# Integral symbolic and adic topologies

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Webinars on Commutative Algebra

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McAdam studied the following interesting set of prime ideals of $R$ containing $I$:

- **Definition.**

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Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

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(iv) For every radical ideal $J$ of $R$ which contains $I$, the topology $\{I^{\langle m\rangle}\}_{m\geq 1}$ is finer than the topology $\{J^{\langle m\rangle}\}_{m\geq 1}$.

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SKETCH OF PROOF. (i) $\Longrightarrow$ (ii) follows from McAdam's Theorem. To prove the conclusion (ii) $\Longrightarrow$ (i), suppose the contrary is true. Then, there exists $\mathfrak{p} \in \bar{Q}^*(I)$ such that $\mathfrak{p} \notin \text{mAss}_R\, R/I$. Then, by Proposition A, there exists an integer $k \geq 1$ such that $I^{\langle m\rangle} \not\subseteq \mathfrak{p}^{\langle k\rangle}$ for all integers $m \geq 1$. Further, in view of assumption (ii), there exists an integer $l \geq 1$ such that $I^{\langle l\rangle} \subseteq \overline{I^k}$. Now, since $\overline{I^k} \subseteq \overline{\mathfrak{p}^k} \subseteq \mathfrak{p}^{\langle k\rangle}$, it follows that $I^{\langle l\rangle} \subseteq \mathfrak{p}^{\langle k\rangle}$, which provides a contradiction.

In order to show (iii) $\Longrightarrow$ (iv), let $l \geq 1$. Then, in view of McAdam's Theorem, there exists an integer $m \geq 1$ such that $I^{\langle m \rangle} \subseteq \overline{I^l}$. Since $I \subseteq J$, we have $I^{\langle m \rangle} \subseteq \overline{J^l}$, and so as $\overline{J^l} \subseteq J^{\langle l \rangle}$ it follows that $I^{\langle m \rangle} \subseteq J^{\langle l \rangle}$, as required.

In order to show (iii) $\Longrightarrow$ (iv), let $l \geq 1$. Then, in view of McAdam's Theorem, there exists an integer $m \geq 1$ such that $I^{\langle m \rangle} \subseteq \overline{I^l}$. Since $I \subseteq J$, we have $I^{\langle m \rangle} \subseteq \overline{J^l}$, and so as $\overline{J^l} \subseteq J^{\langle l \rangle}$ it follows that $I^{\langle m \rangle} \subseteq J^{\langle l \rangle}$, as required.

Finally, in order to prove the conclusion (iv) $\Longrightarrow$ (i), suppose the contrary is true. Then, there is an element $\mathfrak{p} \in \bar{Q}^*(I)$ such that $\mathfrak{p} \notin \mathrm{mAss}_R R/I$. Hence, in view of Proposition B, there exists an integer $k \geq 1$ such that $I^{\langle m \rangle} \nsubseteq \mathfrak{p}^{\langle k \rangle}$ for all integers $m$. Now, since $I \subseteq \mathfrak{p}$ and $\mathfrak{p}$ is a radical ideal, the assumption (iv) provides a contradiction. $\square$

# Comparison of Topologies in Regular Rings



- **Theorem (M. Nagata 1961).** Let $(R, \mathfrak{m})$ be a regular local ring and let $\mathfrak{p}$ be a prime ideal of $R$. Then for every integer $n \geq 1$, $\mathfrak{p}^{(n)} \subseteq \mathfrak{m}^n$.

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• **Corollary.** Let $R$ be a regular ring and let $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals of $R$. Then for every integer $n \geq 1$, $\mathfrak{p}^{\langle n \rangle} \subseteq \mathfrak{q}^{\langle n \rangle}$.

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• **Lemma.** Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be prime ideals of $R$. Then, for every integer $n \geq 1$,

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# Comparison of Topologies in Regular Rings



• **Theorem (M. Nagata 1961).** Let $(R, \mathfrak{m})$ be a regular local ring and let $\mathfrak{p}$ be a prime ideal of $R$. Then for every integer $n \geq 1$, $\mathfrak{p}^{(n)} \subseteq \mathfrak{m}^n$.

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• **Proposition.** Let $R$ be a regular ring and $I$ an ideal of $R$. Let $J$ be an radical ideal of $R$ containing $I$ and let $n \geq 1$ be an integer. Then $I^{\langle n \rangle} \subseteq J^{\langle n \rangle}$.

# Comparison of Topologies in Regular Rings

- **Corollary.** Let $R$ be a regular ring and let $I \subseteq J$ be ideals of $R$. Then for every integer $n \geq 1$, $I^{\langle n \rangle} \subseteq (\mathrm{Rad}(J))^{\langle n \rangle}$.

