepsilon > 0$ and let $r = r(n)$ be a natural number such that $2 \leq r(n) = o(n^{1/3})$. Let $p_c(n, r) = ((r + 1) \log n - r \log r)/(n^{-1})$. As $n \to \infty$,

$$
\mathbb{P} \left( \alpha(G_{p}(n, r, 0)) = \alpha(G(n, r, 0)) = \left(\frac{n - 1}{r - 1}\right) \right) \to \begin{cases} 
1 & \text{if } p \geq (1 + \varepsilon)p_c(n, r) \\
0 & \text{if } p \leq (1 - \varepsilon)p_c(n, r).
\end{cases}
$$

Successively improved by Das, Tran, Balogh, and others. Let $r \geq 4$, $s = 1$.

Pyaderkin, A.M., 2017

W.h.p. $\alpha(G_{1/2}(n, r, s)) = \alpha(G(n, r, s))$.

Of course $1/2$ can be replaced by another function. However, the threshold is unknown.
Random subgraphs of $G(n, r, s)$: independence numbers for $r > 2s + 1$

Let $r \geq 2$, $s = 0$. Then $G(n, r, s)$ is Kneser's graph.

**Bollobás, Narayanan, A.M., 2016**

Fix a real number $\varepsilon > 0$ and let $r = r(n)$ be a natural number such that $2 \leq r(n) = o(n^{1/3})$. Let $p_c(n, r) = ((r + 1) \log n - r \log r) / \binom{n-1}{r-1}$. As $n \to \infty$,

$$
P \left( \alpha(G_p(n, r, 0)) = \alpha(G(n, r, 0)) = \frac{n-1}{r-1} \right) \to \begin{cases} 
1 & \text{if } p \geq (1 + \varepsilon)p_c(n, r) \\
0 & \text{if } p \leq (1 - \varepsilon)p_c(n, r). 
\end{cases}
$$

Successively improved by Das, Tran, Balogh, and others.

Let $r \geq 4$, $s = 1$.

**Pyaderkin, A.M., 2017**

W.h.p. $\alpha(G_{1/2}(n, r, s)) = \alpha(G(n, r, s))$.

Of course $1/2$ can be replaced by another function. However, the threshold is unknown.

No other cases of strong stability are known.
Random subgraphs of $G(n, r, s)$: independence numbers for $r \leq 2s + 1$

Remind that

**Theorem (Bogoliubskiy, Gusev, Pyaderkin, A.M., 2013–2016)**

Let $r, s$ be fixed as $n \to \infty$. If $r \leq 2s + 1$, then w.h.p.

$$\alpha(G_{1/2}(n, r, s)) = \Theta(\alpha(G(n, r, s)) \log n).$$
Random subgraphs of $G(n, r, s)$: independence numbers for $r \leq 2s + 1$

Remind that

**Theorem (Bogoliubskiy, Gusev, Pyaderkin, A.M., 2013–2016)**

Let $r, s$ be fixed as $n \to \infty$. If $r \leq 2s + 1$, then w.h.p.

$$\alpha(G_{1/2}(n, r, s)) = \Theta(\alpha(G(n, r, s)) \log n).$$

If $r = 1$, $s = 0$, then we have already cited the much subtler classical result.
Random subgraphs of \( G(n, r, s) \): independence numbers for \( r \leq 2s + 1 \)

Remind that

**Theorem (Bogoliubskiy, Gusev, Pyaderkin, A.M., 2013–2016)**

Let \( r, s \) be fixed as \( n \to \infty \). If \( r \leq 2s + 1 \), then w.h.p.
\[
\alpha(G_{1/2}(n, r, s)) = \Theta(\alpha(G(n, r, s)) \log n).
\]

If \( r = 1, s = 0 \), then we have already cited the much subtler classical result.

**Theorem**

Let \( p \) be a constant or a function tending to zero and bounded from below by a value \( \frac{c}{n} \), where \( c > 1 \). Let \( d = \frac{1}{1-p} \). Then w.h.p. \( \alpha(G_p(n, 1, 0)) \sim 2 \log_d(np) \).
Random subgraphs of $G(n, r, s)$: independence numbers for $r \leq 2s + 1$

Remind that

**Theorem (Bogoliubskiy, Gusev, Pyaderkin, A.M., 2013–2016)**

Let $r, s$ be fixed as $n \to \infty$. If $r \leq 2s + 1$, then w.h.p.
\[
\alpha(G_{1/2}(n, r, s)) = \Theta(\alpha(G(n, r, s)) \log n).
\]

If $r = 1$, $s = 0$, then we have already cited the much subtler classical result.

