Combinatorial Optimization: a Bridge between Combinatorics and Algebra

Ngo Viet Trung

Institute of Mathematics
Vietnamese Academy of Science and Technology

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Main Theme

Correspondence of Hypergraphs in Combinatorics and Squarefree Monomial Ideals in Algebra

Tool: Combinatorial Optimization

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Correspondence of Hypergraphs in Combinatorics and Squarefree Monomial Ideals in Algebra

Tool: Combinatorial Optimization

This bridge is too big. I am a passenger passing the bridge a few times and could see only a glimpse of its magnificence.

Hypergraphs

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Hypergraphs play an impotant role in Combinatorics and in dealing with real discrete problems.

Example: Social networks.

Matrix presentation

Let $V = \{1, ..., n\}$ and $\Gamma = \{F_1, ..., F_m\}$.

We may identify F_i with the incidence vector $\mathbf{a}_i = (\alpha_{i1}, ..., \alpha_{in})$ (column vector), where

$$\alpha_{ij} = \begin{cases} 1 & \text{if } j \in F_i, \\ 0 & \text{if } j \notin F_i. \end{cases}$$

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Then Γ is uniquely determined by the incidence matrix

$$\mathbf{M} = (\mathbf{a}_1, ..., \mathbf{a}_m).$$

This matrix presentation allows us to use tool of Combinatorial Optimization to study hypergraphs.

Cover

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Let $\mathbf{b}$ be the incidence vector of G and $\mathbf{1}_n := (1, ..., 1) \in \mathbb{N}^n$. Then

$$|G| = \mathbf{1}_n \cdot \mathbf{b}.$$

$$G \cap F_i \neq \emptyset \text{ iff } \mathbf{a}_i \cdot \mathbf{b} \geq 1.$$

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Proposition: $\tau(\Gamma) = \min\{\mathbf{1}_n \cdot \mathbf{b} | \ \mathbf{b} \in \mathbb{N}^n, \mathbf{M}^T \cdot \mathbf{b} \geq \mathbf{1}_m\}$.

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Let $\mathbf{c}$ be the incidence vector of the index set $\{i| \ F_i \in S\}$. Then

$$|S| = \mathbf{1}_m \cdot \mathbf{c},$$

The edges of S are disjoint iff $\sum_{F_i \in S} \mathbf{a}_i \leq \mathbf{1}_n$ iff $\mathbf{M} \cdot \mathbf{c} \leq \mathbf{1}_n$.

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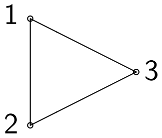
Proposition: $\nu(\Gamma) = \max\{\mathbf{1}_m \cdot \mathbf{c}| \ \mathbf{c} \in \mathbb{N}^m, \mathbf{M} \cdot \mathbf{c} \leq \mathbf{1}_n\}$.

Example

Let $\Gamma = \{\{1,2\},\{1,3\},\{2,3\}\}$:



Γ has three minimal covers $\{1,2\},\{1,3\},\{2,3\}$, hence $\tau(\Gamma) = 2$.

Γ has three maximal matchings of one edge $\{1,2\},\{1,3\},\{2,3\}$, hence $\nu(\Gamma) = 1$.

König property

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Another point of view:

$$\nu(\Gamma) \leq \max\{\mathbf{1}_m \cdot \mathbf{c} | \ \mathbf{c} \in \mathbb{R}_+^m, \mathbf{M} \cdot \mathbf{c} \leq \mathbf{1}_n\}$$
$$= \min\{\mathbf{1}_n \cdot \mathbf{b} | \ \mathbf{b} \in \mathbb{R}_+^n, \mathbf{M}^T \cdot \mathbf{b} \geq \mathbf{1}_m\} \leq \tau(\Gamma).$$

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If $\nu(\Gamma) = \tau(\Gamma)$, one says that Γ has the König property.

König: Bipartite graphs have this property.

Expanding Hypergraphs

To construct new hypergraphs one can expand a vertex v to k vertices as follows:

1. Replacing v by k new vertices $v_1, \ldots, v_k$,
2. Replacing every edge F containing v by k new edges $(F \setminus v) \cup v_1, \ldots, (F \setminus v) \cup v_k$.

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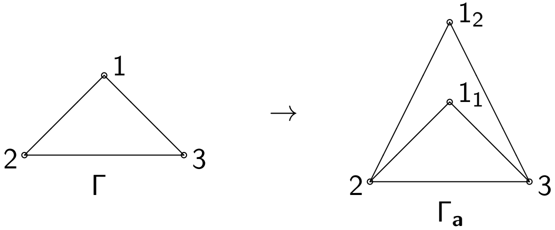
For $\mathbf{a} = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, we define $\Gamma_{\mathbf{a}}$ as the hypergraph obtained from Γ by expanding every vertex i to α_i vertices, $i = 1, ..., n$.

Example

Let $\Gamma = \{\{1,2\},\{1,3\},\{2,3\}\}$ and $\mathbf{a} = (2,1,1)$.

Then Γ_a is obtained by expanding the vertex 1 to two vertices $1_1, 1_2$:



Fractional covering and matching

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Lemma. For all $\mathbf{a} \in \mathbb{N}^m$ we have

$$\nu(\mathbf{a}) = \max\{\mathbf{1}_m \cdot \mathbf{c} |\ \mathbf{c} \in \mathbb{N}^m, \mathbf{M} \cdot \mathbf{c} \leqslant \mathbf{a}\},$$
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The fractional covering and matching numbers is defined by

$$\nu^*(\mathbf{a}) := \max\{\mathbf{1}_m \cdot \mathbf{c} |\ \mathbf{c} \in \mathbb{R}_+^m, \mathbf{M} \cdot \mathbf{c} \leqslant \mathbf{a}\},$$
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Proposition: $\nu(\mathbf{a}) \leq \nu^*(\mathbf{a}) = \tau^*(\mathbf{a}) \leq \tau(\mathbf{a})$.

