CASTELNUOVO-MUMFORD REGULARITY

LE TUAN HOA

ABSTRACT. In this lecture I give some basic properties of the Castelnuovo-Mumford regularity and its applications to bound Hilbert coefficients. The definition of the Castelnuovo-Mumford regularity by using local cohomology is given in the first part, where some basic properties are recalled, such as Castelnuovo's lemma and the relation of the Castelnuovo-Mumford regularities of modules in an exact sequence. The main focus of the second part is to present Mumford's characterization of the Castelnuovo-Mumford regularity for sheaves on projective spaces and its extension by Eisenbud and Goto to modules in terms of the minimal free resolution. The last two parts are devoted to bounding the Castelnuovo-Mumford regularity: in terms of defining degrees of ideals and in terms of Hilbert coefficients. As an application one can get a result on the dependence of Hilbert coefficients, which extends earlier results by Srinivas and Trivedi. In the last section some open problems of current interest are listed.

Table of Contents

1. Definition and some basic properties

In this section, we give the definition of the Castelnuovo-Mumford regularity by using local cohomology. Then we will recall basic properties such as Castelnuovo's lemma and the relation of the Castelnuovo-Mumford regularities of modules in an exact sequence. We also give a N. V. Trung's characterization of the Castelnuovo-Mumford regularity in terms of a-invariants of certain modules associated to a filter regular sequences. We consider a more general situation, when the standard graded algebra is defined over a local ring.

Let $S = \oplus_{i\geq 0} S_i$ be a standard graded algebra over a local ring $(S_0, \mathfrak{n})$ with infinite quotient field $S_0/\mathfrak{n}$, that is S is generated by S_1 over S_0. One can consider S as a quotient ring of a standard graded polynomial ring $R = R_0[X_1, ..., X_n]$ with variables $X_1, ..., X_n$ and $R_0 = S_0$, $\deg(X_1) = \cdots = \deg(X_n) = 1$. In this Lecture Notes, we only consider standard grading.

Let M be a finitely graded S-module. For every integer $i \geq 0$ we set

$$a_i(M) := \max\{j|\ H^i_{S_+}(M)_j \neq 0\},$$

with $a_i(M) = -\infty$ if $H^i_{S_+}(M) = 0$, where $H^i_{S_+}(M)$ denotes the ith local cohomology module of M with respect to the ideal $S_+ = \oplus_{j>0} S_j$. Note that $H^i_{S_+}(M) = 0$ for $i > \dim M/\mathfrak{n}M$.

Definition 1.1. (See [16].) The *Castelnuovo-Mumford regularity* of M is defined by

$$\operatorname{reg}(M) = \max\{a_i(M) + i|\ i \geq 0\}.$$

Given a non-negative integer s, the *Castelnuovo-Mumford regularity of M at and above level s* is defined by

$$\operatorname{reg}^s(M) = \max\{a_i(M) + i|\ i \geq s\},$$

and the *Castelnuovo-Mumford regularity of M at and below level s* is defined by

$$\operatorname{reg}_s(M) = \max\{a_i(M) + i|\ i \leq s\}.$$

We also set

$$a^*_s(M) = \max\{a_i(M)|\ i \leq s\}.$$

Thus, we may consider $a_i(M)$, $a^*_s(M)$, $\operatorname{reg}^s(M), \operatorname{reg}_s(M)$ as partial Castelnuovo-Mumford regularities.

Example: $\operatorname{reg}(R) = 0$, and if $u \in R_a$ is a regular element then $\operatorname{reg}(R/uR) = a - 1$.

If S_0 is an Artinian ring and M is a Cohen-Macaulay module of dimension d, then $\operatorname{reg}(M) = a_d(M) + d$.

Remark 1.2. i) By base change property of local cohomology modules, in the above definition, M can be considered as an R-module as well. We have $\operatorname{reg}(M) = -\infty$ iff $M = 0$.

ii) Let $d = \dim M/\mathfrak{n}M$ and $a = \operatorname{grade}_M(S_+)$. Then

$$d = \min\{t|\ \exists x_1, ..., x_t \text{ s.t. } [(x_1, ..., x_t)M]_i = M_i \text{ for all } i \gg 0\}.$$

Grothendieck's vanishing and non-vanishing theorems can be stated in this case as follows: $H^i_{S_+}(M) = 0$ for $i > d$ and $i < a$, and $H^i_{S_+}(M) \neq 0$ for $i = a, d$[1]. Therefore

$$\operatorname{reg}(M) = \operatorname{reg}^0(M) = \cdots = \operatorname{reg}^a(M) = \operatorname{reg}_d(M) = \cdots = \operatorname{reg}_n(M).$$

iii) $\operatorname{reg}^d(M) \leq \operatorname{reg}^{d-1}(M) \leq \cdots \leq \operatorname{reg}^0(M)$ and $\operatorname{reg}_0(M) \leq \operatorname{reg}_1(M) \leq \cdots \leq \operatorname{reg}_d(M)$.

