**Commutativity pattern of finite nonabelian p-groups determine their orders**

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COMMUTATIVITY PATTERN OF FINITE NON-ABELIAN 
$p$-GROUPS DETERMINE THEIR ORDERS

A. ABDOLLAHI, S. AKBARI, H. DORBIDI, AND H. SHAHVERDI

Abstract. Let $G$ be a non-abelian group and $Z(G)$ be the center of $G$. Associate a graph $\Gamma_G$ (called non-commuting graph of $G$) with $G$ as follows: take $G \setminus Z(G)$ as the vertices of $\Gamma_G$ and join two distinct vertices $x$ and $y$, whenever $xy \neq yx$. Here, we prove that “the commutativity pattern of a finite non-abelian $p$-group determine its order among the class of groups”; this means that if $P$ is a finite non-abelian $p$-group such that $\Gamma_P \cong \Gamma_H$ for some group $H$, then $|P| = |H|$.

1. Introduction and Results

Given a finite non-abelian group $G$, one can associate in many different ways a graph to $G$ (e.g. [3, 11]). Here we consider the non-commuting graph $\Gamma_G$ of $G$: the set of vertices of $\Gamma_G$ is $G \setminus Z(G)$, and two vertices $x$ and $y$ are adjacent if and only if $xy \neq yx$. The non-commuting graph was first considered by Paul Erdős in 1975 [8]. The non-commuting graph of finite groups has been studied by many people (e.g., [1, 7]).

The non-commuting graph of a group is a discrete way to reflect the commutativity pattern of the group. In [1] the following conjecture was formulated:

**Conjecture 1.1** (Conjecture 1.1 of [1]). Let $G$ and $H$ be two finite non-abelian groups such that $\Gamma_G \cong \Gamma_H$. Then $|G| = |H|$.

Conjecture 1.1 was refuted in [7] by exhibiting two groups $G$ and $H$ of orders

$$|G| = 2^{10} \cdot 5^3 \neq 2^3 \cdot 5^6 = |H|$$

with isomorphic non-commuting graphs.

In [1], it is proved that Conjecture 1.1 holds whenever one of the groups in question is a symmetric group, dihedral group, alternative group or a non-solvable AC-group (where by an AC-group we mean a group in which the centralizer of every non-central element is abelian). Recently Darafsheh [5] has proved the validity of Conjecture 1.1 whenever one of the groups $G$ or $H$ is a non-abelian finite simple group.

The main result of the present paper shows that any pair of groups consisting a counterexample for Conjecture 1.1 cannot contain a group of prime power order.

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Key words and phrases. Non-commuting graph; $p$-group; graph isomorphism; groups with abelian centralizers.

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Theorem 1.2. If $P$ is a finite non-abelian $p$-group such that $\Gamma_P \cong \Gamma_G$ for some group $G$, then $|P| = |G|$.

This is a curious general phenomenon for non-abelian groups of prime power order: the order of a prime power order group can be determined among all finite groups by a proper model of its commutativity behavior, i.e., the non-commuting graph.

2. Preliminary Results

It is not hard to prove that the finiteness or the being non-abelian of a group can be transferred under graph isomorphism whenever two groups have the same non-commuting graph. Throughout $P$ denotes a fixed but arbitrary finite non-abelian $p$-group of order $p^n$ whose center $Z(P)$ is of order $p^s$ and $1 < p^{s_1} < p^{s_2} < \cdots < p^{s_k}$ are all distinct conjugacy class sizes of $P$, where $p^{s_i}$ is the size of conjugacy class $g_i^G$ of the element $g_i$. Throughout we also denote by $u$ the common divisor $\gcd(a_1, \ldots, a_k, n-r)$ of $\{a_1, \ldots, a_k, n-r\}$.

Lemma 2.1. Let $G$ be a finite non-abelian group and $H$ be a group such that $\phi : \Gamma_G \to \Gamma_H$ is a graph isomorphism. Then the following hold:

1. $|C_G(h)|$ divides $(|g^G| - 1)(|Z(H)| - |Z(G)|)$, where $h = \phi(g)$.
2. If $|Z(G)| \geq |Z(H)|$ and $G$ contains a non-central element $g$ such that $|C_G(g)|^2 \geq |G| \cdot |Z(G)|$, then $|G| = |H|$.

