On the Diameters of Commuting Graphs

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Abstract

The commuting graph of a ring $\mathcal{R}$, denoted by $\Gamma(\mathcal{R})$, is a graph whose vertices are all non-central elements of $\mathcal{R}$ and two distinct vertices $x$ and $y$ are adjacent if and only if $xy = yx$. Let $D$ be a division ring and $n \geq 3$. In this paper we investigate the diameters of $\Gamma(M_n(D))$ and determine the diameters of some induced subgraphs of $\Gamma(M_n(D))$, such as the induced subgraphs on the set of all non-scalar non-invertible, nilpotent, idempotent, and involution matrices in $M_n(D)$. For every field $F$, it is shown that if $\Gamma(M_n(F))$ is a connected graph, then $\text{diam} \, \Gamma(M_n(F)) \leq 6$. We conjecture that if $\Gamma(M_n(F))$ is a connected graph, then $\text{diam} \, \Gamma(M_n(F)) \leq 5$. We show that if $F$ is an algebraically closed field or $n$ is a prime number and $\Gamma(M_n(F))$ is a connected graph, then $\text{diam} \, \Gamma(M_n(F)) = 4$. Finally, we present some applications to the structure of pairs of idempotents which may prove of independent interest.

Keywords: Commuting graph, Diameter, Division ring, Idempotent.

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1. Introduction

For a ring $\mathcal{R}$, we denote the center of $\mathcal{R}$ by $Z(\mathcal{R})$. If $X$ is either an element or a subset of $\mathcal{R}$, then $C_{\mathcal{R}}(X)$ denotes the centralizer of $X$ in $\mathcal{R}$. For each non-commutative ring $\mathcal{R}$, we associate a graph, with the vertex set $\mathcal{R} \setminus Z(\mathcal{R})$ and join two vertices $x$ and $y$ if and only if $x \neq y$ and $xy = yx$. This graph has been introduced in [2], is called the commuting graph of $\mathcal{R}$, and is denoted by $\Gamma(\mathcal{R})$. If $X$ is a subset of $\mathcal{R}$, then $\Gamma(X)$ denotes...
the induced subgraph of $\Gamma(\mathcal{R})$ on $\mathcal{X} \setminus Z(\mathcal{R})$; that is the subgraph of $\Gamma(\mathcal{R})$ with vertex set $\mathcal{X} \setminus Z(\mathcal{R})$. If $D$ is a division ring and $m, n$ are natural numbers, then we denote the set of all $m \times n$ matrices over $D$ and the ring of all $n \times n$ matrices over $D$ by $M_{m \times n}(D)$ and $M_n(D)$, respectively, and for simplicity we put $D^n = M_{n \times 1}(D)$. We denote the group of all invertible matrices in $M_n(D)$ by $GL_n(D)$. For any $i, j, 1 \leq i, j \leq n$, we denote by $E_{ij}$, that element in $M_n(D)$ whose $(i, j)$-entry is 1 and whose other entries are 0. Also 0, $I$, $0_r$, and $I_r$ denote the zero matrix, the identity matrix, the zero matrix of size $r$, and the identity matrix of size $r$, respectively. A matrix $E \in M_n(D)$ is called idempotent if $E^2 = E$. Also a matrix $T \in M_n(D)$ is called an involution if $T^2 = I$. For any matrix $X \in M_{m \times n}(D)$, we denote the transpose of $X$ by $X^t$. Moreover, for any two matrices $X \in M_{m \times n}(D)$ and $Y \in M_{r \times s}(D)$, we define

$$X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in M_{(m+r) \times (n+s)}(D).$$

For any field $F$ and matrices $A, B, A', B' \in M_n(F)$, a pair $\{A, B\}$ is said to be similar to a pair $\{A', B'\}$ if there is a matrix $P \in GL_n(F)$ such that $A' = PAP^{-1}$ and $B' = PBP^{-1}$. We say that $\{A, B\}$ is triangularizable if there exists a matrix $P \in GL_n(F)$ such that $PAP^{-1}$ and $PBPP^{-1}$ are upper triangular. Also a pair $\{A, B\}$ is said to be irreducible if every invariant subspace of $\{A, B\}$ is equal to $\{0\}$ or $F^n$. In this paper, a matrix $A \in M_n(D)$ is called cyclic if there is a vector $a^t \in D^n$ such that $\{a, aA, \ldots, aA^{n-1}\}$ is a basis for $M_{1 \times n}(D)$ as a left vector space over $D$. Indeed, the representation of $A$ in the above basis has the following form

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix},$$

for some $a_1, \ldots, a_n \in D$. If $a_1 = \cdots = a_n = 0$, then the above matrix is denoted by $J$. For any matrix $A \in M_n(D)$, $L_A$ and $R_A$ denote the left multiplication and the right multiplication transformations of $D^n$ and $M_{1 \times n}(D)$ by $A$, respectively. We use nullity $A$ for $\dim \ker L_A = \dim \ker R_A$. Let $D$ be a division ring with center $F$. Then for any matrix $A \in M_n(D)$, $F[A]$ denotes the $F$-subalgebra generated by $A$.

In a graph $G$, a path $\mathcal{P}$ is a sequence of distinct vertices $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{k+1}$ in which every two consecutive vertices are adjacent. The number $k$ is called the length of $\mathcal{P}$. For two vertices $u$ and $v$ in a graph $G$, the distance between $u$ and $v$, denoted by $d(u, v)$, is the length of the shortest path between $u$ and $v$, if such a path exists; otherwise we define $d(u, v) = \infty$. The diameter of a graph $G$ is defined as

$$\text{diam } G = \sup \{d(u, v) \mid u \text{ and } v \text{ are distinct vertices of } G\}.$$  

Moreover, a graph $G$ is called connected if there exists a path between every two distinct vertices of $G$.
In this article, we denote the set of all non-invertible, nilpotent, idempotent, and involution matrices in $M_n(D)$ by $\mathcal{A}_n$, $\mathcal{N}_n$, $\mathcal{E}_n$, and $\mathcal{I}_n$, respectively. In [3] it is shown that the graphs $\Gamma(\mathcal{A}_n)$, $\Gamma(\mathcal{N}_n)$, $\Gamma(\mathcal{E}_n)$, $\Gamma(\mathcal{I}_n)$ are connected. Here we find the diameters of these graphs as follows.

i) $\text{diam} \Gamma(\mathcal{A}_n) = 4$, for any $n \geq 3$;

ii) $\text{diam} \Gamma(\mathcal{N}_3) = 5$ and $\text{diam} \Gamma(\mathcal{N}_n) = 4$, for each $n \geq 4$;

iii) $\text{diam} \Gamma(\mathcal{E}_n) = 3$, for any $n \geq 3$;

iv) $\text{diam} \Gamma(\mathcal{I}_n) = 3$, for every $n \geq 3$, if $\text{char } D \neq 2$; otherwise, $\text{diam} \Gamma(\mathcal{I}_3) \leq 5$. Also $\text{diam} \Gamma(\mathcal{I}_n) \leq 4$, for every $n \geq 4$.

Note that according to Remarks 2, 3, 4 and 5 of [3], all aforementioned commuting graphs for the case $n = 2$, fail to be connected for every division ring $D$.

2. Non-invertible matrices

In this section we would like to obtain the diameter of the induced subgraph on all non-invertible matrices in $M_n(D)$. We begin with the following lemma.

**Lemma 1.** Let $D$ be a division ring and $n \geq 2$. If $A \in M_n(D)$ is a cyclic matrix of the form (†), then for any matrix $B \in C_{M_n(D)}(A)$, there exists a polynomial $f(x) \in D[x]$ such that $B = f(A)$.