iv) Let J be a nonzero homogeneous ideal of R. Since $H^i_{R_+}(R) = 0$ for $i \neq n$ and $H^i_{R_+}(J) = 0$ for $i > n$, from the exact sequence $0 \longrightarrow J \longrightarrow R \longrightarrow R/J$ we can deduce that $\operatorname{reg}(J) = \operatorname{reg}(R/J) + 1$.

[1]For the vanishing statement one can use induction on d. For non-vanishing part one can reduce to the classical case by using the exact sequence $0 \to \mathfrak{n}M \to M \to M/\mathfrak{n}M \to 0$.

v) Denote by $M(a)$ the graded S-module obtained from M by shifting its degree by a, that is $[M(a)]_i := M_{i+a}$. Then $\operatorname{reg}(M(a)) = \operatorname{reg}(M) - a$.

We call x a S_+*-filter-regular element* w.r.t. M (or M-filter-regular element for short), if $x \notin \mathfrak{p}$ for all associated primes of M such that $S_+ \not\subseteq \mathfrak{p}$. An element $x \in S_1$ can be written as a combination $x = \alpha_1\overline{X_1} + \cdots + \alpha_n\overline{X_n}$, where $\alpha_i \in S_0$. We say that x is generic if there is an open nonempty subset $\mathcal{U} \subset (S/\mathfrak{n})^n$ (w.r.t. Zariski's topology) such that $(\overline{\alpha_1}, ..., \overline{\alpha_n}) \in \mathcal{U}$. So, when we say that "$x \in S$ is generic" we always require that x is M-filter-regular, see the proof of [6, Lemma 16.1.3].

Lemma 1.3. *Let $x \in S_1$ be a generic element and $\dim M/\mathfrak{n}M > 0$. Then for all $s \geq 0$ we have*
(i) $a_{s+1}(M) < a_s(M/xM) \leq \max\{a_s(M),\ a_{s+1}(M) + 1\}$.
(ii) $\operatorname{reg}^{s+1}(M) \leq \operatorname{reg}^s(M/xM) \leq \operatorname{reg}^s(M)$.
(iii) (Cf. [15, Proposition 20.20].) $\operatorname{reg}^s(M) = \max\{a_s(M) + s,\ \operatorname{reg}^s(M/xM)\}$.

Proof. It suffices to prove (i). The exact sequence

$$0 \to [M/(0 :_M x)](-1) \xrightarrow{\cdot x} M \to M/xM \to 0$$

induces the following exact sequences for all $j \geq 0$:

$$H^j_{S_+}(M)_i \to H^j_{S_+}(M/xM)_i \to H^{j+1}_{S_+}(M)_{i-1} \to H^{j+1}_{S_+}(M)_i \to H^{j+1}_{S_+}(M/xM)_i \to \cdots . \tag{1}$$

From this it immediately implies that $a_s(M/xM) \leq \max\{a_s(M),\ a_{s+1}(M) + 1\}$.

For the inequality $a_{s+1}(M) < a_s(M/xM)$, it suffices to note that for all $i > a_s(M/xM)$ we have

$$H^{s+1}_{S_+}(M)_{i-1} \hookrightarrow H^{s+1}_{S_+}(M)_i \hookrightarrow H^{s+1}_{S_+}(M)_{i+1} \hookrightarrow \cdots \hookrightarrow 0.$$

□

Lemma 1.4. (Castelnuovo). *Assume that $H^i_{S_+}(M)_{p+1-i} = 0$ for all $i \geq s$ and for an integer p. The following hold:*
(i) If $s > 0$ then $\operatorname{reg}^s(M) \leq p$.
(ii) ([36, Lemma 2.1]). *If $s = 0$ and $p \geq \Delta(M)$ - the maximal degree of a minimal homogeneous generator of M, then also $\operatorname{reg}(M) \leq p$.*

Proof. Induction on $d = \dim M/\mathfrak{n}M$.

$d = 0$: There is nothing to prove in (i). For (ii), note that $H^0_{S_+}(M) = M$ and $M_{i+1} = S_1M_i$ for all $i \geq \Delta(M)$. Hence $H^0_{S_+}(M)_i = 0$ for all $i \geq p+1$.

