Some relations between rank of a graph and its complement

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Abstract

Let G be a graph of order n and rank(G) denotes the rank of its adjacency matrix. Clearly, n ≤ rank(G) + rank(\overline{G}) ≤ 2n. In this paper we characterize all graphs G such that rank(G) + rank(\overline{G}) = n, n + 1 or n + 2. Also for every integer n ≥ 5 and any k, 0 ≤ k ≤ n, we construct a graph G of order n, such that rank(G) + rank(\overline{G}) = n + k.

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1. Introduction

Throughout this paper all graphs are simple. For a graph G we denote its complement by \overline{G}. The order of G is the number of vertices of G. For any graph G with the vertex set \{v_1, \ldots, v_n\}, the adjacency matrix of G is an n × n matrix A whose entries a_{ij} are given by a_{ij} = 1, if v_i and v_j are adjacent and a_{ij} = 0 if they are not adjacent. The rank and the null space of the adjacency
matrix of $G$ are denoted by rank($G$) and ker($G$), respectively. We say $\lambda$ is an eigenvalue of $G$ if it is an eigenvalue of its adjacency matrix. For a vertex $v$ of $G$, $N(v)$ denotes the neighborhood of $v$. In this paper a cycle of order $n$ is denoted by $C_n$. We denote the complete $k$-partite graph with part sizes $p_1, \ldots, p_k$ by $K_{p_1, \ldots, p_k}$. We also denote the null graph of order $n$ by $\overline{K}_n$. Let $G_1, \ldots, G_n$ be a sequence of disjoint graphs. Denote by $P(G_1, \ldots, G_n)$ the graph obtained from $G_1 \cup \cdots \cup G_n$ (disjoint union of $G_1, \ldots, G_n$) by joining every vertex of $G_i$ to each vertex of $G_{i+1}$, for any $i$, $1 \leq i \leq n - 1$. We also use $G_1 \vee G_2$ for $P(G_1, G_2)$. The square matrix with all entries 1 is denoted by $J$.

It is obvious that for any graph $G$ of order $n$, $n \leq \text{rank}(G) + \text{rank}(\overline{G}) \leq 2n$. The main goal of this paper is the classification of all graphs of order $n$ for which rank($G$) + rank($\overline{G}$) $= n$, $n + 1$ or $n + 2$.

2. Characterization of graphs with rank($G$) + rank($\overline{G}$) $= n$, $n + 1$

In this section we classify all graphs $G$ such that rank($G$) + rank($\overline{G}$) $= n$ or $n + 1$. We start with the following theorem.

**Theorem A** [4, p. 163]. A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.

**Lemma 1** [4, p. 21]. If for every eigenvalue $\lambda$ of a graph $G$, $\lambda \geq -1$, then $G$ is a union of complete graphs.

**Lemma 2.** Given a graph $G$ of order $n$. For any integer $k$, if rank($G$) + rank($\overline{G}$) $= n + k$, then $G$ has at most $k + 1$ eigenvalues not contained in {$-1, 0$}.

**Proof.** Let $W_1 = \text{Ker}(G)$, $W_2 = \text{Ker}(\overline{G})$ and $K = \text{Ker}(J)$, then dim $W_1 + \text{dim } W_2 = n - k$ and dim $K = n - 1$. Since dim $W_2 + \text{dim } K - \text{dim } (W_2 \cap K) = \text{dim } (W_2 + K) \leq n$, dim $(W_2 \cap K) \geq \text{dim } W_2 - 1$. Since $A + \overline{A} = J - I$, therefore $W_1 \cap W_2 = \{0\}$ and for every $x \in W_2 \cap K$, $Ax = -x$. Hence the multiplicity of the eigenvalue $-1$ is at least dim $W_2 - 1$. Since dim $W_1 + \text{dim } W_2 = n - k$, $A$ has at most $k + 1$ eigenvalues not contained in {$-1, 0$}. □

We state two following simple lemmas without proof.

**Lemma 3.** Let $G$ be a graph and $u$, $v$ be two distinct vertices such that $N(u) = N(v)$, then rank($G$) = rank($G\setminus \{v\}$).

**Lemma 4.** Let $G$ be a graph with vertex set $\{v_1, \ldots, v_n\}$ and adjacency matrix $A$. If $N(v_i) \cup \{v_i\} = N(v_j) \cup \{v_j\}$ and $(x_1, \ldots, x_n)$ is a vector in the null space of $A$, then $x_i = x_j$.