**Proof.** Let $\alpha = [1 \ 0 \ \cdots \ 0]$ and $B$ be an element of $C_{M_n(D)}(A)$. Since $\{\alpha, \alpha A, \ldots, \alpha A^{n-1}\}$ is a basis for $M_{1 \times n}(D)$ as a left vector space over $D$, there are $d_0, \ldots, d_{n-1} \in D$ such that $\alpha B = \sum_{i=0}^{n-1} d_i (\alpha A^i)$. We show that $B = \sum_{i=0}^{n-1} d_i A^i$. Since $AB = BA$,

$$ (\alpha A^j)B = (\alpha B)A^j = \sum_{i=0}^{n-1} d_i (\alpha A^j) A^i, $$

for any $j$, $0 \leq j \leq n - 1$. But all entries of $\alpha A^j$ are contained in $Z(D)$ for each $j$, $0 \leq j \leq n - 1$, so we have $(\alpha A^j)B = (\alpha A^j) \sum_{i=0}^{n-1} d_i A^i$. This completes the proof. $\square$

**Lemma 2.** Let $D$ be a division ring and $n \geq 3$. Then $d(J, J^t) = 4$ in $\Gamma(\mathcal{A}_n)$.

**Proof.** We show that if two non-invertible matrices $A \in C_{M_n(D)}(J)$ and $B \in C_{M_n(D)}(J^t)$ commute, then at least one of them is scalar. By Lemma 1, there exist $\alpha_0, \ldots, \alpha_{n-1} \in D$ such that


\[ A = \begin{bmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} & \alpha_{n-1} \\ 0 & \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_0 & \alpha_1 \\ 0 & 0 & \cdots & 0 & \alpha_0 \end{bmatrix}. \]

Since \( A \) is a non-zero non-invertible matrix, there exists the minimum integer \( r \geq 1 \) such that \( \alpha_r \neq 0 \). So we may assume that
\[
A = \begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix},
\]
for some matrix \( U \in GL_{n-r}(D) \). Assume that \( r \geq n/2 \). If the matrix
\[
X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \in M_n(D),
\]
where \( X_{11}, X_{33} \in M_{n-r}(D) \), and \( X_{22} \in M_{2r-n}(D) \), commutes with \( A \), then by an easy calculation, using the invertibility of \( U \), we find that \( X \) has the form
\[
(i) \begin{bmatrix} * & * & * \\ 0_{(2r-n) \times (n-r)} & * & * \\ 0_{n-r} & 0_{(n-r) \times (2r-n)} & * \end{bmatrix}.
\]
If \( r \leq n/2 \), then similarly we obtain that any element of \( C_{M_n(D)}(A) \) has the form
\[
(ii) \begin{bmatrix} * & * & * \\ 0_{(n-2r) \times r} & * & * \\ 0_r & 0_{r \times (n-2r)} & * \end{bmatrix}.
\]
On the other hand, \( B^t \) commutes with \( J \), so Lemma 1 yields that there exist \( \beta_0, \ldots, \beta_{n-1} \in D \) such that
\[
B = \begin{bmatrix} \beta_0 & 0 & \cdots & 0 & 0 \\ \beta_1 & \beta_0 & \cdots & 0 & 0 \\ \vdots & \beta_1 & \ddots & \ddots & \vdots \\ \beta_{n-2} & \cdots & \beta_1 & \ddots & \beta_0 \\ \beta_{n-1} & \beta_{n-2} & \cdots & \beta_1 & \beta_0 \end{bmatrix}.
\]
Now, if \( B \) has one of the forms (i) or (ii), then we have \( \beta_1 = \cdots = \beta_{n-1} = 0 \). This shows that \( d(J, J^t) \geq 4 \) in \( \Gamma(A_n) \). Since \( J - E_{1n} - E_{22} - E_{n1} - J^t \) is a path in \( \Gamma(A_n) \), the proof is complete. \( \square \)

**Theorem 3.** Let \( F \) be a field and \( n \geq 3 \). If \( \Gamma(M_n(F)) \) is a connected graph, then \( \text{diam}(\Gamma(M_n(F))) \geq 4 \).
Proof. We show that \( d(J, J') = 4 \). To get a contradiction, assume that there is a path 
\( J - A - B - J' \) in \( \Gamma(M_n(F)) \). So \( A \) and \( B \) have the forms given in the proof of Lemma 2. Hence two matrices \( A - \alpha I \in C_{M_n(F)}(J) \) and \( B - \beta I \in C_{M_n(F)}(J') \) commute. By Lemma 2, one of them is a scalar matrix, a contradiction. □

Lemma 4. Let \( D \) be a division ring and \( n \geq 2 \). Suppose \( A, B \in M_n(D) \) are two matrices such that \( \text{Ker} \ L_A \cap \text{Ker} \ L_B \neq \{0\} \) and \( \text{Ker} \ R_A \cap \text{Ker} \ R_B \neq \{0\} \). Then \( C_{M_n(D)}(\{A, B\}) \) contains at least one matrix with rank 1.

Proof. By the hypothesis, there are non-zero elements \( X, Y \in D^n \) such that \( AX = BX = 0 \) and \( Y^tA = Y^tB = 0 \). If we put \( M = XY^t \), then we have \( AM = MA = 0 \) and \( BM = MB = 0 \). Since \( X \) and \( Y \) are non-zero, rank \( M = 1 \) and the proof is complete. □

Theorem 5. Let \( D \) be a division ring and \( n \geq 3 \). If \( A_n \) is the set of all non-invertible matrices in \( M_n(D) \), then \( \text{diam}(A_n) = 4 \).

Proof. Suppose that \( A \) and \( B \) are two non-zero matrices in \( A_n \). Since \( A \) is non-invertible, there exist non-zero elements \( X, Y \in D^n \) such that \( AX = Y^tA = 0 \). Let \( A_1 = XY^t \). We have rank \( A_1 = 1 \) and \( AA_1 = A_1A = 0 \). Similarly, we find a matrix \( B_1 \in M_n(D) \) such that rank \( B_1 = 1 \) and \( BB_1 = B_1B = 0 \). Since \( A_1 \) and \( B_1 \) are rank 1 matrices, nullity \( A_1 + \) nullity \( B_1 = 2n - 2 > n \). This implies that \( \text{Ker} \ L_{A_1} \cap \text{Ker} \ L_{B_1} \neq \{0\} \) and \( \text{Ker} \ R_{A_1} \cap \text{Ker} \ R_{B_1} \neq \{0\} \). By Lemma 4, there is a matrix \( M \in C_{M_n(D)}(\{A_1, B_1\}) \) with rank 1. Therefore \( A - A_1 - M - B_1 - B \) is a path in \( \Gamma(A_n) \). Now Lemma 2 completes the proof. □

Theorem 6. Let \( F \) be a field and \( n \geq 3 \). If \( T_n \) is the set of all triangularizable matrices in \( M_n(F) \), then \( \text{diam}(T_n) = 4 \).

Proof. Suppose that \( A \) and \( B \) are two non-scalar matrices in \( T_n \). Since \( A \) and \( B \) are triangularizable matrices, each of them has at least one eigenvalue in \( F \). It means that there are scalars \( \alpha, \beta \in F \) such that \( A - \alpha I \) and \( B - \beta I \) are non-zero non-invertible matrices. Using the proof of Theorem 5, there is a path in \( \Gamma(M_n(F)) \) of length at most 4 between \( A - \alpha I \) and \( B - \beta I \) whose intermediate vertices are rank 1 matrices. Since each matrix of rank 1 is triangularizable, noting Theorem 3, the assertion is proved. □

Corollary 7. Let \( F \) be an algebraically closed field and \( n \geq 3 \). Then \( \text{diam}(M_n(F)) = 4 \).