If $d > 0$, then from (1) it follows that $H^i_{S_+}(M/xM)_{p+1-i} = 0$ for all $i \geq s$. Note that $\Delta(M/xM) \leq \Delta(M)$. By induction, $\operatorname{reg}^s(M/xM) \leq p$. By Lemma 1.3, $\operatorname{reg}^{s+1}(M) \leq p$.

Moreover we also have (even for $s = 0$):

$$0 = H^s_{S_+}(M)_{p+1-s} \longrightarrow H^s_{S_+}M)_{p+2-s} \longrightarrow H^s_{S_+}(M/xM)_{p+1-s} = 0.$$

Hence $H^s_{S_+}(M)_{p+2-s} = 0$. Repeating this arguments we can deduce that $H^s_{S_+}(M)_{i-s} = 0$ for all $i \geq s+1$. □

Remark. - Without the assumption $p \geq \Delta(M)$, (ii) does not hold. An easy example: $M' := M \oplus S_0(-q)$ for any $q > p+1$ satisfies the same condition as M, but $a_0(M') \geq q$.
- If $H^i_{S_+}(M)_{p+1-i} = 0$ for all $i \geq 0$, then M is said to be *weakly p-regular*. If $H^i_{S_+}(M)_{q-i} = 0$ for all $i \geq 0$ and $q \geq p+1$, then M is said to be *p-regular*. The above lemma gives a relationship between p-regularity and weakly p-regularity.

Corollary 1.5. *Let $N = H^0_{S_+}(M)$ (Thus N is the maximal S_+-torsion submodule of M). The following are equivalent:*
(i) M is p-regular.
(ii) M is weakly p-regular and $N_i = 0$ for all $i > p$.

Let
$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$
be a short exact sequence of S-modules. Then we have long exact sequences for all $j \geq 0$:
$$H^j_{S_+}(L) \to H^j_{S_+}(M) \to H^j_{S_+}(N) \to H^{j+1}_{S_+}(L) \to \cdots. \qquad (2)$$
From (2) one can easily show (see [15, Corollary 20.19] and [24, Lemma 3.1]):

Lemma 1.6. *If $0 \to L \to M \to N \to 0$ is a short exact sequence of graded finitely generated S-modules, then*
(i) $\operatorname{reg}(L) \leq \max\{\operatorname{reg}(M),\ \operatorname{reg}(N)+1\}$. *The equality holds if* $\operatorname{reg}(M) \neq \operatorname{reg}(N)$.
(ii) $\operatorname{reg}(M) \leq \max\{\operatorname{reg}(L),\ \operatorname{reg}(N)\}$. *The equality holds if* $\operatorname{reg}(N) \neq \operatorname{reg}(L) - 1$ *or if $L_n = 0$ for $n \gg 0$.*
(iii) $\operatorname{reg}(N) \leq \max\{\operatorname{reg}(L) - 1,\ \operatorname{reg}(M)\}$. *The equality holds if* $\operatorname{reg}(M) \neq \operatorname{reg}(L)$.

The following result stimulates the study of the Castelnuovo-Mumford regularity (see [16] and [6, Theorem 15.3.1]). See also Theorem 2.2 for the case S_0 being a field.

Theorem 1.7. *Let M be a finitely generated graded S-module. Then $\Delta(M) \leq \operatorname{reg}(M)$.*

Proof. Induction on $d = \dim M/\mathfrak{n}M$. The case $d = 0$: $\operatorname{reg}(M) = a(M) \geq \Delta(M)$.
For $d > 0$, let $x \in S_1$ be a generic element. Let $N \subseteq M$ be the submodule generated by all elements of degrees at most $\operatorname{reg}(M)$. By induction hypothesis and Lemma 1.3, $\Delta(M/xM) \leq \operatorname{reg}(M/xM) \leq \operatorname{reg}(M)$. This means $M/xM = (N + xM)/xM$ or $M = N + xM$. If were $M \neq N$, then taking a homogenous element $u \in M \setminus N$ of the smallest possible degree will give a contradiction. □

One can use filter-regular sequences to compute the Castelnuovo-Mumford regularity.

Definition 1.8. Let $\mathbf{x} = \{x_1, ..., x_s\}$ be a sequence of homogeneous elements of S. We call $\mathbf{x}$ a *S_+-filter-regular sequence* w.r.t. M (or M-filter-regular sequence for short), if $x_i \notin \mathfrak{p}$ for all associated primes of $M/(x_1, ..., x_{i-1})M$ such that $S_+ \not\subseteq \mathfrak{p}$, $i = 1, ..., s$.

Let

$$a(\mathbf{x}) := \max\{a((x_1,...,x_{i-1})M : x_i/(x_1,...,x_{i-1})M)|\ i = 1,...,s\}.$$

Then $\mathbf{x}$ is a filter-regular sequence if and only if $a(\mathbf{x}) < \infty$. Note that if $x_1,...,x_s,\ s \le \dim M$, are generic, then $\mathbf{x}$ is a filter-regular sequence.