Proof. (1) Since $\Gamma_G \cong \Gamma_H$, we have

\[ |G| - |Z(G)| = |H| - |Z(H)| \Rightarrow |H| = |G| - |Z(G)| + |Z(H)| \quad (a) \]

and

\[ |C_G(g)| - |Z(G)| = |C_H(h)| - |Z(H)| \Rightarrow |C_H(h)| = |C_G(g)| + |Z(H)| - |Z(G)| \quad (b) \]

As $|C_H(h)|$ divides $|H|$, $|C_H(h)|$ divides

\[ \frac{|G|}{|C_G(g)|} \left| \frac{|C_G(g)| + |Z(H)| - |Z(G)|}{|C_G(g)|} \right| \]

it follows from (a), (b), (c) $|C_H(h)|$ divides

\[ (|g^G| - 1)(|Z(H)| - |Z(G)|). \]

(2) Let $h = \phi(g)$. By part (1), we have $|C_H(h)| = |C_G(g)| + |Z(H)| - |Z(G)|$ divides $\left( \frac{|g^G| - 1}{|C_G(g)| + |Z(H)| - |Z(G)|} \right)$. Now, the inequality $|C_G(g)|^2 \geq |G| \cdot |Z(G)|$ implies that

\[ 0 \leq |C_H(h)| \leq \left( \frac{|g^G| - 1}{|C_G(g)| + |Z(H)| - |Z(G)|} \right) < |C_G(g)| + |Z(H)| - |Z(G)| = |C_H(h)| \]

and this yields $\left( \frac{|g^G| - 1}{|C_G(g)| + |Z(H)| - |Z(G)|} \right) = 0$. Hence $|Z(G)| = |Z(H)|$. \hfill $\square$

Lemma 2.2. Suppose that $H = P_1 \times A$ is a finite group, where $P_1$ is a $p$-group, $A$ is a finite abelian group such that $\gcd(p, |A|) = 1$. If $\Gamma_P \cong \Gamma_H$, then $|P| = |H|$.

Proof. Let $\phi$ be a graph isomorphism from $\Gamma_P$ to $\Gamma_H$. Suppose $h = \phi(g_t)$ for some $1 \leq t \leq k$ and $|P_t| = p^{s_t}, |Z(F_t)| = p^{s_t}, |A| = a$ and $|C_H(h)| = ap^{s_t}$. Since $\Gamma_P \cong \Gamma_H$, we have

\[ |P| - |Z(P)| = p^{s_t} (p^{n-r} - 1) = ap^{s_t} (p^{n-a} - 1) = |H| - |Z(H)|, \]

\[ |P| - |C_P(g_t)| = p^{n-a} (p^{s_t} - 1) = ap^{s_t} (p^{s_t-a} - 1) = |H| - |C_H(h)|, \]
Therefore we have the following equalities since gcd(a, p) = 1.

Therefore a = 1. Since r = ω, |Z(P)| = |Z(H)|. Hence |P| = |H|.

Lemma 2.3. Suppose H = Q × A, where Q is a q-group for some prime q, A is an abelian group and gcd(|A|, q) = 1. If ΓP ≅ ΓH, then |H| = |P|.

Proof. If p = q, then Lemma 2.2 completes the proof. Suppose, for a contradiction, that p ≠ q.

Note that |gp1| = pa1. Let φ be a graph isomorphism from ΓP to ΓH and let φ(g1) = h, |A| = a, |Q| = q̇, |CH(h)| = aq, |Z(H)| = aqω.

It is clear that κ > ν > ω. Since ΓP ≅ ΓH, we have

(1) |CH(h)| − |Z(H)| = aqω(−ν − 1) = p̂(p̂a1 − 1) = |Cp(g1)| − |Z(P)|,
(2) |H| − |CH(h)| = aqν(−ω − 1) = p̂na1 − 1) − |P| − |Cp(g1)|.

Since |gp1| ≤ |gP| for all g ∈ P \ Z(P), hH has the minimum size among all conjugacy classes of non-central elements of H. By considering the conjugacy class equation of H, we have

aqν = aqω + qνω + s

where

|xH| = {gH | g ∈ H \ Z(H)} \ {hH}.