3. Nilpotent matrices
Theorem 8. Let $D$ be a division ring. If $N_n$ is the set of all nilpotent matrices in $M_n(D)$, then $\text{diam} \Gamma(N_3) = 5$ and $\text{diam} \Gamma(N_n) = 4$, for any $n \geq 4$.

Proof. Suppose that $A$ and $B$ are two non-zero matrices in $N_n$. There are two matrices $P, Q \in GL_n(D)$ such that $PAP^{-1}$ and $QBQ^{-1}$ are upper triangular matrices whose diagonal entries are 0. Clearly, $E_{1n}(PAP^{-1}) = (PAP^{-1})E_{1n} = 0$ and $E_{1n}(QBQ^{-1}) = (QBQ^{-1})E_{1n} = 0$. Hence if we put $A' = P^{-1}E_{1n}P$ and $B' = Q^{-1}E_{1n}Q$, then we have $AA' = A'A = 0$ and $BB' = B'B = 0$. Furthermore, rank $A' = \text{rank} B' = 1$ imply that $\text{dim} (\text{Ker} \mathcal{L}_A \cap \text{Ker} \mathcal{L}_B) \geq n - 2$. Assume that $n \geq 3$. Hence there is a matrix $T \in GL_n(D)$ such that the first columns of two matrices $TA'^{-1}$ and $TB'^{-1}$ are zero. So we have $(TA'^{-1})E = (TB'^{-1})E = 0$, where $E = [1 \ 0 \ \ldots \ 0]^t \in D^n$.

First, assume that $n \geq 4$. Since $\text{dim} (\text{Ker} \mathcal{R}_A \cap \text{Ker} \mathcal{R}_B) \geq 2$, there is an element $X \in D^n$ whose first component is 0 and $X'(TA'^{-1}) = X'(TB'^{-1}) = 0$. Let $S = EX'$.

We have $(TA'^{-1})S = S(TA'^{-1}) = 0$ and $(TB'^{-1})S = S(TB'^{-1}) = 0$. Note that $S$ is a non-nilpotent matrix, so $A - A' - T^{-1}ST - B' - B$ is a path in $\Gamma(N_n)$. Now Lemma 2 shows that $\text{diam} \Gamma(N_n) = 4$.

Next, suppose that $n = 3$. Since nullity $TA'^{-1} = \text{nullity} TB'^{-1} = 2$, using the method used for $\text{Ker} \mathcal{R}_A \cap \text{Ker} \mathcal{R}_B$ in the previous case, we find two elements $Y, Z \in D^3$ whose first components are 0, $Y'(TA'^{-1}) = 0$ and $Z'(TB'^{-1}) = 0$. Let $M = EY'$ and $N = EZ'$. We have $(TA'^{-1})M = M(TB'^{-1}) = 0$ and $(TB'^{-1})N = N(TB'^{-1}) = 0$.

On the other hand, it is not hard to see that $M$ and $N$ are non-zero nilpotent matrices and $MN = NM = 0$.

Hence

$$A - A' - T^{-1}MT - T^{-1}NT - B' - B$$

is a path in $\Gamma(N_3)$. Now, we claim that $d(J, J') = 5$ in $\Gamma(N_3)$. By the proof of Lemma 2, every nilpotent matrix that commutes with a matrix $H_1 \in C_{M_3(D)}(J)$ is strictly upper triangular and every nilpotent matrix that commutes with a matrix $H_2 \in C_{M_3(D)}(J')$ is strictly lower triangular. This implies that $d(J, J') \geq 5$ in $\Gamma(N_3)$, so the claim is established. Therefore $\text{diam} \Gamma(N_3) = 5$, and the proof is complete.

Theorem 9. Let $D$ be a division ring and $n \geq 3$. If $M, N \in M_n(D)$ are two non-zero matrices such that $M^2 = N^2 = 0$, then $d(M, N) \leq 2$ in $\Gamma(M_n(D))$.

Proof. Clearly, nullity $M$ and nullity $N$ are more than or equal to $n/2$. If $\text{Ker} \mathcal{L}_M \cap \text{Ker} \mathcal{L}_N \neq \{0\}$ and $\text{Ker} \mathcal{R}_M \cap \text{Ker} \mathcal{R}_N \neq \{0\}$, then Lemma 4 establishes the assertion. So without loss of generality, suppose that $\text{Ker} \mathcal{L}_M \cap \text{Ker} \mathcal{L}_N = \{0\}$ (if $\text{Ker} \mathcal{R}_M \cap \text{Ker} \mathcal{R}_N = \{0\}$, then we consider $M^t$ and $N^t$ instead of $M$ and $N$). It implies that $n = 2r$, for some integer $r \geq 2$, and nullity $M = \text{nullity} N = r$. If $W_1$ and $W_2$ are two bases for $\text{Ker} \mathcal{L}_M$ and $\text{Ker} \mathcal{L}_N$, respectively, then $W_1 \cup W_2$ is a basis for $D^n$. Since $M^2 = N^2 = 0$, then using the representation of $A$ in the basis $W_1 \cup W_2$, we find a matrix $P \in GL_n(D)$ such that

$$PMP^{-1} = \begin{bmatrix} 0 & M_1 \\ 0 & 0 \end{bmatrix}$$

and

$$PNP^{-1} = \begin{bmatrix} 0 & 0 \\ N_1 & 0 \end{bmatrix}.$$
for some $M_1, N_1 \in GL_r(D)$. Now for any non-scalar matrix $X \in C_{M_n(D)}(M_1N_1)$, we have $P^{-1}(X \oplus M_1^{-1}XM_1)P \in C_{M_n(D)}(\{M, N\}) \setminus FI$, as desired. □

4. Idempotent and involution matrices

**Theorem 10.** Let $D$ be a division ring and $n \geq 3$. If $E_n$ is the set of all idempotent matrices in $M_n(D)$, then $\text{diam } \Gamma(E_n) = 3$.

**Proof.** First we prove the assertion for $n = 3$. Let $A, B$ be two non-scalar matrices in $E_3$. Without loss of generality, replacing an idempotent $P$ by $I - P$ if necessary, assume that nullity $A$ and nullity $B$ are equal to 2. Hence $\dim (\ker L_A \cap \ker L_B) \geq 1$. There exists a matrix $Q \in GL_3(D)$ such that $QAQ^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $QBQ^{-1} = \begin{bmatrix} 0 & R \\ 0 & S \end{bmatrix}$, where $S \in M_2(D)$ is a non-scalar idempotent. Clearly, $RS = R$ and we have the path $A - Q^{-1}E_{11}Q - Q^{-1}(I_1 \oplus S)Q - B$.