Theorem 1.9. *Let $\mathbf{x} = \{x_1,...,x_s\} \subset S_1$ be a filter-regular sequence of M. Then we have*

(i) (See [46, Proposition 2.2] and [47, Proposition 2.2].)

$$\begin{aligned} \operatorname{reg}_{s-1}(M) &= a(\mathbf{x}), \\ \operatorname{reg}_s(M) &= \max\{a((x_1,...,x_i)M : S_+/(x_1,...,x_i)M)|\ i = 0,...,s\}. \end{aligned}$$

(ii) (See [48, Corollary 2.6].)

$$a^*_{s-1}(M) = \min\{p|\ [(x_1,...,x_{i-1})M : x_i]_t = [(x_1,...,x_{i-1})M]_t\ \forall t \ge p+i,\ i = 1,...,s\}.$$

Proof. It is easy to show for all $i > 0$ that

$$\begin{aligned} a((x_1,...,x_{i-1})M : x_i/(x_1,...,x_{i-1})M) &= a((x_1,...,x_{i-1})M : S_+/(x_1,...,x_{i-1})M) \\ &= a_0(M/(x_1,...,x_{i-1})M). \end{aligned}$$

By Lemma 1.3(i)

$$a_{i-1}(M) + i - 1 \le a_0(M/(x_1,...,x_{i-1})M) \le \max\{a_j(M) + j|\ j = 0,...,i-1\}.$$

Hence

$$\begin{aligned} \operatorname{reg}_{s-1}(M) &= \max\{a((x_1,...,x_{i-1})M : x_i/(x_1,...,x_{i-1})M)|\ i = 1,...,s\}, \\ \operatorname{reg}_s(M) &= \max\{a((x_1,...,x_i)M : S_+/(x_1,...,x_i)M)|\ i = 0,...,s\}. \end{aligned}$$

Thus (i) is shown. The proof of (ii) is similar. □

Exercise 1.10. 1. Show that all local cohomology modules $H^i_{S_+}(M)$ have a $\mathbb{Z}$-graded structure.

2. Let $M \neq 0$. Show that $\operatorname{reg}(M)$ is always finite.
3. Let J be a nonzero homogeneous ideal of R. Show that $\operatorname{reg}(J) = \operatorname{reg}(R/J)+1$. Give an example to show that this is not true if R is replaced by S.
4. i) Let $N \subset M$. Show that $N_i = 0$ for all $i \gg 0$ if and only if there is m such that $S^m_+ N = 0$.

   ii) Let $x \in S$ be a homogenous element. Show that x is a filter-regular element of M if and only if $[0 :_M x]_i = 0$ for all $n \gg 0$.
5. Let $x \in S_+$ be a homogeneous filter regular element of M. Show that $a(0 :_M x) = a(0 :_M S_+)$.
6. Let $f \in S_a$ be a homogeneous regular element of degree of M. Show that $\operatorname{reg}(M/fM) = \operatorname{reg}(M) + a - 1$.
7. Let $I \subset k[X-1,...,X_n]$ be an ideal generated by square-free monomials (such an ideal is called a *square-free monomial* or *Stanley-Reisner* ideal) and $d = \dim(R/I)$. Use Hochster's formula (see, e.g. [7]) for computing local cohomology modules of R/I to show that $\operatorname{reg}(R/I) \le d$.

2. Characterization of the Castelnuovo-Mumford regularity by syzygies

In this section we consider graded modules over a standard graded ring over a field, that is $R = k[X_1, ..., X_n]$, where k is an infinite field and S is a quotient ring of R. The main focus of the section is to present Mumford's characterization of the Castelnuovo-Mumford regularity for sheaves on projective spaces and its extension by Eisenbud and Goto to modules in terms of the minimal free resolution.

Let M be an S-module. The we often consider it as an R-module. Let $\operatorname{indeg}(M) = \inf\{t|\ M_t \neq 0\}$ provided $M \neq 0$ and $\operatorname{indeg}(0) = +\infty$.