Since gcd(a, q) = 1 and qνω | \sumi=1 |xiH|, it follows that κ − ν ≤ ω. (1)

Equation (1) implies that the largest p-power number possibly dividing a is p̂. Now it follows from Equations (1), (2) and the inequality (1) that

p̂na1 − 1 ≤ qνω − 1 ≤ p̂na1 − 2,

which is a contradiction. This completes the proof. □

Lemma 2.4. Let H be a group such that ΓP ≅ ΓH. Then |Z(H)| divides p̂(p̂u − 1), where u = gcd(a1, ..., ak, n − r).

Proof. (1) Since ΓP ≅ ΓH, \ |P| − |Z(P)| = |H| − |Z(H)| and \ |P| − |Cp(gi)| = |H| − |CH(h)|, for every i ∈ {1, ..., k} and h = φ(g1), where φ : ΓG → ΓH.

Therefore we have the following equalities

p̂(p̂u − 1) = |Z(H)| \ (\frac{|H|}{|Z(H)|} − 1)

p̂na1(p̂a1 − 1) = |CH(hi)| \ (\frac{|H|}{|CH(hi)|} − 1)

for each i ∈ {1, ..., k}. Thus |Z(H)| divides the great common divisors of the left hand side of two latter equalities which is p̂(p̂u − 1).

□
A class of groups arising in the proof of our main result is the class of AC-groups; as we mentioned, a group \( G \) is called an AC-group whenever the centralizer of every non-central element is abelian. AC-groups was studied by many people (e.g., \([10]\)). It is easy to see that \( C_G(x) \cap C_G(y) = Z(G) \) for any two non-central elements \( x, y \in G \) with distinct centralizers. This implies that \[
\mathcal{C}(G) = \{ C_G(x)/Z(G) \mid x \in G \setminus Z(G) \}
\]
is a partition of \( G/Z(G) \); where by a partition for a group \( H \) we mean a collection \( \mathcal{C} \) of proper subgroups of \( H \) such that \( H = \bigcup_{S \in \mathcal{C}} S \) and \( S \cap T = 1 \) for any two distinct \( S, T \in \mathcal{C} \). Each element of \( \mathcal{C} \) is called a component of the partition. If each component is abelian, we call \( \mathcal{C} \) an abelian partition. Thus \( \mathcal{C}(G) \) is an abelian partition for \( G/Z(G) \). The size of \( \mathcal{C}(G) \) is an invariant of the non-commuting graph \( \Gamma_G \), called the clique number; where by definition the clique number of a finite graph is the maximum number of vertices which are pairwise adjacent. The clique number of the non-commuting graph \( \Gamma_H \) of a non-abelian group \( H \) will be denoted by \( \omega(H) \). Thus \( \omega(H) \) is simply the maximum number of pairwise non-commuting elements in the group.

**Lemma 2.5.** Suppose that \( G \) is a finite non-abelian AC-group such that \( G/Z(G) \) is a \( p \)-group. Then \( \omega(G) \equiv 1 \mod p \).

**Proof.** Since \( G \) is an AC-group, \( \omega = \omega(G) = |\mathcal{C}(G)| \), where \[
\mathcal{C}(G) = \{ C_G(x) \mid x \in G \setminus Z(G) \}.
\]
On the other hand, \( C_G(x) \cap C_G(y) = Z(G) \) for any two non-central elements \( x, y \in G \) such that \( C_G(x) \neq C_G(y) \). Therefore
\[
|G| = -(|\omega - 1|Z(G)| + \sum_{S \in \mathcal{C}(G)} |S|).
\]
This completes the proof. \(\square\)

**Lemma 2.6.** \(\text{[Mann 2]: Lemma 39.8, p. 354]}\) Suppose \( C \) is a subgroup of group \( G \) and let \( a \in G \) be such that \( CC^a = C^aC \). Then \( CC^a = C[C, a] \).

**Proof.** We have
\[
CC^a = \bigcup_{c \in C} Cc^a = \bigcup_{c \in C} Cc^{-1}a = \bigcup_{c \in C} C[c, a] \subseteq C[C, a].
\]
Thus \( CC^a \subseteq C[C, a] \). Since all the generators \( [c, a] = c^{-1}a^c (c \in C) \) of \( [C, a] \) belongs to \( CC^a \), we have \( C[C, a] \subseteq CC^a \), since by hypothesis \( CC^a \) is a group. This completes the proof. \(\square\)

In the following proposition we will use this property of any AC-groups \( G \); for any two commuting non-central elements \( x \) and \( y \) of \( G \), we have \( C_G(x) = C_G(y) \).