**Theorem**

Let $p$ be a constant or a function tending to zero and bounded from below by a value $\frac{c}{n}$, where $c > 1$. Let $d = \frac{1}{1-p}$. Then w.h.p.
\[
\alpha(G_p(n, 1, 0)) \sim 2 \log_d(np).
\]

There are only two more cases where the $\Theta$ notation is replaced by the $\sim$ one.
Random subgraphs of $G(n, r, s)$: independence numbers for $r \leq 2s + 1$

Remind that

**Theorem (Bogoliubskiy, Gusev, Pyaderkin, A.M., 2013–2016)**

Let $r, s$ be fixed as $n \to \infty$. If $r \leq 2s + 1$, then w.h.p. 
\[ \alpha(G_{1/2}(n, r, s)) = \Theta(\alpha(G(n, r, s)) \log n). \]

If $r = 1$, $s = 0$, then we have already cited the much subtler classical result.

**Theorem**

Let $p$ be a constant or a function tending to zero and bounded from below by a value $\frac{c}{n}$, where $c > 1$. Let $d = \frac{1}{1-p}$. Then w.h.p. 
\[ \alpha(G_p(n, 1, 0)) \sim 2 \log_d(np). \]

There are only two more cases where the $\Theta$ notation is replaced by the $\sim$ one.

**Theorem**

(Pyaderkin, 2016) W.h.p. 
\[ \alpha(G_{1/2}(n, 3, 1)) \sim 2\alpha(G(n, 3, 1)) \log_2 n. \]
Random subgraphs of $G(n, r, s)$: independence numbers for $r \leq 2s + 1$

Remind that

**Theorem (Bogoliubskiy, Gusev, Pyaderkin, A.M., 2013–2016)**

Let $r, s$ be fixed as $n \to \infty$. If $r \leq 2s + 1$, then w.h.p.

$$\alpha(G_{1/2}(n, r, s)) = \Theta(\alpha(G(n, r, s)) \log n).$$

If $r = 1$, $s = 0$, then we have already cited the much subtler classical result.

**Theorem**

Let $p$ be a constant or a function tending to zero and bounded from below by a value $\frac{c}{n}$, where $c > 1$. Let $d = \frac{1}{1-p}$. Then w.h.p. $\alpha(G_p(n, 1, 0)) \sim 2 \log_d(np)$.

There are only two more cases where the $\Theta$ notation is replaced by the $\sim$ one.

**Theorem**

(Pyaderkin, 2016) W.h.p. $\alpha(G_{1/2}(n, 3, 1)) \sim 2\alpha(G(n, 3, 1)) \log_2 n$.

(Kiselev, Derevyanko, 2017) W.h.p. $\alpha(G_{1/2}(n, 2, 1)) \sim \alpha(G(n, 2, 1)) \log_2 n$. 
Random subgraphs of $G(n, r, s)$: chromatic numbers

Let us skip rather cumbersome cases of arbitrary $r, s$ and concentrate on Kneser's graphs ($r > 1, s = 0$).
Random subgraphs of $G(n, r, s)$: chromatic numbers

Let us skip rather cumbersome cases of arbitrary $r, s$ and concentrate on Kneser’s graphs ($r > 1, s = 0$).

Lovász, 1978: if $r \leq n/2$, then $\chi(G(n, r, 0)) = n - 2r + 2$. 
Random subgraphs of $G(n, r, s)$: chromatic numbers

Let us skip rather cumbersome cases of arbitrary $r, s$ and concentrate on Kneser’s graphs ($r > 1, s = 0$).

Lovász, 1978: if $r \leq n/2$, then $\chi(G(n, r, 0)) = n - 2r + 2$.

Very simply the chromatic number of $G(n, r, 0)$ is not so stable as the independence number: w.h.p. even $\chi(G_{1/2}(n, r, 0)) < n - 2r + 2$. However
Random subgraphs of $G(n, r, s)$: chromatic numbers

Let us skip rather cumbersome cases of arbitrary $r, s$ and concentrate on Kneser’s graphs ($r > 1, s = 0$).

Lovász, 1978: if $r \leq n/2$, then $\chi(G(n, r, 0)) = n - 2r + 2$.