**Lemma 5.** Let $k \geq 1$, $p_1, \ldots, p_k \geq 2$ and $q \geq 0$. If $G$ is isomorphic to $K_{p_1} \cup \cdots \cup K_{p_k} \cup \overline{K}_q$, then rank($G$) $= p_1 + \cdots + p_k$ and rank($\overline{G}$) $= q + k$, if $q + k \geq 2$.

**Proof.** The first part is obvious. For the second part we note that graph $\overline{G}$ is isomorphic to $P(K_q, K_{p_1}, \ldots, p_k)$. Now by Lemma 3, rank($\overline{G}$) = rank($P(K_q, K_{p_1}, \ldots, p_k)$) = rank($K_{q+k}$) = $q + k$. □
Lemma 6. Let \( q \geq 0, k \geq 2 \) and \( p_1 \geq \cdots \geq p_k \geq 1 \). If \( G \) is isomorphic to \( \overline{K}_q \cup K_{p_1, \ldots, p_k} \), then \( \text{rank}(G) = k \). If \( r \) is the largest index such that \( p_r \geq 2 \) (for \( p_1 = 1 \), put \( r = 0 \)), then

\[
\text{rank}(\overline{G}) = \begin{cases} 
q + p_1 + \cdots + p_k & \text{if } r = k, \\
p_1 + \cdots + p_r & \text{if } q = 0, \\
q + p_1 + \cdots + p_r + 1 & \text{if } q \geq 1 \text{ and } r < k.
\end{cases}
\]

Proof. By Lemma 3 we have \( \text{rank}(G) = \text{rank}(K_{1, \ldots, 1}) = k \). The graph \( \overline{G} \) is isomorphic to \( P(K_q, K_{p_1} \cup \cdots \cup K_{p_k}) \). If \( q = 0 \), then obviously \( \text{rank}(\overline{G}) = p_1 + \cdots + p_r \). Suppose that \( q \geq 1 \) and \( r \geq k - 1 \), that is \( p_{k-1} \geq 2 \). Let \( \overline{A} \) be the adjacency matrix of \( \overline{G} \) and \( \mathbf{x} \) be a vector in the null space of \( \overline{A} \). By Lemma 4, \( \mathbf{x} = (z, \ldots, z, y_1, \ldots, y_1, \ldots, y_k, \ldots, y_k)' \), where \( z \) is repeated \( q \) times and \( y_i \) is repeated \( p_i \) times, \( i = 1, \ldots, k \). Now \( \overline{A}\mathbf{x} = \mathbf{0} \) implies that

\[
\begin{pmatrix}
q - 1 & p_1 & \cdots & p_k \\
q & p_1 - 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
q & 0 & \cdots & p_k - 1
\end{pmatrix}
\begin{pmatrix}
z \\
y_1 \\
\vdots \\
y_k
\end{pmatrix}
= \mathbf{0}.
\]

Since \( p_1 \geq \cdots \geq p_{k-1} \geq 2 \), one may easily conclude that \( z = y_1 = \cdots = y_k = 0 \). Therefore \( \text{rank}(\overline{G}) = q + p_1 + \cdots + p_k \). If \( q \geq 1 \) and \( r < k \), then by Lemma 3, \( \text{rank}(\overline{G}) = \text{rank}P(K_q, (K_{p_1} \cup \cdots \cup K_{p_r} \cup K_1)) \) and by previous case we conclude that \( \text{rank}(\overline{G}) = q + p_1 + \cdots + p_r + 1 \). \( \square \)

Theorem 1. Let \( G \) be a graph of order \( n \), then \( \text{rank}(G) + \text{rank}(\overline{G}) = n \geq 2 \) if and only if either \( G \) or \( \overline{G} \) is a complete graph.

Proof. Let \( \text{rank}(G) + \text{rank}(\overline{G}) = n \). By Lemma 2, \( G \) has at most one eigenvalue different from 0, \(-1\). If \( G \) is a null graph, then we are done. So it has exactly one positive eigenvalue. Thus by Lemma 1, \( G \) is a union of finitely many complete graphs. Since \( G \) has exactly one positive eigenvalue, \( G = K_p \cup \overline{K}_q \), where \( p \geq 2 \) and \( q \geq 0 \). If \( q \geq 1 \), then by Lemma 5, \( \text{rank}(G) = p \) and \( \text{rank}(\overline{G}) = q + 1 \), thus \( \text{rank}(G) + \text{rank}(\overline{G}) = n + 1 \) which is impossible. So \( q = 0 \) and we are done. \( \square \)

Theorem 2. Let \( G \) be a graph of order \( n \), then \( \text{rank}(G) + \text{rank}(\overline{G}) = n + 1 \) if and only if either \( G \) or \( \overline{G} \) is isomorphic to \( K_p \cup \overline{K}_q \), where \( p \geq 2, q \geq 1 \).