Squarefree momomial ideals

Let $K[X] = K[x_1, ..., x_n]$ be a polynomial ring over a field K.
For $\mathbf{a} = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, set $x^{\mathbf{a}} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

We call $x^{\mathbf{a}}$ squarefree if $x^{\mathbf{a}}$ is not divided by any square. In this case, $\mathbf{a} \in \{0, 1\}^n$, and we may associate with $\mathbf{a}$ the set $F = \{i | \alpha_i = 1\}$.

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Let $I = (x^{\mathbf{a}_1}, ..., x^{\mathbf{a}_m})$ be a squarefree monomials, i.e. $x^{\mathbf{a}_1}, ..., x^{\mathbf{a}_m}$ are squarefree. Let $F_1, ..., F_m$ be the sets associated with $\mathbf{a}_1, ..., \mathbf{a}_m$. Then I is determined by the hypergraph $\Gamma = \{F_1, ..., F_m\}$.
We call I the edge ideal of Γ.

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This gives a correspondence between squarefree ideals and hypergraphs.

Example

Let $I = (x_1x_2, x_1x_3, x_2x_3) = (x^{\mathbf{a}_1}, x^{\mathbf{a}_2}, x^{\mathbf{a}_3})$,
where $\mathbf{a}_1 = (1, 1, 0)$, $\mathbf{a}_2 = (1, 0, 1)$, $\mathbf{a}_3 = (0, 1, 1)$.

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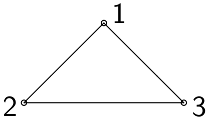
Then $F_1 = \{1,2\}$, $F_2 = \{1,3\}$, $F_3 = \{2,3\}$.
Hence I is the edge ideal of the graph $\Gamma = \{F_1, F_2, F_3\}$:



Symbolic powers

Let I be the edge ideal of a hypergraph Γ.

Let $C_1, \ldots, C_s$ be the minimal covers of Γ.
Then I has the decomposition

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The k-th symbolic power of I is the ideal

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We always have $I^k \subseteq I^{(k)}$.

Mengerian hypergraph

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There are the following membership criteria:

Lemma: $x^{\mathbf{a}} \in I^k$ iff $\nu(\mathbf{a}) \geq k$.

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One calls Γ Mengerian if $\nu(\mathbf{a}) = \tau(\mathbf{a})\ \forall\ \mathbf{a} \in \mathbb{N}^n$.

Herzog-Hibi-Tr-Zheng:
$I^k = I^{(k)}$ for all $k \geq 1$ iff Γ is a Mengerian hypergraph.

Integral closures

Let I be an ideal in a ring R. The integral closure of I is defined as the ideal

$$\overline{I} := \{f \in R | \ \exists \ f^d + g_1 y^{d-1} + \cdots + g_d = 0, g_j \in I^j\}.$$

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For a squarefree monomial ideal I, we have $I^k \subseteq \overline{I^k} \subseteq I^{(k)}$.

Problem: When do we have equality in the above inequalities?

Fulkersonian hypergraph

Recall that $\nu(\mathbf{a}) \leq \nu^*(\mathbf{a}) = \tau^*(\mathbf{a}) \leq \tau(\mathbf{a})$.

Lemma. $x^{\mathbf{a}} \in \overline{I^k}$ iff $\tau^*(\mathbf{a}) \geq k$.

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One calls Γ Fulkersonian if $\tau^*(\mathbf{a}) = \tau(\mathbf{a})\ \forall\ \mathbf{a} \in \mathbb{N}^n$.

Tr: $\overline{I^k} = I^{(k)}$ for all $k \geq 1$ iff Γ is Fulkersonian.

One may expect that
$I^k = \overline{I^k}$ for all $k \geq 1$ iff $\nu(\mathbf{a}) = \tau^*(\mathbf{a})\ \forall\ \mathbf{a} \in \mathbb{N}^n$?

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This is not true. So what is the condition for $I^k = \overline{I^k}$.

Integer round-down property

Let $\lfloor \nu^*(\mathbf{a}) \rfloor$ denote the integer round-down of $\nu^*(\mathbf{a})$. Then

$$\nu(\mathbf{a}) \leq \lfloor \nu^*(\mathbf{a}) \rfloor \leq \nu^*(\mathbf{a}).$$

Lemma: $x^{\mathbf{a}} \in \overline{I^k}$ iff $\tau^*(\mathbf{a}) \geq k$ iff $\lfloor \nu^*(\mathbf{a}) \rfloor \geq k$.

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Lemma: $x^{\mathbf{a}} \in \overline{I^k}$ iff $\tau^*(\mathbf{a}) \geq k$ iff $\lfloor \nu^*(\mathbf{a}) \rfloor \geq k$.

We say that Γ has the integer round-down property if $\nu(\mathbf{a}) = \lfloor \nu^*(\mathbf{a}) \rfloor$ for all $\mathbf{a} \in \mathbb{N}^n$.

Tr: $I^k = \overline{I^k}$ for all $k \geq 1$ iff Γ has the integer round-down property.

Applications

Combinatorics: The above classes of hypergraphs were studied already in the 70' by Berge, Fulkerson, Lovasz, Schrijver, Seymour, Trotter, etc.

Algebra: The corresponding properties of monomial ideals have been studied only since the 90'.

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Algebra: The corresponding properties of monomial ideals have been studied only since the 90'.

Consequences of the relationship:

1. Several new results on monomial ideals can be recovered by earlier results on hypergraphs.
2. New classes of monomial ideals or hypergraphs can be discovered by means of combinatorics or algebra, respectively.