Very simply the chromatic number of $G(n, r, 0)$ is not so stable as the independence number: w.h.p. even $\chi(G_{1/2}(n, r, 0)) < n - 2r + 2$. However

Theorem (Kupavskii, 2016)

For many different $n, r, p$, w.h.p. $\chi(G_p(n, r, 0)) \sim n - 2r + 2$. 
Random subgraphs of $G(n, r, s)$: chromatic numbers

Let us skip rather cumbersome cases of arbitrary $r, s$ and concentrate on Kneser’s graphs ($r > 1, s = 0$).

Lovász, 1978: if $r \leq n/2$, then $\chi(G(n, r, 0)) = n - 2r + 2$.

Very simply the chromatic number of $G(n, r, 0)$ is not so stable as the independence number: w.h.p. even $\chi(G_{1/2}(n, r, 0)) < n - 2r + 2$. However

Theorem (Kupavskii, 2016)

For many different $n, r, p$, w.h.p. $\chi(G_p(n, r, 0)) \sim n - 2r + 2$.

For example, if $g(n)$ is any growing function and $r$ is arbitrary in the range between 2 and $\frac{n}{2} - g(n)$, then for any fixed $p$, $\chi(G_p(n, r, 0)) \sim n - 2r + 2$. 

Random subgraphs of $G(n, r, s)$: chromatic numbers

Let us skip rather cumbersome cases of arbitrary $r, s$ and concentrate on Kneser’s graphs ($r > 1, s = 0$).

Lovász, 1978: if $r \leq n/2$, then $\chi(G(n, r, 0)) = n - 2r + 2$.

Very simply the chromatic number of $G(n, r, 0)$ is not so stable as the independence number: w.h.p. even $\chi(G_{1/2}(n, r, 0)) < n - 2r + 2$. However

Theorem (Kupavskii, 2016)

For many different $n, r, p$, w.h.p. $\chi(G_p(n, r, 0)) \sim n - 2r + 2$.

For example, if $g(n)$ is any growing function and $r$ is arbitrary in the range between 2 and $\frac{n}{2} - g(n)$, then for any fixed $p$, $\chi(G_p(n, r, 0)) \sim n - 2r + 2$.

Many improvements by Kupavskii and by Alishahi and Hajiabolhassan.
Random subgraphs of $G(n, r, s)$: chromatic numbers
Random subgraphs of $G(n, r, s)$: chromatic numbers

**Theorem (Kiselev, Kupavskii, 2019+)**

If $r \geq 3$, then w.h.p.

$$n - c_1 2^{r-2} \sqrt{\log_2 n} \leq \chi(G_{1/2}(n, r, 0)) \leq n - c_2 2^{r-2} \sqrt{\log_2 n}.$$
Random subgraphs of $G(n, r, s)$: chromatic numbers

**Theorem (Kiselev, Kupavskii, 2019+)**

If $r \geq 3$, then w.h.p.

$$n - c_1 2^{r-2} \sqrt{\log_2 n} \leq \chi(G_{1/2}(n, r, 0)) \leq n - c_2 2^{r-2} \sqrt{\log_2 n}.$$ 

If $r = 2$, then w.h.p.

$$n - c_1 2 \sqrt{\log_2 n \cdot \log_2 \log_2 n} \leq \chi(G_{1/2}(n, r, 0)) \leq n - c_2 2^{r-2} \sqrt{\log_2 n \cdot \log_2 \log_2 n}.$$
A general result
A general result

**Theorem (A.M., 2017)**

Let $G_n = (V_n, E_n)$, $n \in \mathbb{N}$, be a sequence of graphs. Let $N_n = |V_n|$, $\alpha_n = \alpha(G_n)$. Let $\gamma_n$ be the maximum number of vertices of $G_n$ that are non-adjacent to both vertices of a given edge. Assume that the quantities $N_n, \alpha_n, \gamma_n$ are monotone increasing to infinity and there exists a function $\beta_n$ such that

1. $\beta_n > \gamma_n$ and $\beta_n = o(\alpha_n)$;
2. $\log_2 N_n = o\left(\frac{\alpha_n}{\beta_n}\right)$;
3. $\log_2 N_n = o\left(\beta_n - \gamma_n\right)$.

Then w.h.p. $\alpha(G_n, 1/2) \sim \alpha(G_n)$. 