Proof. One part of theorem is obvious by Lemma 5. So assume that \( \text{rank}(G) + \text{rank}(\overline{G}) = n + 1 \). By Lemma 2, \( G \) has at most two eigenvalues different from 0, \(-1\). Since \( G \) is not a null graph, it has at least one positive eigenvalue. Now two cases can be considered.

Case 1. The graph \( G \) has exactly one positive eigenvalue. By Theorem A, \( G \) is isomorphic to the union of a null graph of order \( q \) and a \( k \)-partite graph \( K_{p_1, \ldots, p_k} \), where \( q \geq 0, k \geq 1, p_1 \geq \cdots \geq p_k \geq 1 \) and \( q + p_1 + \cdots + p_k = n \). If \( k = 1 \), then we are done. So we may assume that \( k \geq 2 \). Let \( r \) be the largest index such that \( p_r \geq 2 \). By Lemma 6, \( \text{rank}(G) + \text{rank}(\overline{G}) = n + 1 \) only if \( q = 0, r = 1 \) or \( q \geq 1, r = 0 \). In the first case \( \overline{G} \) and in the second case \( G \) is the union of a complete graph and a null graph.
Case 2. The graph \( G \) has exactly two positive eigenvalues. In this case for every eigenvalue \( \lambda \) of \( G \), \( \lambda \geq -1 \), therefore by Lemma 1, \( G \) is a union of finitely many complete graphs. Since \( G \) has exactly two positive eigenvalues, therefore \( G \) is the union of the complete graphs \( K_{p_1}, K_{p_2} \) and the null graph of order \( q \), where \( p_1, p_2 \geq 2 \) and \( q \geq 0 \). Hence by Lemma 5, \( \text{rank}(G) = p_1 + p_2 \) and \( \text{rank}(\overline{G}) = q + 2 \), which implies that \( \text{rank}(G) + \text{rank}(\overline{G}) = n + 2 \), a contradiction. \( \square \)

3. Characterization of graphs with \( \text{rank}(G) + \text{rank}(\overline{G}) = n + 2 \)

In this section we would like to determine all graphs \( G \) which \( \text{rank}(G) + \text{rank}(\overline{G}) = n + 2 \).

Theorem B [5]. The second smallest eigenvalues of both graphs \( G \) and \( \overline{G} \) are at least \(-1\) if and only if at least one of these graphs belongs to one of the following families of graphs:

(a) \( P(K_m, \overline{K}_n, \overline{K}_p, K_q) \) \((m \geq 1, n \geq 1, p \geq 1, q \geq 1)\);
(b) \( P(\overline{K}_m, K_n, K_p, \overline{K}_q) \) \((m = n = 1, p \geq 1, q \geq 1)\) or \( m = p = 1, n \geq 1, q \geq 1 \) or \( m = 1, n = p = 2, q \geq 1 \) or \( m = n = p = q = 2 \);
(c) \( P(\overline{K}_m, K_n, \overline{K}_p, K_q) \) \((1 \leq m \leq 2, n \geq 1, p \geq 1, q \geq 1)\) or \( m = 3, n \geq 1, 1 \leq p \leq 2, q \geq 1 \) or \( m \geq 4, n \geq 1, p = 1, q \geq 1 \);
(d) \( P(\overline{K}_m, \overline{K}_n, \overline{K}_p, K_q) \) \((m = n = 2, p = 3, q \geq 1)\) or \( m = p = 2, n \geq 1, q \geq 1 \);
(e) \( P(K_m, \overline{K}_n, \overline{K}_p) \) \((m \geq 2, n \geq 1, p \geq 1)\);
(f) \( P(K_m, K_n, \overline{K}_p) \) \((m \geq 2, n \geq 1, p \geq 1)\);
(g) \( P(K_m, \overline{K}_n, K_p) \) \((m \geq 2, n \geq 1, p \geq 1)\);
(h) \( P(K_m, K_n, K_p) \) \((m \geq 2, n \geq 1, p \geq 1)\);
(i) \( P(\overline{K}_m, K_n) \) \((m \geq 1, n \geq 1)\);
(j) \( P(\overline{K}_m, \overline{K}_n) \) \((m \geq 1, n \geq 1)\).

