Minimum Flow Number of Complete Multipartite Graphs *

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Abstract

Let G be a directed graph and λ be a positive integer. By a nowhere-zero λ-flow, we mean an edge assignment using the set $\{1, \ldots, \lambda - 1\}$ such that at each vertex the sum of the values of all outgoing edges equals to the sum of values of all incoming edges modulo λ. Given a graph G, let $\Lambda = \Lambda(G)$ denote the smallest integer Λ for which G has a nowhere-zero Λ-flow, in some orientation of $E(G)$. Let m and n $(m, n \geq 2)$ be two natural numbers. It was proved that $\Lambda(K_{m,n}) \leq 3$ and $\Lambda(K_{2l}) = 3$, $(l \geq 3)$. In this paper we present a short proof for these results. Moreover, we show that is $\Lambda(K_{m_1,\ldots,m_k}) \leq 3$ if $m_i \in \mathbb{N}$, for $i = 1, 2, \ldots, k$ and $k \geq 3$.

* *Key Words*: Nowhere-zero λ-flow, Minimum flow number, Complete multipartite graphs

1 Introduction

Given a graph $G(V,E)$ with vertex set $V(G)$ and edge set $E(G)$, where multiple edges are allowed, let (D,f) be an ordered pair where D is an orientation of $E(G)$ and $f : E(G) \rightarrow \mathbb{Z}$ be an integer-valued function called a *flow*. An oriented edge of G is called an *arc*. For a vertex $v \in V(G)$, let $E^+(v) = \{$all arcs of $D(G)$ with their tails at $v\}$ and $E^-(v) = \{$all arcs of $D(G)$ with their heads at $v\}$.
A λ-*flow* of a graph G is a flow f such that $|f(e)| < \lambda$ for every edge $e \in E(G)$ and for every vertex $v \in V(G)$

$$\sum_{e\in E^+(v)} f(e) \equiv \sum_{e\in E^-(v)} f(e) \qquad (mod\ \lambda).$$

A *nowhere-zero* λ-*flow* of a graph G is an ordered pair (D,f) such that for every edge $e \in E(G)$, $f(e) \in \{1,\ldots,\lambda-1\}$ and for every vertex $v \in V(G)$,

$$\sum_{e\in N_D^+(v)} f(e) \equiv \sum_{e\in N_D^-(v)} f(e) \qquad (mod\ \lambda).$$

The *biwheel* on n vertices which is obtained by joining a cycle on $n-2$ vertices and K_2 and is shown by B_n. The complete graph of order n and the complete k-partite graph with part sizes $m_1,\ldots,m_k$ are denoted by K_n and $K_{m_1,\ldots,m_k}$, respectively. Also, the cycle with n vertices is denoted by C_n.
The *join* $G \vee H$ of disjoint graphs G and H is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. Let G be a graph and H and K be two subgraphs of G such that the edges of H and K partition the edges of G. Then we write $G = H \oplus K$.
A problem of interest about flows is the following: Given a graph G, what is the smallest integer λ for which G has a nowhere-zero λ-flow, i.e., an integer λ for which G admits a nowhere-zero λ-flow, but it does not admit a $(\lambda-1)$-flow. Let $\Lambda = \Lambda(G)$ denote this minimum λ. In this paper, we show that if G admits a nowhere-zero 3-flow,

then

$$\Lambda(G \vee K_2) = 3,\ \Lambda(K_{2l}) = 3 \quad (l \geq 3),\ \Lambda(K_{m,n}) \leq 3 \quad (m, n \geq 2),$$
$$\Lambda(K_{m_1,\ldots,m_k}) \leq 3 \quad (k \geq 3),\ \Lambda(B_n) \leq 3 \quad (n \geq 5).$$

In [4], it was shown that $\Lambda(K_n)$ and $\Lambda(K_{m,n})$ do not exceed 3. In the next section, we present a short proof for K_{2n}. In Section 3, we present a short proof for $K_{m,n}$ and then we extend this result to the complete multipartite graphs. For more information on nowhere-zero λ-flows, the interested reader is referred to [1], [3], [5] and [7].

The following theorem characterizes all graphs which admit a nowhere-zero 2-flow see [6, p. 308].

Theorem A. A graph has a nowhere-zero 2-flow if and only if it is an even graph.

2 The Minimum Flow Number of the Complete Graphs

In this section, we investigate the problem of minimum nowhere-zero λ-flow for the complete graph K_{2n}. Since K_{2n+1} is an even graph by Theorem A, $\Lambda(K_{2n+1}) = 2$

Theorem 1. $\Lambda(K_{2n}) = 3$ *for* $n \geq 3$.