Now, suppose that $n \geq 4$ and $A$ and $B$ are two non-scalar matrices in $E_n$. There are two matrices $P, Q \in GL_n(D)$ such that $A_1 = PAP^{-1} = I_r \oplus 0_{n-r}$ and $QBQ^{-1} = I_s \oplus 0_{n-s}$, for some $r, s \geq 1$. Thus $B$ and $Q^{-1}E_{11}Q$ commute. So it is enough to prove that $C_{M_n(D)}(\{A_1, B_1\})$ contains at least one non-central idempotent, where $B_1 = P(Q^{-1}E_{11}Q)P^{-1}$. Assume that

$$B_1 = \begin{bmatrix} X & Y \\ Z & T \end{bmatrix},$$

where $Y$ is an $r \times (n-r)$ matrix. Since $\text{rank } B_1 = 1$, $\text{rank } X$ and $\text{rank } T$ are at most 1. First assume that both of $X$ and $T$ are nilpotent. Hence $X^2 = 0$ and $T^2 = 0$. Idempotency of $B_1$ implies that $XY + YT = Y$. Thus $XY = Y(I - T)$ and since $I - T$ is invertible, $Y = XY(I - T)^{-1} = X^2Y(I - T)^{-2}$. Now since $X^2 = 0$, we have $Y = 0$. Similarly we obtain that $Z = 0$. Therefore $B_1^2 = 0$, a contradiction. Without loss of generality, we may assume that $X$ is not nilpotent. First, suppose that $r \geq 2$. Since $\text{rank } X = 1$, there is a matrix $U \in GL_r(D)$ which $UXU^{-1} = \lambda E_{rr}$, for some $\lambda \in D \setminus \{0\}$. Using similarity with the matrix $V = U \oplus I_{n-r}$, it is enough to show that there exists a non-scalar idempotent matrix which commutes with both $A_1$ and $B_1'$, where

$$B_1' = VB_1V^{-1} = \begin{bmatrix} \lambda E_{rr} & UY \\ ZU^{-1} & T \end{bmatrix}.$$
Since rank $B'_1 = 1$, the first row of $UY$ and the first column of $ZU^{-1}$ are zero. This implies that $V^{-1}E_{11}V \in C_{M_n(D)}(\{A_1, B_1\})$, as desired. Now, assume that $r = 1$. If $T$ is not nilpotent, then since $n - 1 \geq 2$ by a similar argument we prove the assertion. Thus suppose that $T$ is nilpotent. Since rank $T \leq 1$, there is a matrix $U' \in GL_{n-1}(D)$ which $U'TU'^{-1} = \mu E_{1(n-1)}$, for some $\mu \in D$. Using similarity with the matrix $V' = I_1 \oplus U'$, it is enough to show that there exists a non-scalar idempotent matrix which commutes with both $A_1$ and $B'_1$, where

$$B''_1 = V'B_1V'^{-1} = \begin{bmatrix} X & YU'^{-1} \\ U'Z & \mu E_{1(n-1)} \end{bmatrix}.$$  

If $\mu = 0$, since rank $B''_1 = 1$, at most one of two matrices $Y$ and $Z$ is non-zero. Without loss of generality, suppose that $Y = 0$. Now, if $S \in M_{n-1}(D)$ is a non-zero idempotent matrix such that $SU'Z = 0$, then $0_1 \oplus S \in C_{M_n(D)}(\{A_1, B''_1\})$, as desired. If not, since rank $B''_1 = 1$, it is not hard to see that the third row and the third column of $B''_1$ are zero. This yields that $V'^{-1}E_{33}V' \in C_{M_n(D)}(\{A_1, B_1\})$, as desired.

To complete the proof, for each $n \geq 3$ we should find two matrices $A$ and $B$ whose distance in $\Gamma(En)$ is equal to 3. Let

$$R = \sum_{i \text{ is odd}} E_{ii}, \quad S_1 = \sum_{i < n \text{ is odd}} E_{i(i+1)}, \quad \text{and} \quad S_2 = \sum_{i < n \text{ is even}} E_{i(i+1)}.$$ 

If we put $A = R + S_1$ and $B = R - S_2$, then with an easy calculation we find that $A$ and $B$ are idempotents and $A - B = S_1 + S_2 = J$. Assume that $M$ is an idempotent matrix commutes with both $A$ and $B$. Then $M$ is also commutes with $J$ and by Lemma 1, $M$ is a polynomial in $J$. Thus $M$ is an upper triangular matrix with the same diagonal entries. Hence all eigenvalues of $M$ are the same and so $M = 0$ or $I$. This shows that $C_{M_n(D)}(\{A, B\})$ contains no non-scalar idempotent, so the proof is complete. \hfill \Box

**Theorem 11.** Let $D$ be a division ring and $n \geq 3$. If $A, B \in M_n(D)$ are two non-scalar idempotent matrices, then $d(A, B) \leq 2$ in $\Gamma(M_n(D))$.

**Proof.** We have $A(I - A) = (I - A)A = 0$, so one of nullity $A$ or nullity $(I - A)$ is at least $n/2$. Since $I - A$ is idempotent, without loss of generality, we may assume that nullity $A \geq n/2$ and similarly nullity $B \geq n/2$. If $\text{Ker} \mathcal{L}_A \cap \text{Ker} \mathcal{L}_B \neq \{0\}$ and $\text{Ker} \mathcal{R}_A \cap \text{Ker} \mathcal{R}_B \neq \{0\}$, then using Lemma 4, we find a non-scalar matrix in $C_{M_n(D)}(\{A, B\})$, as desired. So without loss of generality, suppose that $\text{Ker} \mathcal{L}_A \cap \text{Ker} \mathcal{L}_B = \{0\}$ (if $\text{Ker} \mathcal{R}_A \cap \text{Ker} \mathcal{R}_B = \{0\}$, then we consider $A'$ and $B'$ instead of $A$ and $B$). This implies that $n = 2r$, for some integer $r \geq 2$, and nullity $A =$ nullity $B = r$. If $W_1$ and $W_2$ are two bases for $\text{Ker} \mathcal{L}_A$ and $\text{Ker} \mathcal{L}_B$, respectively, then $W_1 \cup W_2$ is a basis for $D^n$. Since $D^n = \text{Ker} \mathcal{L}_A \oplus \text{Im} \mathcal{L}_A$, then for any $\omega \in W_2$, there are vectors $a \in \text{Ker} \mathcal{L}_A$ and $a' \in \text{Im} \mathcal{L}_A$ such that $\omega = a + a'$. So $A\omega = a' = -a + \omega$. Using the representation of $A$ in the basis $W_1 \cup W_2$, we find a matrix $P \in GL_n(D)$ such that

$$PAP^{-1} = \begin{bmatrix} 0 & A' \\ 0 & I_r \end{bmatrix},$$

where
for some $A' \in M_r(D)$, and by a similar method, we conclude that

\[ PBP^{-1} = \begin{bmatrix} I_r & 0 \\ B' & 0 \end{bmatrix}, \]

for some $B' \in M_r(D)$. Now, if $A'B' \neq B'A'$, then $P^{-1}(A'B' \oplus B'A')P$ is a non-scalar element of $C_{M_n(D)}(\{A, B\})$. So assume that $A'B' = B'A'$. Hence there is a non-scalar matrix $S \in M_r(D)$ commuting with $A'$ and $B'$ and therefore $P^{-1}(S \oplus S)P$ is a non-scalar element of $C_{M_n(D)}(\{A, B\})$, and the proof is complete.

**Remark 12.** The previous theorem shows that if $D$ is a division ring with center $F$ and $n \geq 3$, then $M_n(D)$ can be generated by no two idempotents as an $F$-algebra. This fact has been proved in [7, Theorem 4] and the above gives a new proof for it.

**Theorem 13.** Let $D$ be a division ring and $n \geq 3$. If $\mathcal{I}_n$ is the set of all involutions in $M_n(D)$, then the following hold.

i) If $\text{char } D \neq 2$, then $\text{diam } \Gamma(\mathcal{I}_n) = 3$.

ii) If $\text{char } D = 2$, then $\text{diam } \Gamma(\mathcal{I}_3) \leq 5$ and $\text{diam } \Gamma(\mathcal{I}_n) \leq 4$, for any $n \geq 4$.