**Proposition 2.7.** Let \( G \) be a nilpotent AC-group of nilpotency class greater than 2, then the set \( \mathcal{C} \) of all centralizers of non-central elements of \( G \) has exactly one normal member \( T \) in \( G \). In particular, \( T \) is a characteristic subgroup of \( G \). Moreover, the latter normal subgroup \( T \) has the maximum order among all members of \( \mathcal{C} \).

**Proof.** Let \( x \) be any element of \( Z_2(G) \setminus Z(G) \). Then \( C_G(x) \) is a normal subgroup of \( G \) containing \( G' \); for the map \( \phi \) defined on \( G \) by \( g^0 = [x, g] \) for all \( g \in G \) is a group homomorphism and its image is contained in \( Z(G) \) and its kernel is \( C_G(x) \).
Since $G$ is of nilpotency class greater than 2, there exists an element $g \in G \setminus Z(G)$. Since $[Z_2(G), G'] = 1$, the remark preceding the proposition implies that

$$C_G(x) = C_G(g) \text{ for all } x \in Z_2(G) \setminus Z(G).$$

Now suppose that $N = C_G(y)$ is a normal centralizer of $G$ for some non-central element $y$. Then there exists an element $t \in (N \cap Z_2(G)) \setminus Z(G)$, since $Z(G) \leq N$. Thus $gt = ty$, it follows from $\triangle$ that $C_G(t) = C_G(g) = C_G(y)$. Hence, we have so far proved that $C$ has exactly one normal member in $G$. This implies that $C_G(x)$ is a characteristic subgroup of $G$.

Now, we prove $C_G(x)$ has the maximum order among all members of $C$. Suppose that $C = C_G(h)$ for some $h \in G \setminus Z(G)$. We may assume that $C$ is not normal in $G$. Thus there exists an element $a \in N_G(N_C(C)) \setminus N_C(C)$, since $C$ is nilpotent.

Then $C^a \neq C$, and $C^a$ is a subgroup of $N_G(C)$. Let $A = CC^a$. By Lemma 2.6, we have

$$CC_G(x) \supseteq CG'Z(G) \supseteq C[C,a]Z(G) = CC^aZ(G) = CC^a = A.$$

It follows that

$$\frac{|C||C_G(x)|}{|Z(G)|} = |CC_G(x)| \geq |A| = |CC^a| = \frac{|C|^2}{|Z(G)|}.$$

Thus $|C_G(x)| \geq |C|$. This completes the proof.

The proof of existence of unique normal centralizer is due to Rocke [9, Lemma 3.8]; the argument to prove the existence of a normal centralizer of maximal order is due to Mann [2, Theorem 39.7, p. 354]. He has proved among all abelian subgroups of maximal order in a metabelian $p$-group, there exists a normal subgroup. The latter was first proved by Gillam [4].

Lemma 2.8. Let $P$ be of nilpotency class 2. Then $a_i \leq r$ for every $i$.

Proof. Since $P$ is of nilpotency class 2, for every $x \in P \setminus Z(P)$ with class size $p^{a_i}$, the conjugacy class of $x$ is contained in $xp^r \subseteq xZ(P)$. Hence $p^{a_i} \leq p^r$. This completes the proof.

Now we will need the following two well known results about Frobenius groups.

Proposition 2.9. (1) (see e.g., Theorem 6.7 of [6]) Let $N$ be a normal subgroup of a finite group $G$, and suppose that $C_G(n) \subseteq N$ for every non-identity element $n \in N$. Then $N$ is complemented in $G$, and if $1 < N < G$, then $G$ is a Frobenius group with kernel $N$.

(2) (see e.g., Lemma 6.1 of [6]) Let $H$ be a Frobenius group with the kernel $F$ and a complement $K$, then $|K|$ divides $|F| - 1$.

Lemma 2.10. Let $H = KF$ be a Frobenius group with the kernel $F$ and a complement $K$. Suppose $1 \subset F_1 \subset F$ is a normal subgroup of $H$. Then $H_1 =KF_1$ is a Frobenius group with the kernel $F_1$ and a complement $K$.

Proof. For every non-identity element $f_1$ of $F_1$,

$$C_{H_1}(f_1) = C_H(f_1) \cap H_1 \subseteq F \cap H_1 = F \cap F_1K = F_1,$$

by the Dedekind modular law. Therefore $H_1$ is a Frobenius group with the kernel $F_1$. It is clear that $K$ is a complement for $F_1$ in $H_1$. 

\[\square\]
3. Proof of the Main Result

In this section we prove our main result, Theorem 1.2.