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- **Theorem (J. Lipman and A. Sathaye 1981).** Let $R$ be a regular ring. Let $I$ be any ideal of $R$ that is generated by $s$ elements. Then for any integer $n \geq 0$,

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- **Theorem C.** Let $R$ be a regular ring. Then the topology defined by $\{I^{\langle m \rangle}\}_{m \geq 1}$ is equivalent to the $I$-adic topology.

# Comparison of Topologies in Locally Quasi-Unmixed Rings

- **Proposition.** Let

$$\bar{Q}^*(I) = \mathrm{mAss}_R\, R/I \text{ and } \bar{Q}^*(J) = \mathrm{mAss}_R\, R/J.$$

Further, suppose that

$$\mathrm{mAss}_R\, R/IJ = \mathrm{mAss}_R\, R/I \bigcup \mathrm{mAss}_R\, R/J.$$

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- **Theorem D.** Let $R$ be a locally quasi-unmixed ring and let $I$ be a set-theoretic complete intersection ideal. Then topologies $\{I^{\langle n \rangle}\}_{n \geq 1}$ and $\{\overline{I^n}\}_{n \geq 1}$ are equivalent.

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- **Theorem D.** Let $R$ be a locally quasi-unmixed ring and let $I$ be a set-theoretic complete intersection ideal. Then topologies $\{I^{\langle n \rangle}\}_{n \geq 1}$ and $\{\overline{I^n}\}_{n \geq 1}$ are equivalent.

- Recall that $I$ is called a set theoretic complete intersection ideal if it is the radical of an ideal generated by height $I$ elements.

SKETCH OF PROOF. In view of Theorem A, it will suffice to show that $\bar{Q}^*(I) = \mathrm{mAss}_R\, R/I$. For this, let $\mathfrak{p} \in \bar{Q}^*(I)$. Since $\bar{Q}^*(I) \subseteq \bar{A}^*(I)$ by [3, Lemma 2.1], it follows that $\mathfrak{p} \in \bar{A}^*(I)$, where

$$\bar{A}^*(I) := \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \in \operatorname{Ass} R/\overline{I^n} \text{ for all large } n\}.$$

So, as $R$ is locally quasi-unmixed, it follows from McAdam's result [2, Proposition 4.1] that $\operatorname{height} \mathfrak{p} = \ell(IR_{\mathfrak{p}})$. Now, as $I$ is a set-theoretic complete intersection ideal, without loss of generality we may assume that $\mathrm{Rad}(I) = I$ and $I$ is generated by $\operatorname{height} I$ elements. As, at least $\ell(\mathfrak{a})$ elements are needed to generate $\mathfrak{a}$, for any ideal $\mathfrak{a}$ in a commutative Noetherian ring $A$, we have $\ell(IR_{\mathfrak{p}}) \leq \operatorname{height} I$, and so $\operatorname{height} \mathfrak{p} = \operatorname{height} I$. That is $\mathfrak{p} \in \mathrm{mAss}_R\, R/I$, as required.

# Hartshorne and Zariski's results



• **Theorem (R. Hartshorne 1970).** Let $R$ be a complete local ring and $\mathfrak{p}$ be prime ideal of $R$ such that $\dim R/\mathfrak{p} = 1$. Then the $\mathfrak{p}$-adic and $\mathfrak{p}$-symbolic topologies are equivalent.

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(i) The topologies defined by $\{\mathfrak{p}^{\langle n \rangle}\}_{n \geq 1}$ and $\{\overline{\mathfrak{p}^n}\}_{n \geq 1}$ are equivalent.
(ii) For all $z \in \mathrm{mAss}_{\hat{R}} \hat{R}$, there exists $\mathfrak{q} \in \mathrm{Spec}(\hat{R})$ such that $z \subseteq \mathfrak{q}$ and $\mathfrak{q} \cap R = \mathfrak{p}$.

# Hartshorne and Zariski's results



SKETCH OF PROOF. For (i) $\Longrightarrow$ (ii), let $z \in \mathrm{mAss}_{\hat{R}}\, \hat{R}$. In view of Theorem A, $\bar{Q}^*(\mathfrak{p}) = \{\mathfrak{p}\}$, and so $\mathfrak{m} \notin \bar{Q}^*(\mathfrak{p})$. Hence $\mathfrak{m}\hat{R}$ is not minimal over $\mathfrak{p}\hat{R} + z$. Let $\mathfrak{q}$ be a minimal over $\mathfrak{p}\hat{R} + z$. Then $\mathfrak{p} \subseteq \mathfrak{q} \cap R$. Now, since $\dim R/\mathfrak{p} = 1$, it is easily seen that $\mathfrak{q} \cap R = \mathfrak{p}$, and $z \subseteq \mathfrak{q}$.