**Proof.** First, assume that $\text{char } D \neq 2$. Indeed, the matrix $A \in M_n(D)$ is a non-scalar involution if and only if $(A + I)/2$ is a non-scalar idempotent matrix. Hence by Theorem 10, the assertion given in (i) is proved.

Next, suppose that $\text{char } D = 2$. For any non-scalar $B \in \mathcal{I}_n$, we have $(B + I)^2 = 0$ and so $B + I$ is a non-scalar nilpotent matrix. Moreover, for any non-zero nilpotent matrix $N$, we know that there is a natural number $k$ such that $N^k = 0$ and $N^{k-1} \neq 0$. If $s$ is the least integer such that $2^s \geq k$, then $(N^{2^{s-1}} + I)^2 = I$. Therefore if we have a path in $\Gamma(N_n)$, then we can find a path in $\Gamma(\mathcal{I}_n)$. Hence Theorem 8 completes the proof.

**5. Invertible matrices**

The following theorems have been proved in [1] and [3], respectively.

**Theorem A.** Let $F$ be a field and $n \geq 3$. The graph $\Gamma(M_n(F))$ is connected if and only if for each cyclic matrix $A \in M_n(F)$, $F[A] \setminus FI$ contains at least one non-cyclic matrix.

**Theorem B.** Let $D$ be a division ring with center $F$ and $|F| \geq 3$, and let $n$ be a natural number. Then $\Gamma(M_n(D))$ is a connected graph if and only if $\Gamma(GL_n(D))$ is a connected graph.
Let $D$ be a division ring with center $F$, and let $n$ a natural number. The matrix $A \in M_n(D)$ is called totally transcendental over $F$ if for any non-zero polynomial $f(t) \in F[t]$, $f(A)$ is an invertible matrix.

Now, we would like to obtain some relations between the diameter of commuting graph of invertible matrices and the diameter of commuting graph of the full matrix ring.

**Theorem 14.** Let $D$ be a division ring with center $F$ such that $|F| \geq 3$ and $n$ be a natural number. Then

$$
\text{diam } \Gamma(GL_n(D)) \leq \text{diam } \Gamma(M_n(D)) \leq \text{diam } \Gamma(GL_n(D)) + 2.
$$

Furthermore, if $D = F$ and $n \geq 3$, then

$$
4 \leq \text{diam } \Gamma(GL_n(F)) \leq \text{diam } \Gamma(M_n(F)) \leq \text{diam } \Gamma(GL_n(F)) + 1.
$$

**Proof.** If $n = 1$, then there is nothing to prove. So we may assume that $n \geq 2$. By Theorem B, if $\Gamma(M_n(D))$ is non-connected, then so is $\Gamma(GL_n(D))$. In this case $\text{diam } \Gamma(GL_n(D)) = \text{diam } \Gamma(M_n(D)) = \infty$ and the result follows. So we may suppose that both of them are connected graphs. We show that for any non-invertible matrix $A$, there exists a polynomial $f(x)$ over $F$ such that $f(A)$ is a non-scalar invertible matrix. First, suppose that $A$ is not algebraic over $F$, then by [5, Proposition 8.3.1], $A$ is similar to a matrix with form $A_0 \oplus A_1$, where $A_0$ is algebraic and $A_1$ is totally transcendental over $F$. By the fact that $A$ is a non-invertible matrix, we have $A_0 \oplus A_1 \neq A_1$, and since $A$ is not algebraic over $F$, $A_0 \oplus A_1 \neq A_0$. Let $g(x)$ be the minimal polynomial of $A_0$ over $F$. Thus $g(A_1) + I$ and so $g(A) + I$ is a non-scalar invertible matrix. Now, suppose that $A$ is algebraic over $F$. Thus $F[A]$ is an Artinian ring. If there exists a nilpotent matrix $C \in F[A]$, then $I + C$ is a non-scalar invertible matrix. Otherwise, since the Jacobson radical of $F[A]$ is a nilpotent ideal, it is zero. Therefore by [4, Theorem 8.7, p. 90], $F[A]$ is a direct product of finitely many fields. Since $|F| \geq 3$, it is easily seen that there exists a non-scalar invertible matrix in $F[A]$, as desired. This shows that $\text{diam } \Gamma(GL_n(D)) \leq \text{diam } \Gamma(M_n(D))$. Now, suppose that $B, C \in M_n(D) \backslash F I$ are arbitrary. There are $h_1(x), h_2(x) \in F[x]$ such that $h_1(B)$ and $h_2(C)$ are non-scalar invertible matrices. Therefore $d(B, C) \leq \text{diam } \Gamma(GL_n(D)) + 2$.

Next, suppose that $D$ is commutative. By the proof of Theorem 3, $d(I + J, I + J^t) \geq 4$ in $\Gamma(GL_n(F))$. So, by the first part of the theorem, to prove the second part it is suffices to show that $\text{diam } \Gamma(M_n(F)) \leq \text{diam } \Gamma(GL_n(F)) + 1$. Assume that $E, G \in M_n(F) \backslash F I$ are arbitrary. If both of them are non-invertible, then by Theorem 5, $d(E, G) \leq 4$. If both of them are invertible, then the result clearly follows. So we may assume that one of them, for example $E$, is non-invertible. Since $\Gamma(M_n(F))$ is a connected graph, by Theorem A, there exists $H \in F[G]$ which is a non-cyclic non-scalar matrix. Assume that $H_1 \oplus \cdots \oplus H_k$ is the rational form of $H$, where for any $i$, $1 \leq i \leq k$, $H_i \in M_{n_i}(F)$ and $n_1 \geq \cdots \geq n_k$. Since $0 \oplus \cdots \oplus 0 \oplus I_k$ commutes with $H_1 \oplus \cdots \oplus H_k$, we find a matrix $K \in M_n(F)$ such that $\text{rank } K \leq n/2$ and $d(G, K) \leq 2$. On the other hand, since $E$ is a non-invertible
matrix, by the proof of Theorem 5, it commutes with a rank 1 matrix, say $L$. Since $n \geq 3$, Ker $L \cap \ker L \neq \{0\}$ and Ker $R \cap \ker R \neq \{0\}$. Hence by Lemma 4, there exists a matrix $M \in C_{M_n(F)}(K) \cap C_{M_n(F)}(L)$ such that rank $M = 1$. So we have the path $G - H - K - M - L - E$ and the proof is complete. \qed

**Theorem 15.** Let $D$ be a division ring with center $F$ and $n \geq 2$. If $|F| > n$, then

$$\text{diam } \Gamma(GL_n(D)) = \text{diam } \Gamma(M_n(D)).$$

**Proof.** Since $|F| > n$, [5, Theorem 8.2.3, p. 377] implies that for any matrix $X \in M_n(D)$, there is a scalar $\lambda_X \in F$ such that $X - \lambda_X I$ is invertible.

Now, suppose that $R$ and $S$ are two arbitrary distinct vertices of $\Gamma(M_n(D))$. If $\mathcal{P}$ is a path between $R - \lambda_R I$ and $S - \lambda_S I$ in $\Gamma(GL_n(D))$, then by replacing the vertices $R - \lambda_R I$ and $S - \lambda_S I$ in $\mathcal{P}$ with $R$ and $S$, respectively, we conclude that $\text{diam } \Gamma(M_n(D)) \leq \text{diam } \Gamma(GL_n(D))$ and Theorem 14 completes the proof. \qed

**6. Full matrix rings**

The following theorem has been proved in [1].