We argue by induction on the order of $P$. If $|P| = p^3$, then $|P| = |G|$ by Proposition 3.20 of [1]. If $P$ is not an AC-group, there exists a non-central element $x \in P$ such that $C_P(x)$ is non-abelian. If $y = \phi(x)$, then $\Gamma_{C_P(x)} \cong \Gamma_{C_G(y)}$. Now induction hypothesis implies that $|C_P(x)| = |C_G(y)|$ and since $|P| - |C_P(x)| = |G| - |C_G(y)|$, we have $|P| = |G|$. Thus, we may assume that $P$ is an AC-group and so $G$ is also an AC-group. By Proposition 3.14 of [1], we may assume that $G$ is solvable. Therefore by the classification of non-abelian solvable AC-groups in [10], $G$ is isomorphic to one the following groups $H_i$ ($i = 1, \ldots, 5$):

1. $H_1$ is non-nilpotent and it has an abelian normal subgroup $N$ of prime index and $\omega(H_1) = |N : Z(H_1)| + 1$.
2. $H_2/Z(H_2)$ is a Frobenius group with the Frobenius kernel and complement $F/Z(H_2)$ and $K/Z(H_2)$, respectively and $F$ and $K$ are abelian subgroups of $H_2$ and $\omega(H_2) = |F : Z(H_2)| + 1$.
3. $H_3/Z(H_3) \cong S_4$ and $V$ is a non-nilpotent subgroup of $H_3$ such that $V/Z(H_3)$ is the Klein 4-group of $H_3/Z(H_3)$ and $\omega(H_3) = 13$, where $S_4$ is the symmetric group of 4 letters.
4. $H_4 = A \times Q$, where $A$ is an abelian subgroup and $Q$ is an AC-group of prime power order.
5. $H_5/Z(H_5)$ is a Frobenius group with the Frobenius kernel and complement $F/Z(H_5)$ and $K/Z(H_5)$, respectively and $K$ is an abelian subgroup of $H$. $Z(F) = Z(H_5)$, and $F/Z(H_5)$ is of prime power order and $\omega(H_5) = |F : Z(H_5)| + \omega(F)$.

By Lemmas 3.11 and 3.12 of [1] and Lemma 2.3, we may assume that $G$ is isomorphic to either $H_1$ or $H_5$. Suppose that $G \cong H_1$. Then, obviously $\Gamma_P \cong \Gamma_{H_1}$. Since $N$ is abelian, there exists $h \in H_1 \setminus Z(H_1)$ such that $C_{H_1}(h) = N$. As $P$ is an AC-p-group, it follows from Lemma 2.5 that

$$\omega(P) \equiv 1 \mod p.$$ 

Since $\Gamma_P \cong \Gamma_{H_1}$, we have

$$\omega(H_1) = |C_{H_1}(h) : Z(H_1)| + 1 \equiv 1 \mod p,$$

and so $p \not| C_{H_1}(h) : Z(H_1))$. On the other hand Lemma 2.1(1) implies that, $|C_{H_1}(h)|$ divides $(p^\alpha - 1)(p^\beta - |Z(H_1)|)$, where $g_1$ maps to $h$ under a graph isomorphism from $\Gamma_P$ to $\Gamma_{H_1}$. Thus $p$ divides $|Z(H_1)|$ and so $p^2 \not| C_{H_1}(h)$). This follows that $p^2$ divides $|Z(H_1)|$. By continuing this latter process, one obtains that $p^r$ divides $|Z(H_1)|$ and so $|Z(H_1)| \geq |Z(F)|$. Now, let $y \in H_1 \setminus C_{H_1}(h)$ such that $H_1 = C_{H_1}(h)C_{H_1}(y)$ and

$$|H_1||Z(H_1)| = |C_{H_1}(h)||C_{H_1}(y)| \leq \max\{|C_{H_1}(h)|^2, |C_{H_1}(y)|^2\}.$$ 

Now, Lemma 2.1(2) implies that $|P| = |H_1|$.