Proof. Since the degree of each vertex of K_{2n} is odd, $\Lambda(K_{2n}) \geq 3$. Partition the vertices of K_{2n} into two sets X and Y such that $|X| = |Y| = n$, and consider the subgraph of K_{2n} induced by all edges with one endpoint in X and the other in Y. Choose three 1-factors of these edges, say M_1, M_2 and M_3. Now, orient the edges of M_1 with tails in X all labeled by 2, while orient the edges of M_2 and M_3 with tails in Y having label 1.

Let $H = K_{2n} \setminus \bigcup_{i=1}^{3} M_i$. Clearly, H is a $(2n-4)$-regular graph and by Theorem A, it admits a nowhere-zero 2-flow. Thus K_{2n} has a nowhere-zero 3-flow. $\square$

The above proof shows that exactly n edges are labeled by 2. Indeed, this is a nowhere-zero 3-flow using the minimum number of edges labeled by 2. The proof follows here.

Theorem 2. *In every nowhere-zero 3-flow of K_{2n}, at least n arcs will be labeled by 2.*

Proof. By contradiction, suppose that no more than $n-1$ arcs are labeled by 2.
Then, by pigeonhole principle, there exists at least one vertex with no incident arc with label 2. Since K_{2n} is $(2n-1)$-regular, then it is not possible to maintain the proper flow at this vertex using only the label 1. $\square$

3 The Minimum Flow Number of the Complete Multipartite Graphs

In this section, we investigate the problem of minimum nowhere-zero λ-flow for the complete bipartite $K_{m,n}$ and also the complete multipartite graph $K_{m_1,\dots,m_k}$.
In order to prove our main theorem, here we present a few lemmas which will be used frequently in the proof of the main theorem.

Lemma 1. *Let G be a graph. If G belongs to the following set*

$$\{K_{2,2}, K_{2,3}, K_{3,3}, K_{1,1,1}, K_{1,1,2}, K_{1,1,3}, K_{1,2,2}, K_{1,2,3}, K_{1,3,3}, K_{1,1,1,1,2}\},$$

then $\Lambda(G) \leq 3$.

Proof. If $G \in \{K_{2,2}, K_{1,1,1}, K_{1,1,3}, K_{1,3,3}\}$, then by Theorem A, $\Lambda(G) = 2$. Here we show a nowhere-zero 3-flow for the remaining graphs:

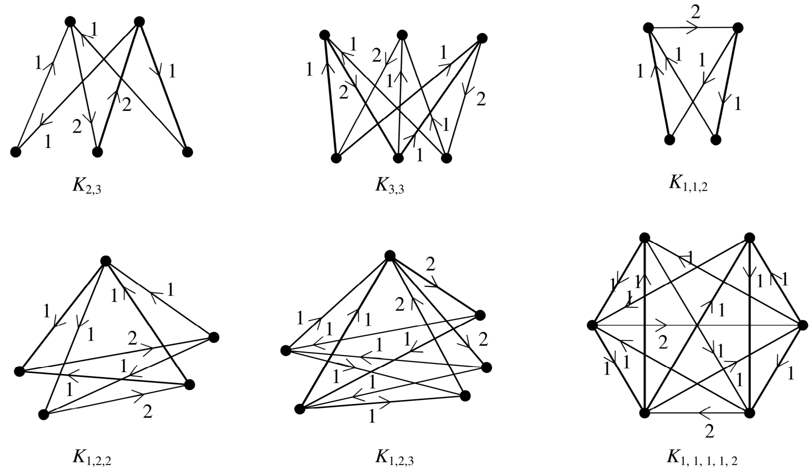

Figure 1: Some small graphs with minimum nowhere-zero 3-flow

□

Lemma 2. *Let $m, n \geq 2$ be two natural numbers. Then we have,* $\Lambda(K_{m,n}) \leq 3$.

Proof. If $m, n \leq 3$, then by Lemma 1 we are done. Without loss of generality, assume that $m \geq 4$. We proceed using induction on $m + n$. We have $K_{m,n} = K_{2,n} \oplus K_{m-2,n}$. By induction hypothesis, $\Lambda(K_{2,n}) \leq 3$ and $\Lambda(K_{m-2,n}) \leq 3$. Therefore, $\Lambda(K_{m,n}) \leq 3$. □

Remark 1. By Theorem A, if $m, n \geq 2$ and at least one of them is odd, then $\Lambda(K_{m,n}) = 3$

Lemma 3. $\Lambda(K_{1,r,n}) \leq 3$*, for* $r, n \in \mathbb{N}$.

