Castelnuovo-Mumford regularity of ideal powers

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Local cohomology

Let S be a noetherian ring and $\mathfrak{m}$ an ideal of S.
Let M be finitely generated S-module M.
Let $\Gamma_{\mathfrak{m}}(M) := \{v \in M | \ \mathfrak{m}^t v = 0 \text{ for some } t \gg 0\}$.

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The vanishing of $H^i_{\mathfrak{m}}(M)$ can be used to classify modules.

Example. If S is a local ring with maximal ideal $\mathfrak{m}$, then M is Cohen-Macaulay iff $H^i_{\mathfrak{m}}(M) = 0$ for $i < \dim M$.

Graded case

Let R be a finitely generated graded algebra over a field and $\mathfrak{m}$ the maximal graded ideal of R.
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The graded local cohomology is related to the sheaf cohomology in Algebraic Geometry.

Grothendieck-Serre correspondence:
Let $X = \operatorname{Proj} R$ and $\tilde{M}$ the coherent sheaf associated with M.
Let $H^i(X, \tilde{M}(t))$ denote the sheaf cohomology of $\tilde{M}(t)$, $t \in \mathbb{Z}$.
There are an exact sequence

$$\to H^0_{\mathfrak{m}}(M)_t \to M_t \to H^0(X, \tilde{M}(t)) \to H^1_{\mathfrak{m}}(M)_t \to 0$$

and the isomorphisms $H^i(X, \tilde{M}(t)) \cong H^{i+1}_{\mathfrak{m}}(M)_t$ for $i > 0$.

Geometric regularity

Mumford: $\tilde{M}$ is called *s-regular* if $M_t \to H^0(X, \tilde{M}(t))$ is surjective and $H^i(X, \tilde{M}(t-i)) = 0$ for all $t > s$ and $i \geq 1$.

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For every graded module E we set

$$a(E) := \begin{cases} \sup\{t \mid E_t \neq 0\} \text{ if } E \neq 0, \\ -\infty \text{ if } E = 0, \end{cases}$$

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Geometric meaning: $\tilde{M}$ is s-regular iff $s > \text{g-reg}(M)$.

Algebraic regularity

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But $\mathrm{reg}(M)$ captures better the structure of M.

Let R be a factor ring of a polynomial ring S.
Consider a minimal graded free resolution of M over S:

$$0 \to F_s \to \cdots \to F_1 \to F_0 \to M \to 0$$

Let $b_i(M)$ denote the maximum degree of the generators of F_i.

Eisenbud-Goto 1984: $\mathrm{reg}(M) = \max\{b_i(M) - i |\ i = 0, \ldots, s\}$.

Regularity of ideal powers

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Yes if $\dim S/I \leq 1$.

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Yes if $\dim S/I \le 1$.

Sturmfels 2000: There exist monomial ideals I such that $\operatorname{reg}(I^2) > 2 \operatorname{reg}(I)$.

Linear bounds

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This problem was inspired by a result in algebraic geometry.

Bertram-Ein-Lazarsfeld 1991:

Let $X \subset \mathbb{P}^s$ be a smooth variety and let $\mathcal{I}_X$ be the ideal sheaf of the embedding of X. Let d_X denote the minimum degree d such that X is a scheme-theoretic intersection of hypersurfaces of degree at most d. There is a number e such that

$$H^i(\mathbb{P}^s, \mathcal{I}_X^n(t)) = 0 \text{ for all } n > 0,\ t \geq nd_X + e,\ i \geq 1.$$

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Swanson 1997: Yes, there exists a number D such that $\operatorname{reg}(I^n) \leq nD$ for all $n > 0$.

Asymptotic behaviour

It turns out that $\operatorname{reg}(I^n)$ is asymptotically a linear function.

Cutkosky-Herzog-Trung 1999, Kodiyalam 2000: There exist numbers d, e, n_0 such that $\operatorname{reg}(I^n) = nd + e$ for all $n \geq n_0$.

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The slope d can be described explicitly.

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Geometric meaning: If I is the defining ideal of a projective variety X, then d is the minimum degree such that X is a scheme-theoretic intersection of hypersurfaces of degree at most d.

Bigraded Rees algebra

Basic idea: to consider the Rees algebra $S[It] = \bigoplus_{n \geq 0} I^n t^n \subseteq S[t]$.
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If $S = k[x_1, ..., x_r]$ and $I = (f_1, ..., f_s)$, there is a presentation $S[It] = k[x_1, .., x_r, y_1, .., y_s]/Q$, where $k[x_1, .., x_r, y_1, .., y_s]$ is a bigraded polynomial ring with $\deg x_i = (1, 0)$, $\deg y_j = (\deg f_j, 1)$, and Q is a bihomogeneous ideal.

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The asymptotic linearity of $\mathrm{reg}(I^n)$ follows from the fact that $S[It]$ has a minimal bigraded resolution over $k[x_1, .., x_r, y_1, .., y_s]$, which provides resolutions for all I_n.

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Cutkosky-Ein-Lazarsfeld 2001: $\lim_{n\to\infty} \operatorname{reg}(\tilde{I^n})/n$ exists and equals the Seshadri constant.

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We know $\operatorname{reg}(I^n) = dn + e$ for $n \geq n_0$, where d is well-determined. But what about e and n_0?

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If $\dim S/I = 0$, S/I has finite length, and $\mathrm{reg}(I) = a(S/I) + 1$, where $a(S/I)$ denotes the largest non-vanishing degree of S/I.

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If I is generated by forms of degree d, one can modify the bigrading of $S[It]$ by letting $\deg ft^n = (\deg f - nd, n)$ for all $f \in I^n$. Then $S[It]$ is standard bigraded, i.e. it is generated by forms of degree $(1, 0)$ and $(0, 1)$.

Regularity of standard graded algebras

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Let R be a standard graded algebra (not neccessarily over a field). For every finitely generated graded R-module M one define

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The Rees algebra $S[It]$ is a standard graded algebra over S with $S[It]_n = I^n t^n$ for $n \geq 0$. Hence one can define $\operatorname{reg}(S[It])$.

Estimate for n_0

Let n_0 be the minimal number s. t. $\operatorname{reg}(I^n) = dn + e$ for $n \geq n_0$.

Eisenbud-Ulrich 2012: Assume that $\dim S/I = 0$ and I is generated by forms of the same degree. Then $n_0 \leq \operatorname{reg}(S[It])$.

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Trung (to be published): Assume that I is generated by forms of the same degree with $\dim S/I$ arbitrary. Then

$$n_0 \leq \max\left\{\operatorname{reg}\left(S[It]/(x_1,..,x_i)S[It]\right) \mid i = 0,..,r\right\},$$

where $x_1, ..., x_r$ are generic variables.

Estimate for e

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For every relevant homogeneous prime ideal P of $k[I_d]$, we have the homogeneous localization $S[It]_{(P)}$, which is a standard graded algebra over $k[I_d]_{(P)}$. Hence we can define $\operatorname{reg}(S[It]_{(P)})$.

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Geometric meaning: e is the maximum of the regularity of the fibers of the linear projection $\operatorname{Proj} S[It] \to \mathbb{P}^{s-1}$.

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Berlekamp 2012: The answer is no.