Thus, it remains to deal with the case $G \cong H_5$. Let $H = H_5$ and note that $\Gamma_P \cong \Gamma_{H_1}$. We need to introduce some new notation for the group $H$. Since $F/Z(F)$ is a $q$-group for some prime $q$, we set $|F| = bq^\alpha$, for some positive integer $b$ such that $\gcd(b, q) = 1$ and therefore $|Z(H)| = bq^\alpha$ and $|C_F(f_i)| = |C_H(f_i)| = bq^\alpha$ for some $f_i \in F \setminus Z(F)$. (Recall that $Z(F) = Z(H)$ in this case) Since $F$ is nilpotent and non-abelian, we have $1 \leq \omega < \ell_i < \kappa$. Since $\gcd(|K|/|Z(H)|, |F/Z(H)|) = 1$, we have $|C_H(h)| = |K| = ar^\omega$ for some $h \in H \setminus F$ and for some positive integer $a$. 


It is clear that $b \mid a$ and $\gcd(a, q) = 1$. Therefore $|H| = ap^\kappa$. Suppose that under a graph isomorphism from $\Gamma_H$ to $\Gamma_P$, $h$ maps to $g_i$ for some integer $1 \leq t \leq k$ and $f_i$ maps to $g_i$, where $1 \leq i \leq k$ and $i \neq t$. Here note that $f_i$ is not defined. Suppose further that $\beta = a_t$.

We need to prove the following (a), (b), (c) and (d).

(a) $p \neq q$.

(b) if $p^t$ divides $a$, for some integer $l$, then $p^t$ divides $b$ and $p^{t+1}$ does not divide $a$. This simply means that the largest $p$-power part of $a$ and $b$ are the same and $p^r$ is the largest $p$-power possibly dividing $a$.

(c) $\Gamma_P$ is a regular graph so that there exists integers $\nu$ and $\alpha$ such that $\nu_1 = \nu$ and $a_i = \alpha$ for all $1 \leq i \leq k$ and $i \neq t$.

(d) $\nu \leq 2\omega$ and $\kappa \leq 3\omega$.

Proof of (a) Suppose $p = q$. Since $\Gamma_P \cong \Gamma_H$, $aq^\alpha p^{n-\omega} - 1 = p^{n-\beta}(p^{\beta} - 1)$ and $bp^\alpha p^{n-\omega} - 1 = p^\beta(p^{n-a_1-r} - 1)$. Therefore $n - \beta = \omega = r$, a contradiction.

Proof of (b) Since $\Gamma_P \cong \Gamma_H$, we have

$$\frac{(a - b)q^\omega}{p^\beta(p^{n-\beta-r} - 1)}.$$ 

Thus $p^{\beta} \mid a - b$. This proves the first part of (b) for all $l \in \{1, \ldots, r\}$. Now, suppose $t > r$ and $p^t$ divides $a$ and $p^{t+1}$ divides $b$. Equation (3) shows $p^{t+1} \mid b$. Now let $i \in \{1, \ldots, k\}$ such that $i \neq t$. Then by the graph isomorphism, we have

$$p^{n-a_i} - p^{n-a_1} = aq^\omega - bq^\nu.$$ 

(\*)

Since $r + 1 \geq n - a_1$ and $r + 1 \geq n - a_i$, it follows from (\*) that $p^{t+1}$ divides $b$, a contradiction. Now Equation (3) implies that $p^{t+1} \mid a$ and since $b$ divides $a$, the proof of part (b) follows.

Proof of (c) Suppose $\Gamma_P$ is not regular. Therefore $F$ has two centralizers $C_H(f_1)$ and $C_P(f_2)$ of order $bq^\nu_1$ and $bq^\nu_2$, respectively, where $\nu_1 \neq \nu_2$. We may assume that the conjugacy class of $f_1$ in $F$ is of minimum size among all conjugacy classes of non-central elements of $F$. We distinguish two cases to reach a contradiction.

(I) Suppose that $\nu_1 - \nu_2 \leq \omega$.

(4) $p^{n-a_2} - p^r = bq^\nu_2 - bq^\nu$

(5) $p^{n-a_1} - p^{n-a_2} = bq^\nu_1 - bq^\nu_2$

Now it follows from Equations (4), (5) and part (b) that

$$p^{n-a_2-r}|q^{\nu_1 - \nu_2} - 1 \leq q^\omega - 1 \leq p^{n-a_2-r} - 2,$$

a contradiction.

(II) Suppose that $\nu_1 - \nu_2 > \omega$.