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In order to prove (ii) $\Longrightarrow$ (i), in view of Theorem A, it will suffice to show that $\bar{Q}^*(\mathfrak{p}) = \{\mathfrak{p}\}$. To this end, let $\mathfrak{q} \in \bar{Q}^*(\mathfrak{p})$. Then $\mathfrak{p} \subseteq \mathfrak{q} \subseteq \mathfrak{m}$. Since $\dim R/\mathfrak{p} = 1$, we have $\mathfrak{q} = \mathfrak{p}$ or $\mathfrak{q} = \mathfrak{m}$. If $\mathfrak{q} = \mathfrak{p}$, the claim is true. Hence, let $\mathfrak{q} = \mathfrak{m}$. Then $\mathfrak{m} \in \bar{Q}^*(\mathfrak{p})$, and so there is $z \in \mathrm{mAss}_{\hat{R}} \hat{R}$ such that $\mathrm{Rad}(\mathfrak{p}\hat{R} + z) = \mathfrak{m}\hat{R}$. Therefore, in view of assumption (ii), there exists $\mathfrak{q} \in \mathrm{Spec}(\hat{R})$ such that $z \subseteq \mathfrak{q}$ and $\mathfrak{q} \cap R = \mathfrak{p}$. Hence $\mathfrak{q} \subseteq \mathfrak{p}\hat{R}$, and so $\mathrm{Rad}(\mathfrak{p}\hat{R}) = \mathfrak{m}\hat{R}$. Whence, $\dim \hat{R}/\mathfrak{p}\hat{R} = \dim R/\mathfrak{p} = 0$, which is a contradiction.

# Hartshorne and Zariski's results



• **Theorem (O. Zariski 1951).** Let $R$ be a Noetherian domain which is analytically irreducible at all prime ideals containing $\mathfrak{p}$, i.e. $\hat{R}_{\mathfrak{q}}$ is an integral domain for all primes $\mathfrak{q}$ with $\mathfrak{q} \supseteq \mathfrak{p}$. Then the $\mathfrak{p}$-adic and $\mathfrak{p}$-symbolic topologies are equivalent.

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• **Theorem F.** Suppose that $\mathrm{mAss}_{\hat{R}_{\mathfrak{p}}} \hat{R}_{\mathfrak{p}}$ consists of a single prime ideal $z$, for all $\mathfrak{p} \in \bar{A}^*(I)$. Then the topologies $\{I^{\langle n \rangle}\}_{n \geq 1}$ and $\{\overline{I^n}\}_{n \geq 1}$ are equivalent.

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Recall: $\bar{A}^*(I) := \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \in \operatorname{Ass} R/\overline{I^n} \text{ for all large } n\}$.

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SKETCH OF PROOF. In view of Theorem A, it will suffice to show that $\bar{Q}^*(I) = \mathrm{mAss}_R\, R/I$. To do this, suppose the contrary is true. Then, as $\mathrm{mAss}_R\, R/I \subseteq \bar{Q}^*(I)$, there exists $\mathfrak{p} \in \bar{Q}^*(I)$ such that $\mathfrak{p} \notin \mathrm{mAss}_R\, R/I$. Since $\mathfrak{p} \in V(I)$, it follows that there exists $\mathfrak{q} \in \mathrm{mAss}_R\, R/I$ such that $\mathfrak{q} \subsetneqq \mathfrak{p}$. Moreover, it is well known that $\bar{Q}^*(I) \subseteq \bar{A}^*(I)$, hence $\mathrm{mAss}_{\hat{R}_\mathfrak{p}}\, \hat{R}_\mathfrak{p} = \{z\}$. Therefore, $\mathrm{Rad}(I\hat{R}_\mathfrak{p} + z) = \mathfrak{p}\hat{R}_\mathfrak{p}$. Now, let $\mathfrak{q}^*$ be a minimal prime over $\mathfrak{q}\hat{R}_\mathfrak{p}$. Then $I\hat{R}_\mathfrak{p} \subseteq \mathfrak{q}\hat{R}_\mathfrak{p} \subseteq \mathfrak{q}^*$. Now, since $z \subseteq \mathfrak{q}^*$ it follows that $\mathfrak{p}\hat{R}_\mathfrak{p} \subseteq \mathfrak{q}^*$, and hence $\mathfrak{p}\hat{R}_\mathfrak{p} \subseteq \mathfrak{q}^* \cap R_\mathfrak{p}$. On the other hand, since $\mathfrak{q}^*$ is a minimal prime over $\mathfrak{q}\hat{R}_\mathfrak{p}$, we can therefore deduce from the Going-Down Theorem that $\mathfrak{q}^* \cap R_\mathfrak{p} = \mathfrak{q}R_\mathfrak{p}$. Hence $\mathfrak{q}R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$, and so $\mathfrak{q} = \mathfrak{p}$, which is a contradiction.

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# Thanks for your attention