**Theorem C.** Let $F$ be a field and $n \geq 3$. The graph $\Gamma(M_n(F))$ is connected if and only if every field extension of degree $n$ over $F$ contains at least one proper intermediate field.

**Lemma 16.** Let $A \in M_n(F)$ and $B \in M_m(F)$ be two matrices such that the minimal polynomial of $A$ divides the minimal polynomial of $B$. Then the equation $AX = XB$ has at least one non-zero solution in $M_{n \times m}(F)$.

**Proof.** Suppose that $E$ is the algebraic closure of $F$. Since the minimal polynomial of $A$ divides the minimal polynomial of $B$, $A$ and $B$ have at least one common eigenvalue in $E$. Since $X = 0$ is a solution of the equation $AX = XB$, by [8, Theorem 27.5.1], this equation has infinitely many solutions over $E$. Now, since $AX -XB = 0$ is a system of linear equations with coefficients in $F$ which has a non-zero solution over $E$, it should have a non-zero solution over $F$. The proof is complete. \qed

**Theorem 17.** Let $F$ be a field and $n \geq 3$. If $\Gamma(M_n(F))$ is a connected graph, then $\text{diam } \Gamma(M_n(F)) \leq 6$.

**Proof.** By Theorem 9, it is enough to show that for every vertex $A$ of $\Gamma(M_n(F))$, there is a vertex $C$ such that $C^2 = 0$ and $d(A, C) \leq 2$. Since $\Gamma(M_n(F))$ is a connected graph, by Theorem A, there exists a non-cyclic matrix $B$ in $F[A] \setminus FI$. Assume that $B_1 \oplus \cdots \oplus B_k$
is the rational form of $B$, where for any $i$, $1 \leq i \leq k$, $B_i$ is a cyclic matrix of size $n_i$ and $n_1 \geq \cdots \geq n_k$. Since the minimal polynomial of $B_2$ divides the minimal polynomial of $B_1$, by Lemma 16, there exists a non-zero matrix $B' \in M_{n_1 \times n_2}(F)$ such that $B_1B' = B'B_2$. So the matrix

$$C = \begin{bmatrix} 0 & B' \\ 0 & 0 \end{bmatrix} \oplus 0_{n-n_1-n_2}$$

commutes with $B$ and its square is zero, as desired. □

**Conjecture 18.** Let $F$ be a field. If $\Gamma(M_n(F))$ is a connected graph, then its diameter is at most 5.

In the next theorem we show that the conjecture is true, when $n$ is a prime number.

**Theorem 19.** Let $F$ be a field and $p \geq 3$ a prime number. If $\Gamma(M_p(F))$ is a connected graph, then $\text{diam } \Gamma(M_p(F)) = 4$.

**Proof.** Let $M$ be an arbitrary matrix in $M_p(F)\setminus FI$. We show that $M$ is adjacent to a matrix whose nullity is at least $(p+1)/2$. If $M$ is a non-cyclic matrix, then using the rational form of $M$, we find a matrix with the desired property. So we may assume that $M$ is a cyclic matrix. Let $f(x)$ be the minimal polynomial of $M$. Since $\Gamma(M_p(F))$ is a connected graph, by Theorem C, $f(x)$ is reducible. So there are non-scalar polynomials $f_1(x)$ and $f_2(x)$ in $F[x]$ such that $f(x) = f_1(x)f_2(x)$. Since $f_1(M)f_2(M) = 0$ and $f_1(M)$ and $f_2(M)$ are non-zero matrices, the nullity of at least one of them is not less than $(p+1)/2$.

Suppose that $A, B \in M_p(F)\setminus FI$ are two arbitrary matrices. There are $A', B' \in M_p(F)\setminus FI$ such that $AA' = A'A$ and $BB' = B'B$ and their nullities are at least $(p+1)/2$. Then $\text{Ker } L_{A'} \cap \text{Ker } L_{B'} \neq \{0\}$ and $\text{Ker } R_{A'} \cap \text{Ker } R_{B'} \neq \{0\}$. By Lemma 4, we find a common neighbor for $A'$ and $B'$, say $S$. So $A - A' - S - B' - B$ is a path in $\Gamma(M_p(F))$. Now Theorem 3 completes the proof. □

**Theorem 20.** Let $\mathbb{H}$ be the division ring of real quaternions. Then $\text{diam } \Gamma(M_2(\mathbb{H})) \leq 6$ and $\text{diam } \Gamma(M_n(\mathbb{H})) \leq 4$, for all $n \geq 3$.

**Proof.** Suppose that $A$ and $B$ are two vertices of $\Gamma(M_n(\mathbb{H}))$. By [10, Theorem 1], there are two matrices $P, Q$ in $M_n(\mathbb{H})$ such that $PAP^{-1}$ and $QBQ^{-1}$ are contained in $M_n(\mathbb{C})$. Using the proof of Theorem 5, the vertices $PAP^{-1}$ and $QBQ^{-1}$ have neighbors of rank 1 in $\Gamma(M_n(\mathbb{C}))$. Hence there are two matrices $A_1, B_1 \in M_n(\mathbb{H})$ with rank 1 that commute with $A, B$, respectively. If $\text{Ker } L_{A_1} \cap \text{Ker } L_{B_1} \neq \{0\}$ and $\text{Ker } R_{A_1} \cap \text{Ker } R_{B_1} \neq \{0\}$, then by Lemma 4, there is a non-scalar matrix $M$ that commutes with both $A_1$ and $B_1$. Therefore $A - A_1 - M - B_1 - B$ is a path in $\Gamma(M_n(\mathbb{H}))$, as desired. So without loss of generality, assume that $\text{Ker } L_{A_1} \cap \text{Ker } L_{B_1} = \{0\}$ (if $\text{Ker } R_{A_1} \cap \text{Ker } R_{B_1} = \{0\}$, then we consider $A_1^t$ and
instead of $A_1$ and $B_1$). Since rank $A_1 = \text{rank } B_1 = 1$, \text{dim} (\ker \mathcal{L}_{A_1} \cap \ker \mathcal{L}_{B_1}) \geq n - 2$ and hence $n = 2$. Moreover, there is a matrix $U \in \text{GL}_2(\mathbb{H})$ such that 

$$UA_1U^{-1} = \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix} \quad \text{and} \quad UB_1U^{-1} = \begin{bmatrix} b_1 & 0 \\ b_2 & 0 \end{bmatrix},$$

for some $a_1, a_2, b_1, b_2 \in \mathbb{H}$. Let $D = d_1I$, for some $d_1 \in C_{\mathbb{H}}(\{a_1, a_2\}) \setminus \mathbb{R}$, if $a_1a_2 = a_2a_1$; and otherwise, let $D = \text{diag} (a_1a_2a_1^{-1}, a_2)$. Also let $D' = d_2I$, for some $d_2 \in C_{\mathbb{H}}(\{b_1, b_2\}) \setminus \mathbb{R}$, if $b_1b_2 = b_2b_1$; and otherwise, let $D' = \text{diag} (b_1, b_2b_1b_2^{-1})$. Now,

$$A \to A_1 \to U^{-1}DU \to U^{-1}E_{11}U \to U^{-1}D'U \to B_1 \to B,$$

is a path in $\Gamma (M_2(\mathbb{H}))$. This completes the proof. \hfill \Box

7. On the structure of pairs of idempotents

In this section we would like to obtain simple representations for pairs of idempotents in $M_n(F)$, for any field $F$ and each integer $n \geq 2$. We start with three well-known results; we include short proofs for completeness.