We claim that the nilpotency class of $F$ is greater than 2. If not, then Lemma 2.8 implies that $\kappa - \nu_2 \leq \omega$.

Since $\nu_1 - \nu_2 > \omega$, $\kappa$ is a contradiction. Therefore the nilpotency class of $F$ is greater than 2. Since $H$ is an AC-group, $F$ is also an AC-group. Therefore every maximal abelian subgroup of $F$ is centralizer of non-central element of $F$. 

By Proposition 2.7, $F$ has a characteristic centralizer $C_F(f_j)$ of order $bq^\nu_j = bq^{\nu_1}$ having the maximum order among the proper centralizers. Thus $\nu_j = \nu_1$ and so $a_1 = a_j$. Since $F$ is normal subgroup of $H$, $C_F(f_j)$ is normal in $H$. Since $H/Z(H)$ is Frobenius group, by Lemma 2.10 $K/Z(H)C_F(f_j)/Z(H)$ is a Frobenius group with the kernel $C_F(f_j)/Z(H)$ and a complement $K/Z(H)$. Thus

$$\frac{a}{b}q^{\alpha_1 - \omega} - 1.$$ \hfill \(\heartsuit\)

By the graph isomorphism, we have

$$bq^\nu(q^{\nu_1 - \omega} - 1) = p^\nu(p^n - a_{11} - r - 1),$$

$$bq^{\nu_1} \left(\frac{a}{b}q^{\kappa - \nu_1} - 1\right) = p^{n-a_{11}}(p^{\alpha_1} - 1).$$

Since $\gcd(\frac{a}{b}, p) = 1$, Equations $\heartsuit$ and (6) imply that $\frac{a}{b}q^\omega|p^{n-a_{11} - r} - 1$. Equation (7) imply that $p^{n-a_{11} - r}|\frac{a}{b}q^{\kappa - \nu_1} - 1$ and by the conjugacy class equation $\kappa - \nu_1 \leq \omega$. Therefore $\frac{a}{b}q^\omega < \frac{a}{b}q^\omega$, a contradiction.

**Proof of (d)** Since $\Gamma_F$ is regular and $F$ is an AC-group, we have $\omega(F) = \frac{p^{\kappa}}{p^{\omega} - 1}$. Therefore $\nu - \omega$ divides $\kappa - \omega$. Now, by considering the conjugacy class equation of $F$, we find that $\nu - \omega \leq \omega$ and $\kappa \leq 3\omega$.

Now we have two different possibilities on the centralizer orders of $H$:

(I) $bq^\nu > aq^\omega$. Since $\Gamma_P \cong \Gamma_H$, we have

$$p^{n-\alpha} - p^{n-\beta} = bq^\nu - aq^\omega,$$

where $\beta = a_1$. It follows from the latter equation, Lemma 2.4 and parts (b),(d) that

$$p^{n-\beta-r}|q^{\nu-\omega} - \frac{a}{b} < q^\omega|p^n - 1 < p^{n-\beta-r},$$

a contradiction.

(II) $aq^\omega > bq^\nu$.

Since $\Gamma_P \cong \Gamma_H$, we have

$$aq^\omega - bq^\nu = p^{n-\beta} - p^{n-\alpha}.$$  

We consider two cases:

(i) $u < n - \alpha - r$. Since $u | n - \alpha - r$, $2u \leq n - \alpha - r$. Since $H/Z(H)$ is a Frobenius group, $\frac{|K/Z(H)|}{a/b}$ divides $|F/Z(F)| - 1$. Now it follows from parts (b) and (d), Lemma 2.4(1) and Equation (8), we have

$$p^{n-\alpha-r}\left|\frac{a}{b} - q^{\nu-\omega} \leq q^\kappa - \omega - 1 - q^{\nu-\omega} < q^{2\omega}(p^u - 1)^2 < p^{2u},$$

a contradiction.

(ii) $u = n - \alpha - r$. Since $u | n - \beta - r$, $n - \beta - r \geq 2u$. By the graph isomorphism

$$p^n - p^{n-\beta} = aq^\omega(q^{\kappa-\omega} - 1).$$

This latter equation, Lemma 2.4 and parts (b) and (d) imply that

$$p^{n-\beta-r}|q^{\kappa-\omega} - 1 < q^{2\omega}|(p^u - 1)^2 < p^{2u},$$

a contradiction.
This completes the proof. □

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