The proof of the next lemma is similar to the proof of Lemma 6.

Lemma 7. Let \( G \) be a graph. Then the following hold:

(a) If \( G \) is isomorphic to \( P(K_m, \overline{K}_n, \overline{K}_p, K_q) \), \( m, n, p, q \geq 1 \), then \( \text{rank}(G) = m + q + 2 \) and \( \text{rank}(\overline{G}) = n + p + 2 \).
(b) If \( G \) is isomorphic to \( P(\overline{K}_m, K_n, K_p, \overline{K}_q) \), \( m, n, p, q \geq 1 \), then \( \text{rank}(G) = n + p + 2 \) and \( \text{rank}(\overline{G}) = m + q + 2 \).
(c) If \( G \) is isomorphic to \( P(\overline{K}_m, K_n, \overline{K}_p, K_q) \), \( m, n, p, q \geq 1 \), then \( \text{rank}(G) = n + q + 2 \) and \( \text{rank}(\overline{G}) = m + p + 2 \).
(d) If \( G \) is isomorphic to \( P(\overline{K}_m, \overline{K}_n, \overline{K}_p, K_q) \), \( m = n = 2, p = 3, q \geq 1 \) or \( m = p = 2, n \geq 1, q \geq 1 \), then \( \text{rank}(G) = q + 3 \) and \( \text{rank}(\overline{G}) = m + n + p + 1 \).
(e) If \( G \) is isomorphic to \( P(K_m, \overline{K}_n, \overline{K}_p) \), \( m \geq 2, n, p \geq 1 \), then \( \text{rank}(G) = m + 2 \) and \( \text{rank}(\overline{G}) = n + p + 1 \).
(f) If \( G \) is isomorphic to \( P(K_m, K_n, \overline{K}_p) \), \( m \geq 2, n, p \geq 1 \), then \( \text{rank}(G) = m + n + 1 \) and \( \text{rank}(\overline{G}) = p + 1 \).
(g) If \( G \) is isomorphic to \( P(K_m, \overline{K}_n, K_p) \), \( m \geq 2, n, p \geq 1 \), then \( \text{rank}(G) = m + p + 1 \) and \( \text{rank}(\overline{G}) = n + 2 \).
(h) If $G$ is isomorphic to $P(K_m, K_n, K_p)$, $m \geq 2$, $n$, $p \geq 1$, then $\text{rank}(G) = m + n + p$ and $\text{rank}(\overline{G}) = 2$.

(i) If $G$ is isomorphic to $P(K_m, K_n)$, $m \geq 2$, $n \geq 1$, then $\text{rank}(G) = n + 1$ and $\text{rank}(\overline{G}) = m$.

(j) If $G$ is isomorphic to $P(K_m, K_n)$, $m, n \geq 2$, then $\text{rank}(G) = 2$ and $\text{rank}(\overline{G}) = m + n$.

The following corollary is an easy consequence of Lemma 7.

**Corollary 1.** Suppose that a graph $G$ of order $n$ belongs to one of ten families of graphs given in Theorem B. If $\text{rank}(G) + \text{rank}(\overline{G}) = n + 2$, then $G$ belongs to one of the following families:

(i) $P(K_m, K_p, K_q)$ $(m \geq 2$, $p, q \geq 1)$;
(ii) $P(K_m, K_p)$ $(m \geq 2$, $p, q \geq 1)$;
(iii) $P(K_m, K_p, K_q)$ $(m \geq 2, p \geq 1)$.

**Theorem 3.** Let $G$ be a graph of order $n$, then $\text{rank}(G) + \text{rank}(\overline{G}) = n + 2$ if and only if either $G$ or $\overline{G}$ is isomorphic to one of the following families of graphs:

(i) $P(K_m, K_p, K_q)$ $(m \geq 2$, $p, q \geq 1)$;
(ii) $P(K_m, K_p, K_q)$ $(m \geq 2, p, q \geq 2)$;
(iii) $P(K_m, K_p, K_q)$ $(m \geq 2, p, q \geq 1)$.

**Proof.** One part of theorem is obvious by Lemmas 5 and 7. So assume that $\text{rank}(G) + \text{rank}(\overline{G}) = n + 2$. By Lemma 2 both $G$ and $\overline{G}$ have at most three eigenvalues different from 0, $-1$. Clearly $G$ and $\overline{G}$ have at least one positive eigenvalue. Now we consider three cases.