First, by the local duality theorem $\operatorname{Hom}(H^i_{R_+}(M), k) \cong \operatorname{Ext}^{n-i}_R(M, R)(-n)$. Hence

Proposition 2.1. *For all $i \geq 0$ we have*

$$a_i(M) = \max\{p|\ [\operatorname{Ext}^{n-i}_R(M, R)]_{-p-n} \neq 0\}.$$

In particular,

$$\operatorname{reg}_s(M) = -\min\{\operatorname{indeg}(\operatorname{Ext}^i_R(M, R)) + i|\ i \geq n - s\}.$$

Let

$$0 \to F_p \to \cdots \to F_1 \to F_0 \to M \to 0$$

be a minimal free resolution of M over R. Let b_i be the maximum degree of the generators of F_i, i.e. $F_i = \oplus_j R(-b_{ij})$ with $b_i = \max_j\{b_{ij}\}$. Then

Theorem 2.2. *We have*

(i) (See [16], [15, Proposition 20.16], [41, Theorem 3.9], [49, Theorem 3.1].)

$$\operatorname{reg}_s(M) = \max\{b_i - i|\ i \geq n - s\} = \max\{a(\operatorname{Tor}^R_i(k, M)) - i||n - s \leq i \leq n\}.$$

(ii) (See [49, Theorem 3.1].) $a^*_s(M) = \max\{b_i|\ i \geq n - s\} - n$.

Proof. Let $F^*_i(M) = \operatorname{Hom}_R(F_i, R)$. Then $[F^*_i]_p = 0$ for all $p < -b_i$. Since $\operatorname{Ext}^i_R(M, R)$ is a subquotient of F^*_i, by Proposition 2.1,

$$\begin{aligned}\operatorname{reg}_s(M) &= \max\{p|\ [\operatorname{Ext}^i_R(M, R)]_{-p-i} \neq 0 \text{ for some } i \geq n - s\}\\ &\leq \max\{b_i - i|\ i \geq n - s\} =: b.\end{aligned}$$

For the converse, let $j \geq n - s$ be the largest number such that $b_j - j = b$. Then F^*_j has $R(b + j)$ as a summand. However F^*_{j+1} has no summand of the form $R(p)$ with $p > b + j$. We have

$$F^*_{j-1} = \oplus_{t\leq b+j-1} R(t) \to F^*_j = R(b + j) \oplus (\oplus_{u\leq b+j} R(u)) \to F^*_{j+1} = \oplus_{v\leq b+j} R(v).$$

If $\{e_i\}$ and $\{e'_{i'}\}$ are bases of F^*_j and F^*_{j+1}, respectively, and A is the matrix of the map $\varphi_j :\ F^*_j \to F^*_{j+1}$, then $\varphi_j(e_1) = \sum a_{i'1} e'_{i'}$, where e_1 is the generator of $R(b + j)$ which has degree $-(b+j)$. By the minimality of MFR, $\deg(a_{i'1}) > 0$ if $a_{i'1} \neq 0$. Since $\deg(e'_{i'}) \geq -(b + j)$ it follows that $\varphi_j(e_1) = 0$. Comparing degrees we also conclude that e_1 cannot be the image of an element of F^*_{j-1}. Hence e_1 gives a nonzero element in $\operatorname{Ext}^j_R(M, R)$ which has degree $-b - j$. Hence by Proposition 2.1, $\operatorname{reg}_s(M) \geq b$.

The proof of (ii) is similar. □

Example. Using Koszul complex, by the above theorem, we see that $\operatorname{reg}(f_1, ..., f_s) = \deg(f_1) + \cdots + \deg(f_s) - s + 1$, provided $f_1, ..., f_s$ is a homogeneous regular sequence of S.

Definition 2.3. For a monomial ideal $\mathbf{X}^A$ in R we denote by $m(\mathbf{X}^A)$ the maximum index j such that X_j appears in $\mathbf{X}^A$. A monomial ideal I is said to be *stable*[2], *if* for any monomial $\mathbf{X}^A \in I$ and index $j < m(\mathbf{X}^A)$, the monomial $X_j\mathbf{X}^A/X_m$ again belongs to I.

Now we recall the construction of a MFR of a stable monomial ideal I given in [17]. Denote by $G(I)$ the minimal monomial generating set of I. Define a symbol $e(i_1, ..., i_q; u)$ to be *admissible* if the following three conditions are satisfied:
(i) $u \in G(I)$,
(ii) $1 \leq i_1 < \cdots < i_q \leq n$,
(iii) $i_q \leq m(u)$.
In this definition, q may be 0. If $q = 0$, we consider that (ii) and (iii) are satisfied as void conditions. Now, let $L_q = L_q(I)$ be the free R-module on the set of all admissible symbols $e(i_1, , ...i_q; u)$ for fixed $q \geq 0$. The grading on L_q is defined by

$$\deg(e(i_1, , ...i_q; u)) = \deg(u) + q.$$

We define the map of R-modules $\alpha : L_0 \to I$ by $\alpha(e(u)) = u$.