Lemma 21. Let $F$ be an algebraically closed field and $n \geq 3$. Then every pair of idempotents in $M_n(F)$ has a non-trivial common invariant subspace.

Proof. Assume that $\{A, B\}$ is a pair of idempotents in $M_n(F)$. By Theorem 11, there exists a non-scalar matrix $M$ that commutes with both $A$ and $B$. Since $F$ is algebraically closed, there is $\lambda \in F$ such that $M - \lambda I$ is not invertible. Clearly, $\ker \mathcal{L}_{M - \lambda I}$ is an invariant subspace under $A$ and $B$. This completes the proof. \hfill \Box

Lemma 22. Let $F$ be a field and $\{A, B\}$ an irreducible pair of idempotents in $M_2(F)$. Then there is an element $t \in F \setminus \{0, 1\}$ such that $\{A, B\}$ is similar to

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} t & t \\ 1 - t & 1 - t \end{bmatrix} \right\}.$$

Proof. Without loss of generality, we may assume that 

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

for some $a, b, c, d \in F$. By irreducibility, we have $bc \neq 0$. Since $B$ is not scalar, rank $B = 1$. This implies that $a + d = 1$ and $ad \neq 0$. Using the similarity effected by $\text{diag} (a, b)$, we obtain that 

$$B = \begin{bmatrix} a & a \\ c' & 1 - a \end{bmatrix},$$

for some $c' \in F$. Since rank $B = 1$, we have $c' = 1 - a$, and the proof is complete. \hfill \Box
Corollary 23. Let $F$ be an algebraically closed field and $n \geq 2$. If $A$ and $B$ are two idempotents in $M_n(F)$, then there exists an integer $k \geq 0$ such that $\{A, B\}$ is similar to a pair of block upper triangular form matrices with diagonal blocks $\{A_i, B_i\}$, where for any $i$, $1 \leq i \leq k$,

$$A_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_i = \begin{bmatrix} t_i & t_i \\ 1 - t_i & 1 - t_i \end{bmatrix}$$

are matrices in $M_2(F)$ for some scalars $t_i \neq 0, 1$, and $\{A_i, B_i\} \subseteq \{0, 1\}$, for each $i \geq k+1$.

**Proof.** If $n = 2$, then using Lemma 22, we are done. So assume that $n \geq 3$. By Lemma 21, $\{A, B\}$ has a non-trivial invariant subspace. Thus there are the idempotents $A_1, A_2, B_1, B_2$ whose sizes are less than $n$ and $\{A, B\}$ is similar to $\{A_1 \ast 0, B_1 \ast 0\}$. Now, by induction the proof is complete. \[ \square \]

Lemma 24. Let $F$ be an algebraically closed field and $n \geq 2$. If $\{A, B\}$ is a pair of idempotents in $M_n(F)$ such that $d(A, B) = 3$ in $\Gamma(E_n)$, then there exist a scalar $\lambda \in F$ and a nilpotent matrix $N$ such that $(A - B)^2 = \lambda I + N$.

**Proof.** Clearly, $S = (A-B)^2$ commutes with both $A$ and $B$. Since $F[S] \subseteq C_{M_n(F)}(\{A, B\})$ and $d(A, B) = 3$ in $\Gamma(E_n)$, $F[S]$ has no non-trivial idempotent. By [4, Theorem 8.7, p. 90], $F[S]$ is a local ring. Since $F$ is an algebraically closed field, there exists a scalar $\lambda \in F$ such that $S - \lambda I$ is not invertible. Because $F[S]$ is an Artinian local ring and $S - \lambda I$ is contained in the Jacobson radical of $F[S]$, by [4, Corollary 8.2, p. 89], $S - \lambda I$ is a nilpotent matrix. This implies that $S = \lambda I + N$, for some nilpotent matrix $N$. \[ \square \]

Lemma 25. Let $F$ be a field and $\{A, B\}$ a pair of idempotents in $M_n(F)$, where $n \geq 2$. If $A - B$ is a nilpotent matrix, then $\{A, B\}$ is triangularizable.

**Proof.** By McCoy’s Theorem [9, Theorem 1.3.4, p. 8], we may assume that $F$ is an algebraically closed field. For any $t \in F \setminus \{1\}$, the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} t & t \\ 1 - t & 1 - t \end{bmatrix}$$

is not nilpotent. Since $A - B$ is a nilpotent matrix, so all of the diagonal blocks $A_i$ and $B_i$ appearing in Corollary 23, are 0 or 1. This yields that $\{A, B\}$ is triangularizable, as desired. \[ \square \]

Corollary 26. Let $F$ be an algebraically closed field and $n \geq 3$ an odd integer. Then every pair of idempotents in $M_n(F)$ with distance 3 in $\Gamma(E_n)$ is triangularizable.
Proof. Without loss of generality, we may assume that nullity $A$ and nullity $B$ are at least $(n + 1)/2$. Thus $\ker L_A \cap \ker L_B \neq \{0\}$ and therefore $A - B$ is not invertible. By Lemma 24, $A - B$ is nilpotent and so Lemma 25 completes the proof. \[ \square \]

Theorem 27. For every field $F$, the following are equivalent.

i) $F$ is an algebraically closed field.

ii) For any $n \geq 3$, every pair of idempotents in $M_n(F)$ has a non-trivial common invariant subspace.

iii) For any $n \geq 1$, every non-triangularizable pair of idempotents in $M_{2n}(F)$ with distance 3 in $\Gamma(\mathcal{E}_{2n})$, is similar to

\[
\left\{ \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} M & M \\ I-M & I-M \end{bmatrix} \right\},
\]

where $M = \lambda I + J \in M_n(F)$ and $\lambda \neq 0, 1$.

Proof. By Lemma 21, (i) implies (ii). For the other direction, suppose that $F$ is not algebraically closed. Thus there is an irreducible polynomial $p(x)$ of degree $m \geq 2$ in $F[x]$. Let $S \in M_m(F)$ be the companion matrix of $p(x)$. Then we claim that the following pair of idempotents

$E_1 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ and $E_2 = \begin{bmatrix} S & S \\ I-S & I-S \end{bmatrix}$

has no non-trivial common invariant subspace. Assume that $V \subseteq F^{2m}$ is a non-trivial common invariant subspace of $E_1$ and $E_2$. Let

\[
\begin{bmatrix} \alpha \\ \beta \end{bmatrix}
\]

be a non-zero vector in $V$, where $\alpha, \beta \in F^m$. We know that

$E_1 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$ and $(I - E_1) \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$

are two vectors in $V$. Thus without loss of generality, we may assume that $\alpha \neq 0$. For any $f(x) \in F[x]$, we have

\[
f(E_1E_2) \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} f(S)\alpha \\ 0 \end{bmatrix} \in V,
\]

and since $F^{2m}$ is irreducible as $F[S]$-module,

\[
\begin{bmatrix} x \\ 0 \end{bmatrix} \in V,
\]

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for each \( x \in F^m \). Now for any \( x, y \in F^m \), we have

\[
\begin{bmatrix}
  x \\
  y 
\end{bmatrix} = 
\begin{bmatrix}
  x - S(I - S)^{-1}y \\
  0
\end{bmatrix} + E_2 \begin{bmatrix}
  (I - S)^{-1}y \\
  0
\end{bmatrix} \in V,
\]

a contradiction.