**Case 1.** The graph $G$ or $\overline{G}$, say $G$, has exactly one positive eigenvalue. By Theorem A, $G$ is isomorphic to the union of the null graph $\overline{K}_q$ and the $k$-partite graph $K_{p_1}, ..., p_k$, where $q \geq 0$, $k \geq 2$, $p_1 \geq \cdots \geq p_k \geq 1$. Let $r$ be the largest index such that $p_i \geq 2$. By Lemma 6, $\text{rank}(G) + \text{rank}(\overline{G}) = n + 2$ only if $r = k = 2$ or $q = 0$, $r = 2$ or $q \geq 1$, $r = 1$. In the first case $\overline{G}$ is isomorphic to $P(K_{p_1}, K_q, K_{p_2})$, $p_1, p_2 \geq 2, q \geq 0$, in the second case $\overline{G}$ is isomorphic to $K_{p_1} \cup K_{p_2} \cup \overline{K}_m, p_1, p_2 \geq 2, m \geq 1$ and in the third case $\overline{G}$ is isomorphic to $P(K_{p_1}, K_q, K_{m}), p_1 \geq 2, q, m \geq 1$.

**Case 2.** Both $G$ and $\overline{G}$ have exactly two positive eigenvalues. In this case the second smallest eigenvalues of both graphs $G$ and $\overline{G}$ are at least $-1$, therefore at least one of them belongs to one of ten families of graphs given in Theorem B. Since $\text{rank}(G) + \text{rank}(\overline{G}) = n + 2$, either $G$ or $\overline{G}$ belongs to one of the families given in Corollary 1.

**Case 3.** The graph $G$ or $\overline{G}$, say $G$, has exactly three positive eigenvalues. In this case for every eigenvalue $\lambda$ of $G$ we have $\lambda \geq -1$, therefore by Lemma 1, $G$ is a union of finitely many complete graphs. Since $G$ has exactly three positive eigenvalues, $G$ is the union of three complete graphs $K_{p_1}, K_{p_2}, K_{p_3}$ and the null graph $\overline{K}_q$, where $p_1, p_2, p_3 \geq 2$ and $q \geq 0$. Hence by Lemma 5, $\text{rank}(G) = p_1 + p_2 + p_3$ and $\text{rank}(\overline{G}) = q + 3$ which implies that $\text{rank}(G) + \text{rank}(\overline{G}) = n + 3$, a contradiction. □
4. A solution for the equation $\text{rank}(G) + \text{rank}(\overline{G}) = n + k$ ($1 \leq k \leq n$)

In [1] it has been proved that for any regular graph $G$, $\text{rank}(G \vee K_1) = \text{rank}(G) + 1$. The next lemma is a generalization of this result.

**Lemma 8.** Let $G$ and $H$ be regular graphs. Then $\text{rank}(G \vee H) = \text{rank}(G) + \text{rank}(H)$.  

**Proof.** By [4, Theorem 2.8] the number of zero eigenvalues of $G \vee H$ is equal to sum of the number of zero eigenvalues of $G$ and $H$. □

**Remark 1.** It is well known that if $G$ is an $r$-regular graph of order $n$ with eigenvalues $r, \lambda_2, \ldots, \lambda_n$, then the eigenvalues of $\overline{G}$ are $n - r - 1, -1 - \lambda_2, \ldots, -1 - \lambda_n$, see [2, p. 20].

**Remark 2.** If $n$ is not divisible by 4, then we have $\text{rank}(C_n) = n$ (see [2, p. 17]). Also by considering the eigenvalues of $C_n$ and using Remark 1 if $n$ is not divisible by 3, then $\text{rank}(\overline{C_n}) = n$.

**Lemma 9.** For any integer $n \geq 5$ ($n \neq 6, 8$), there exists a 2-regular graph $G$ of order $n$ such that both $G$ and $\overline{G}$ have full ranks.

**Proof.** If $n$ is not divisible by 3 and 4, then by Remarks 1 and 2, set $G = C_n$. Otherwise, noting that $C_{n-k} \cup C_k$ is a regular graph and using Remarks 1 and 2 define $G$ as follows:

1. If $n \equiv 0, 3, 4, 6$ (mod 12), then set $G = C_{n-5} \cup C_5$.  
2. If $n \equiv 8, 9$ (mod 12), then set $G = C_{n-7} \cup C_7$. □

**Lemma 10.** For any integer $n \geq 3$ and every $k$, $0 \leq k \leq \lfloor n/2 \rfloor$, there exists a graph $G$ of order $n$ such that $\text{rank}(G) + \text{rank}(\overline{G}) = n + k$.