Theorem 2.4. (See [17, Theorem 2.1]) *Let I be a stable ideal and m the maximal index such that X_m appears in a generator of $G(I)$. There are differential maps $d_q : L_q \to L_{q-1}$ such that*

$$0 \to L_m \to L_{m-1} \to \cdots \to L_1 \to L_0 \to I,$$

is a MFR of I.

As an immediate consequence of this theorem and Theorem 2.2, we get

Corollary 2.5. *Let I be a stable ideal. Then* $\operatorname{reg}(I)$ *is equal to the maximal degree of generators from $G(I)$.*

We end this section by giving a simpler form of Theorem 1.9 in the case $S_0 = k$ is a field,

Theorem 2.6. (Criterion for n-regularity, see [2, Theorem 1.10].) *Assume that I is a homogeneous ideal of $R = k[X_1, ..., X_n]$. Let $\Delta(I) \leq p$. Then* $\operatorname{reg}(I) \leq p$ *if and only if there exists a filter-regular sequence $x_1, ..., x_s \in R_1$ for some s such that*

$$[(I, x_1, ..., x_{i-1}) : x_i)]_p = [(I, x_1, ..., x_{i-1})]_p \quad \text{for} \quad i = 1, ..., s,$$

and

$$[(I, x_1, ..., x_s)]_p = R_p.$$

[2]Some times in the literature this notion is called "strongly stable", while a stable ideal means an ideal stable under the action of the Borel subgroup of upper-triangular matrices.

Proof. By Theorem 1.9 it suffices to show the sufficiency. We do induction on s. If $s = 0$, then $I_p = R_p$ which implies $\text{reg}(I) = \text{reg}(R/I) + 1 = a(R/I) + 1 = p$.

If $s > 0$, (I, x_1) is p-regular by induction, and $(I : x_1)_p = I_p$ by hypothesis. Note that $x_2, ..., x_s$ is a filter-regular sequence w.r.t. (I, x_1).

We first show that $(I : x_1)$ is generated in degrees $< p$.

Indeed, choose a minimal set of generators for I of the form

$$f_1, ..., f_r, x_1 f_{r+1}, ..., x_1 f_m,$$

where $f_1, ..., f_r$ and x_1 are minimal set of generators for (I, x_1). If $f \in (I : x_1)$, then

$$x_1 f = g_1 f_1 + \cdots + g_r f_r + x_1(g_{r+1} f_{r+1} + \cdots + g_m f_m),$$

for some $g_1, ..., g_m \in S$. Thus

$$(f - g_{r+1} f_{r+1} + \cdots + g_m f_m) x_1 - g_1 f_1 - \cdots - g_r f_r = 0$$

is a syzygy of (I, x_1). Conversely, any syzygy of (I, x_1) yields in this way an element of $(I : x_1)$. Because (I, x_1) is p-regular, each syzygy of (I, x_1) can be expressed in terms of syzygies of (I, x_1) of degree $\leq p + l$. By expressing the above syzygy in this way,

$$f - g_{r+1} f_{r+1} + \cdots + g_m f_m$$

can be expressed in terms of elements of $(I : x_1)$ of degree $\leq m$. Since $f_{r+1}, ..., f_m$ also belong to $(I : x_1)$, and have degrees $\leq p$, $(I : x_1)$ can be generated by elements of degree $\leq p$. The claim is proved.

Now we come back to the proof of the theorem. Since $(I : x_1)_p = I_p$ and both I and $(I : x_1)$ are generated by elements of degrees $\leq p$, we have $(I : x_1)_q = I_q$ for all $q \geq p$. This implies $a_0(R/I) \leq p - 1$. By Lemma 1.3(iii), $\text{reg}(R/I) \leq p - 1$. □

Thus, in order to estimate $\text{reg}(I)$, using Theorem 2.6 we have to check $s+1$ inclusions $[(I, x_1, ..., x_{i-1}) : x_i)]_p \subseteq [(I, x_1, ..., x_{i-1})]_p$, $i = 1, ..., s$, and $[(I, x_1, ..., x_s)]_p \supseteq S_p$ for a fixed component $p \geq \Delta(I)$, while by Theorem 1.9(i) we have to check these inclusions for all components $q \geq p$. This explains why Theorem 2.6 is very important in many applications.

Exercise 2.7.

1. Using Macaulay or CoCoa, compute the Castelnuovo-Mumford regularity of 5 homogeneous ideals!
2. Let

$$F : 0 \to F_p \to \cdots \to F_1 \to F_0 \to M \to 0$$

be a minimal free resolution of M over $S = k[x_1, ..., x_n]$, where $F_i = \oplus_j R(-b_{ij})$ with $a_i = \min_j\{b_{ij}\}$ and $b_i = \max_j\{b_{ij}\}$. Show that $a_p > a_{p-1} > \cdots > a_0$. If the dual

$$F^* : 0 \to F_0^* \to F_1^* \cdots \to F_p^*$$

is also exact, then show that $b_p > b_{p-1} > \cdots > b_0$. Show that this condition is satisfied when $\text{depth}(M) = \dim(M)$.