Proof. We proceed by induction on $r+n$. If $r, n \leq 3$, then by Lemma 1, we are done. Now, assume $n \geq 4$. Notice that $K_{1,r,n} = K_{1,r,2} \oplus K_{1+r,n-2}$. By induction hypothesis and using Lemma 2 we find that $\Lambda(K_{1,r,2}) \leq 3$ and $\Lambda(K_{1+r,n-2}) \leq 3$. Therefore, $\Lambda(K_{1,r,n}) \leq 3$. $\square$

Now, we are in a position to prove our main result.

Theorem 3. *Let $m_i \in \mathbb{N}$, for $i = 1, \ldots, k$ and $k \geq 3$. Then $\Lambda(K_{m_1,\ldots,m_k}) \leq 3$.*

Proof. We break the proof into the following:

Case 1: Assume $m_i \geq 2$ for all i. For $i \neq j$, $1 \leq i, j \leq k$, the edges with endpoints in sets of size m_i and m_j form a K_{m_i,m_j}. By Lemma 2, $\Lambda(K_{m_i,m_j}) \leq 3$. The proof of this case is now complete.

Case 2: Now, suppose $m_i = 1$ for some i, $1 \leq i \leq k$. Without loss of generality, assume that $m_1 = \cdots = m_r = 1$, $m_{r+1}, \ldots, m_k \geq 2$. Note that in this case $r \geq 1$. Our proof now is divided into the following four subcases.

Case 2.1: Consider the case in which $r \geq 3$, $r \neq 4$. The sets m_1, through m_r can be viewed as the complete graph K_r. By Theorem A, $\Lambda(K_{2n+1}) = 2$ and by Theorem 1, $\Lambda(K_r) \leq 3$. Notice that $K_{m_1,\ldots,m_k} = K_r \oplus K_{r,m_{r+1},m_{r+2},\ldots,m_k}$. By Case 1, we conclude that $\Lambda(K_{m_1,\ldots,m_k}) \leq 3$.

Case 2.2: Consider the case $r = 4$. Clearly, K_4 has no nowhere-zero 3-flow. In the following we prove that $K_{1,1,1,1,m_5}$ has a nowhere-zero 3-flow.
By induction on m_5, we prove that $\Lambda(K_{1,1,1,1,m_5}) \leq 3$. The case $m_5 = 2$ follows from Lemma 1. The case $m_5 = 3$ follows from Theorem A. Since for $m_5 \geq 4$ we have $K_{1,1,1,1,m_5} = K_{1,1,1,1,2} \oplus K_{4,m_5-2}$, by induction hypothesis and Lemma 2, $\Lambda(K_{1,1,1,1,m_5}) \leq 3$.
Obviously, we have $K_{1,1,1,1,m_5,m_6,\ldots,m_k} = K_{1,1,1,1,m_5} \oplus K_{4+m_5,m_6,\ldots,m_k}$.

Following Case 1 and the above result, we find that

$$\Lambda(K_{1,1,1,1,m_5,m_6,\dots,m_k}) \leq 3.$$

Case 2.3: Consider the case $r = 1$. We further break the case into the following:
If $k = 3$, then by Lemma 3, we are done. If $k \geq 4$, then we write $K_{1,m_2,m_3,\dots,m_k} = K_{1,m_2,m_3} \oplus K_{1+m_2+m_3,m_4,m_5,\dots,m_k}$. The above can be handled similarly by Lemmas 2, 3 and Case 1.

Case 2.4: Let $r = 2$. If $k = 3$, then by Lemma 3, we are done. Thus assume that $k \geq 4$. Notice that $K_{1,1,m_3,\dots,m_k} = K_{1,1,m_3} \oplus K_{2+m_3,m_4,\dots,m_k}$. The right hand side of the above can similarly be handled by Lemmas 2, 3 and Case 1. The proof is now complete. $\square$

Remark 2. If $\sum_{\substack{i=1 \\ i \neq j}}^{k} m_i$ is even for every j, $1 \leq j \leq k$, then since the complete k-partite graph $K_{m_1,\dots,m_k}$ is an even graph, we obtain $\Lambda(K_{m_1,\dots,m_k}) = 2$.

Corollary 1. $\Lambda(B_n) \leq 3$*, for* $n \geq 5$*.*

Proof. We have $B_n = C_{n-2} \oplus K_{1,1,n-2}$. By Lemma 3, we have $\Lambda(K_{1,1,n-2}) \leq 3$. $\square$

In [2], the authors proved that if G admits a nowhere-zero 3-flow and if the number vertices of G is even, then $\Lambda(G \vee K_2) = 3$. The following corollary shows that the even condition is redundant and the proof is complete.

Corollary 2. *If* G *admits a nowhere-zero* 3*-flow, then we have* $\Lambda(G \vee K_2) \leq 3$*.*

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