**Proof.** If $k = 0$, then consider the complete graph $K_n$. If $k = 1$, then consider the complete bipartite graph $K_{1,n-1}$. For each $k$, $2 \leq k \leq \lfloor n/2 \rfloor$ consider a complete $k$-partite graph such that each part has at least two vertices. Such a graph has rank $k$ and its complement has full rank, as desired. □

**Lemma 11.** For every integer $n \geq 9$, there exists a graph $G$ of order $n$ such that $\text{rank}(G) + \text{rank}(\overline{G}) = n + 8$ and for any $n \geq 7$, there exists a graph $G$ of order $n$ such that $\text{rank}(G) + \text{rank}(\overline{G}) = n + 6$.

**Proof.** Set $G = C_7 \cup K_{n-7}$. Thus $G$ has full rank and $\overline{G} = \overline{C_7} \vee K_{n-7}$. But we observe that $\text{rank}(\overline{G}) = \text{rank}(\overline{C_7} \vee K_1)$. Since by Remark 2 $\overline{C_7}$ has full rank, Lemma 8 implies that $\text{rank}(\overline{G}) = 8$. Thus $\text{rank}(G) + \text{rank}(\overline{G}) = n + 8$, as desired. For the second part set $H = C_5 \cup K_{n-5}$. Now similar to the previous case we conclude that $\text{rank}(H) + \text{rank}(\overline{H}) = n + 6$ and the proof is complete. □

**Lemma 12.** For any integer $n \geq 2$, we define a graph $G$ as follows: consider the complete graph $K_n$ with the vertex set $\{v_1, \ldots, v_n\}$ and add $n$ new vertices $u_1, \ldots, u_n$ and for any $i$, $1 \leq i \leq n$, join each $u_i$ to $v_i$. Then $G$ and $\overline{G}$ have full ranks.
Proof. By Observation 3 of [1], $G$ has full rank. To show that $\overline{G}$ has full rank we note that the adjacency matrix of $\overline{G}$ is of the form

\[
\begin{pmatrix}
J - I & J - I \\
J - I & 0
\end{pmatrix}.
\]

So it has full rank. □

We close this paper with the following theorem.

Theorem 4. For every integer $n \geq 5$ and any $k$, $0 \leq k \leq n$, there exists a graph $G$ of order $n$ such that $\text{rank}(G) + \text{rank}(\overline{G}) = n + k$.

Proof. We have the following six cases.

Case 1. $n \geq 16$. If $0 \leq k \leq 8$, then by Lemma 10 we have done. If $k \geq 9$, then by Lemma 9 there is a regular graph $G_k$ with $k$ vertices such that both $G_k$ and $\overline{G}_k$ have full ranks. Now set $G = G_k \vee K_{n-k}$. By Lemma 8 we have $\text{rank}(G) + \text{rank}(\overline{G}) = n + k$. So for $n \geq 16$ we are done.

Case 2. $12 \leq n \leq 15$. For $0 \leq k \leq 6$, we can construct the desired graph by Lemma 10. For $k \geq 9$ and $k = 7$, set $G = G_k \vee K_{n-k}$ as before. For $k = 8$ use Lemma 11.

Case 3. $8 \leq n \leq 11$. For any $0 \leq k \leq 4$, by Lemma 10 there is nothing to prove. For $k \geq 9$ and $k = 5, 7$ set $G = G_k \vee K_{n-k}$ as before. Apply Lemma 11 for $k = 6$. For $n \neq 8$ and $k = 8$ we again use Lemma 11. Finally for $n = 8$ and $k = 8$ consider the graph defined in Lemma 12 for $n = 4$.

Case 4. $n = 7$. By Lemma 10 there is nothing to prove for $0 \leq k \leq 3$. For $k = 5, 7$ set $G = G_k \vee K_{n-k}$ as before. Apply Lemma 11 for $k = 6$. For $k = 4$ see (14–710) in [3, p. 223].

Case 5. $n = 6$. Apply Lemma 10 for $0 \leq k \leq 3$. For $k = 5$ set $G = G_k \vee K_{n-k}$ as before. For $k = 6$ consider the graph given in Lemma 12 for $n = 3$. For $k = 4$ see (No. 3.2) in [4, p. 293].

Case 6. $n = 5$. For $0 \leq k \leq 2$, apply Lemma 10. Set $G = C_5$ for $k = 5$. For $k = 4$ see (No. 1.16) in [4, p. 273]. Finally for $k = 3$ see (No. 1.18) in [4, p. 273]. □

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