Next, we prove that (i) implies (iii). Suppose that \( \{A, B\} \) is a pair of non-triangularizable idempotents in \( M_{2n}(F) \) such that \( d(A, B) = 3 \) in \( \Gamma(E_{2n}) \). We claim that \( \text{Ker} \mathcal{L}_A \cap \text{Ker} \mathcal{L}_B = \{0\} \). To get a contradiction assume that \( \text{Ker} \mathcal{L}_A \cap \text{Ker} \mathcal{L}_B \neq \{0\} \). So, Lemma 24 implies that \( (A - B)^2 \) is nilpotent. Now by Lemma 25, \( \{A, B\} \) is triangularizable, a contradiction. Therefore \( \text{Ker} \mathcal{L}_A \cap \text{Ker} \mathcal{L}_B = \{0\} \) and similarly \( \text{Ker} \mathcal{L}_A \cap \text{Ker} \mathcal{L}_{I - B} = \{0\} \), \( \text{Ker} \mathcal{L}_{I - A} \cap \text{Ker} \mathcal{L}_B = \{0\} \) and \( \text{Ker} \mathcal{L}_{I - A} \cap \text{Ker} \mathcal{L}_{I - B} = \{0\} \). So we conclude that \( \text{rank } A = \text{rank } B = n \). Without loss of generality, we can assume that

\[
A = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} X & Y \\ Z & T \end{bmatrix},
\]

where \( X, Y, Z, T \in M_n(F) \). We claim that \( Y \) is an invertible matrix. Suppose otherwise. Since \( B \) is idempotent, \( XY + YT = Y \) and so \( \text{Ker} \mathcal{L}_Y \) is invariant under \( T \). Thus \( \{A, B\} \) is similar to

\[
\begin{cases}
  \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \begin{bmatrix} X & Y_1 & 0 \\ Z_1 & T_{11} & 0 \\ Z_2 & T_{21} & T_{22} \end{bmatrix},
\end{cases}
\]

where \( T_{22} \) is an idempotent matrix. Thus at least one of the subspace \( \text{Ker} \mathcal{L}_A \cap \text{Ker} \mathcal{L}_B \) and \( \text{Ker} \mathcal{L}_{I - A} \cap \text{Ker} \mathcal{L}_{I - B} \) is non-zero, a contradiction. By a similar argument, one can prove that \( Z \) is invertible. Now, using similarity by the matrix \( Y^{-1} \oplus I_n \), we may assume that \( Y = I \). Since \( B^2 = B \), we find that

\[
B = \begin{bmatrix} X & I \\ X - X^2 & I - X \end{bmatrix}.
\]

Also, noting that \( X \) is necessarily invertible, and using similarity by the matrix \( X \oplus I_n \), we may assume that \( B \) is equal to the matrix

\[
\begin{bmatrix} X & X \\ I - X & I - X \end{bmatrix}. 
\]

By Lemma 24, there are \( \lambda \in F \) and nilpotent matrix \( N \) such that

\[
(A - B)^2 = \begin{bmatrix} I - X & 0 \\ 0 & I - X \end{bmatrix} = \lambda I + N.
\]

This yields that \( X = (1 - \lambda)I + N' \), for some nilpotent matrix \( N' \in M_n(F) \). Since \( \{A, B\} \) is not triangularizable and \( X \) is invertible, by Lemma 25, we have \( \lambda \neq 0, 1 \). To complete the proof, we must show that \( N' \) is a cyclic matrix. If \( N' \) is not cyclic, then using the
rational form of \( N' \), we find a non-scalar idempotent \( E \in C_{M_n(F)}(N') \). Now, by the form of \( B \) given in (\( \star \)), it is easily seen that \( E \oplus E \) is a non-scalar idempotent matrix which commutes with both \( A \triangleq B \), and this contradicts \( d(A, B) = 3 \) in \( \Gamma(\mathcal{E}_{2m}) \).

Finally, we prove that (iii) implies (i). To get a contradiction, suppose that \( F \) is not algebraically closed. We show that \( \{E_1, E_2\} \) which was defined in the first step of the proof, is not triangularizable and \( d(E_1, E_2) = 3 \) in \( \Gamma(\mathcal{E}_{2m}) \). Since \( F[S] \) is a field, 

\[
(E_1E_2 - E_2E_1)^2 = \begin{bmatrix} S^2 - S & 0 \\ 0 & S^2 - S \end{bmatrix}
\]

is not a nilpotent matrix and so \( \{E_1, E_2\} \) is not triangularizable. Moreover, if \( d(E_1, E_2) \leq 2 \) in \( \Gamma(\mathcal{E}_{2m}) \), then there exists a non-scalar idempotent matrix with the form \( X \oplus Y \) commuting with \( E_2 \). Since \( S \) is a cyclic matrix, \( C_{M_{2m}(F)}(S) = F[S] \) is a field. Thus \( X, Y \) are two idempotents in \( F[S] \) and so \( \{X, Y\} = \{0, I_m\} \). Since \( XS = SY \), we conclude that \( X = Y \), a contradiction. Now, we show that \( \{E_1, E_2\} \) is not similar to

\[
\left\{ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} M & M \\ I - M & I - M \end{bmatrix} \right\},
\]

where \( M = \lambda I + J \in M_n(F) \) and \( \lambda \in F \setminus \{0, 1\} \). Indeed if it is, then there exists a matrix \( P \in GL_n(F) \) such that \( M = PSP^{-1} \). This yields that \( x - \lambda \) divides \( p(x) \), which contradicts the fact that \( p(x) \) is an irreducible polynomial of degree \( m \geq 2 \).

\[\square\]

**Remark 28.** Note that each of the three statements in the previous theorem is equivalent to the assertion that every matrix \( A \in M_n(F) \) has a non-trivial invariant subspace.

**Theorem 29.** Let \( F \) be an algebraically closed field and \( n \geq 2 \). If \( A \) and \( B \) are two idempotents in \( M_n(F) \), then there is an integer \( k \geq 1 \) such that \( \{A, B\} \) is similar to

\[
\{A_1 \oplus \cdots \oplus A_k, B_1 \oplus \cdots \oplus B_k\},
\]

such that for every \( i, 1 \leq i \leq k \), the pair \( \{A_i, B_i\} \) is either upper triangularizable or equal to

\[
\left\{ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} M_i & M_i \\ I - M_i & I - M_i \end{bmatrix} \right\},
\]

where for each \( i \), \( n_i \geq 1 \), \( I = I_{n_i} \), \( M_i = \lambda_i I + J \in M_{n_i}(F) \) and \( \lambda_i \in F \setminus \{0, 1\} \).

**Proof.** By Lemma 22, the case \( n = 2 \) is easily verified. Assume that \( n \geq 3 \). If \( d(A, B) = 1 \) in \( \Gamma(\mathcal{E}_n) \), then \( \{A, B\} \) is triangularizable and we are done. Also if \( d(A, B) = 3 \) in \( \Gamma(\mathcal{E}_n) \), then by Corollary 26 and Theorem 27, the assertion is proved. Thus assume that \( d(A, B) = 2 \) in \( \Gamma(\mathcal{E}_n) \). This means that there exists a non-scalar idempotent matrix \( E \) commuting with both \( A \) and \( B \). Indeed, \( E \) is similar to \( I_r \oplus 0_{n-r} \), for some \( r \geq 1 \). Hence \( A \) and \( B \) are similar to \( A_1 \oplus A_2 \) and \( B_1 \oplus B_2 \), respectively, where \( A_1, B_1 \in M_r(F) \) and \( A_2, B_2 \in M_{n-r}(F) \) are idempotents. Now by induction, the proof is complete. \[\square\]
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