Linear Algebraic Techniques in Combinatorics/Graph Theory

Linear Algebra and Matrix Theory provide one of the most important tools sometimes the only tool—in Combinatorics and Graph Theory. Even though the ideas used in applications of linear algebra to combinatorics may be very simple, the results obtained can be very strong and surprising. A famous instance is the Graham-Pollak theorem which asserts that if the complete graph of order n is partitioned into m complete bipartite subgraphs, then m is at least $n-1$ ($n-1$ arises naturally by recursively deleting a star at a vertex, but there are many other ways to achieve $n-1$). The only known proofs of this theorem use some form of linear algebra. How does linear algebra enter into this and other combinatorial problems?

Almost all combinatorial objects can be described by incidence matrices (e.g. combinatorial designs) or adjacency matrices (e.g. graphs and digraphs). Sometimes the Laplacian and Seidel matrices are also used. Therefore one basic approach is to investigate combinatorial objects by using linear algebraic parameters (rank, determinant, spectrum, etc.) of their corresponding matrices. Two of the great pioneers of this approach were H.J. Ryser and D.R. Fulkerson. Strong characterization and non-existence results can be obtained in this way. The application of linear algebra to combinatorics works in the reverse order as well. Many linear algebraic issues can be refined using combinatorial or graph-theoretic ideas. A classical instance of this is contained in the Perron-Frobenius theory of nonnegative matrices where use of an associated digraph gives more detailed information on the spectrum of the matrix.

Another more recent, but now classical, instance is the use of an associated digraph to refine the classical eigenvalue inclusion region of Gershgorin. Another classical instance of the use of linear algebra to get combinatorial information is the theorem of Bruck and Ryser which rules out the existence of finite projective planes of certain orders. A further instance is the Friendship Thoerem which states that a graph in which every pair of vertices have exactly one common neighbor is a bunch of triangles, glued together in one single vertex. The important class of highly structured graphs known as strongly regular (connected) graphs have a linear algebraic characterization: they are the graphs whose adjacency matrix have exactly three distinct eigenvalues. The Graham-Pollak theorem mentioned above can be further generalized to graphs in general giving a bound on the integer parameter m in terms of the spectrum of the adjacency matrix.

Sometimes the lines between linear algebra and combinatorics are blurred. A simple example is the Cayley-Hamilton theorem that asserts that a matrix satisfies its characteristic polynomial. This theorem can be formulated and proved as a theorem in graph theory. Sign-nonsingularity of a matrix, where nonsingularity depends only on the signs $(+, -, 0)$ of its entries and not on its magnitudes, and the resulting theory and application to linear systems, is a both a linear algebraic issue and a combinatorial issue. The existence of Hadamard matrices of all orders $4m$ can be viewed as both a linear algebraic problem and a combinatorial design problem. Related to this basic combinatorial problem are bounds for the determinant of matrices of 1s and -1s. There are many more examples that can be given.

It seems to be the case and it has been conjectured that almost all graphs are determined by their spectrum (the adjacency spectrum as well as the Laplacian spectrum). This would mean that the spectrum can be used as a kind of fingerprint for a graph. Especially for large networks this is an interesting property which, for example, makes it possible to order these networks (in almost all cases) in a systematic way. In another direction it is shown that the null space of the incidence matrices of directed graphs, undirected graphs, and inclusion matrices of designs have correspondence with flows in graphs, zero-sum flows, and trades in Latin squares, respectively.