3. Let I be a monomial ideal, and F be the least common multiple of its generating monomials. Use the Taylor's resolution (see, e.g., [15, Exercise 17.11]) to prove that $\mathrm{reg}(I) \le \deg(F)$.
4. We say that a module M generated in some degree a has a *linear free resolution* if it has a free resolution of the form

$$0 \to \oplus S(-a-p)^{\beta_p} \to \cdots \to \oplus S(-a-1)^{\beta_1} \to \oplus S(-a)^{\beta_0} \to M \to 0.$$

i) Suppose M is a module of finite length. Show that M has a linear free resolution iff $M \cong k(a)^{\beta}$.

ii) Suppose that $I \subset S$ is a homogeneous ideal such that S/I has finite length. Show that S/I has a linear free resolution iff I is a power of the maximal ideal.

5. Let $I \subset R = k[X_1, ..., X_n]$ be a stable monomial ideal of dimension d. Show that $X_n, ..., X_{n-d+1}$ is a R/I-filter-regular sequence.
6. Use Theorem 2.6 to give another proof of Corollary 2.5.

3. Some bounds on Castelnuovo-Mumford regularity

In this section, $R = k[X_1, ..., X_n]$ is a polynomial ring over an infinite field k and $I \subset R$ is a homogeneous ideal of dimension d. Let $c = n - d$. Note that c is the true codimension of I if I does not contain a linear form. Let $\mathfrak{m} = (X_1, ..., X_n)$ denote the maximal homogeneous ideal of R and set $S = R/I$. Let $H_S(t) = \dim_k S_t$ and $P_S(t)$ denote the Hilbert function and the Hilbert polynomial of S, respectively. The leading coefficient of $P_S(t)$, multiplied by $(d-1)!$, is called the degree of S and denoted by $\deg S$. We also denote $\deg S$ by $e(S)$, or just by e.

Bounding the Castelnuovo-Mumford regularity is the most important problem in the study of this invariant. The most famous problem is

Eisenbud-Goto Conjecture (see [16]). Assume that k is an algebraically closed field and $\mathfrak{p}$ is a prime ideal of R containing no linear form. Then

$$\mathrm{reg}(\mathfrak{p}) \le \deg(R/\mathfrak{p}) - c + 1.$$

The most important result is a theorem of Gruson, Lazarsfeld and Peskine [19] which proved the conjecture for projective curves. There are some other results supporting this conjecture, see [30, 38, 42, 23, 22, 37, 1, 29]. However, the conjecture was recently disproved by J. McCullough and I. Peeva [32]. They constructed large classes of prime ideals with huge regularities.

Counterexamples of McCullough and I. Peeva [32, Counterexamples 1.8]. The counterexamples in (1) and (2) hold over any field k.

(1) For $r \ge 1$, Koh constructed in [28] an ideal I_r generated by $22r-3$ quadrics and one linear form in a polynomial ring with $n = 22r - 1$ variables, and such that $\Delta(Syz_1(I_r)) \ge 2^{2^{r-1}}$. His ideals are based on the Mayr-Meyer construction in

[33]. By [32, Theorem 1.6], I_r leads to a homogeneous prime ideal P_r (in a standard graded polynomial ring R_r) whose multiplicity and maxdeg are:

$$\begin{array}{ll} \deg(R_r/P_r) & \leq 4 \cdot 3^{22r-3} < 4^{22r-2} < 2^{50r}, \\ \operatorname{reg}(P_r) & \geq \Delta(P_r) \geq 2^{2^r-1} + 1 > 2^{2^r-1}. \end{array}$$

Therefore, the Eisenbud-Goto Conjecture predicts

$$2^{2^r-1} + 1 \leq 4 \cdot 3^{22r-3},$$

which fails for $r \geq 9$. Moreover,

$$\operatorname{reg}(P_r) > 2^{2^r-1} > 2^{50r \cdot 2^{r/2}} > (\deg(P_r))^{2^{n/44}} > (\deg(P_r))^{2^{d/44}},$$

for all $r \gg 0$.

(2) Alternatively, we can use the Bayer-Stillman example in [3, Theorem 2.6] instead of Kohs example. For $r \geq 1$, they constructed a homogeneous ideal I_r generated by $7r + 5$ forms of degree at most 5 in a polynomial ring P_r with $n = 10r + 11$ variables and such that $\Delta(Syz_1(I_r)) \geq 3^{2^r-1}$. The example is based on the Mayr-Meyer construction in [33]. By [32, Theorem 1.6], I_r leads to a homogeneous prime ideal P_r whose multiplicity is

$$\deg(R_r/P_r) \leq 2 \cdot 6^{7r+5},$$

and

$$\operatorname{reg}(P_r) \geq \Delta(P_r) > 3^{2^{r-1}} + 1.$$

Therefore, the Eisenbud-Goto conjecture fails for $r \geq 8$.

(3) There are two examples of 3-dimensional projective varieties in $\mathbb{P}^5$ for which the Eisenbud-Goto conjecture fails.

The above examples say that in the worst case, the Castelnuovo-Mumford regularity of a prime ideal must be a polynomial of the multiplicity of degree which is exponential in the dimension. In this section we will present an upper bound of this type given in [21]. Namely,

Theorem 3.1. *Let k be an arbitrary infinite field and $\mathfrak{p}$ a prime ideal of $R = k[x_1, ..., x_n]$. Assume that dimension $d \geq 2$. Then*

$$\operatorname{reg}(\mathfrak{p}) \leq (\deg(\mathfrak{p}))^{2^{d-1}}.$$

The arithmetic degree is defined as follows:

$$\operatorname{adeg} S = \operatorname{adeg} I = \sum_{\mathfrak{p} \in \operatorname{Ass}(R/I)} \ell(H^0_{\mathfrak{m}_\mathfrak{p}}(R_\mathfrak{p}/I_\mathfrak{p})) \deg(R/\mathfrak{p}),$$

(see [1, Definition 3.4] and [51, Definition 9.13]). The number $\ell(H^0_{\mathfrak{m}_\mathfrak{p}}(R_\mathfrak{p}/I_\mathfrak{p}))$ is the multiplicity of the component $\mathfrak{p}$ with respect to I. In this definition $\mathfrak{p}$ runs over all associated primes of S, while the usual degree $\deg S$ can be computed by a similar formula, but the sum is only taken over primes of the highest dimension. Thus

$$\operatorname{adeg} S \geq \deg S,$$

and the equality holds if and only if S is a pure-dimensional ring.

Following Brodmann and Sharp [6], the function

$$h^i_S(t) := \dim_K(H^i_{\mathfrak{m}}(S)_t)$$

is called the i-th *Hilbert homological function* of S. We will often use the Grothendieck-Serre formula

$$P_S(t) - H_S(t) = \sum_{i=0}^{d} (-1)^{i+1} h^i_S(t). \tag{3}$$

Lemma 3.2. *Let S be an one-dimensional Cohen-Macaulay ring. Then*

$$h^1_S(0) + \cdots + h^1_S(\operatorname{reg} S - 1) \le e(e-1)/2.$$

Proof. Since $P_S(t) = e$, from the Grothendieck-Serre formula (3) we have

$$h^1_S(t) = e - H_S(t).$$

Let $r = \operatorname{reg} S$. Since S is a Cohen-Macaulay ring, its Hilbert-Poincare series can be written in the form

$$HP_S(z) := \sum_{i \ge 0} H_S(i) z^i = \frac{1 + h_1 z + \cdots + h_r z^r}{1-z},$$

where $h_1, ..., h_r$ are positive integers (see, e.g., [51, p. 240]). From this it follows that

$$H_S(t) = 1 + h_1 + \cdots + h_t \ge t + 1$$

for all $t \le r$. Moreover, under the Cohen-Macaulay assumption, $r \le e - 1$. Hence

$$h^1_S(0) + \cdots + h^1_S(\operatorname{reg} S - 1) \le re - (1 + \cdots + r) = r(2e - r - 1)/2 \le e(e-1)/2.$$

□

Lemma 3.3. *Assume that $S = R/I$ is a reduced ring of dimension at least two. Then*

$$h^1_S(-1) \le \operatorname{adeg} I - e.$$

Proof. Since S is reduced, one may write $I = J \cap Q$, where J is the intersection of all associated primes of R/I of dimension at least 2, and Q is the intersection of all associated primes of R/I of dimension 1. By [23, Lemma 1], we have $h^1_{R/J}(-1) = 0$. Thus if $Q = R$, then $h^1_S(-1) = 0$. Assume that $Q \ne R$. Since $J \ne R$ and R/I has no embedded primes, $J + Q$ is an $\mathfrak{m}$-primary ideal, i.e. $\dim R/(J+Q) = 0$. The exact sequence

$$0 \to S \to R/J \oplus R/Q \to R/(J+Q) \to 0$$

implies

$$h^1_S(-1) = h^1_{R/J}(-1) + h^1_{R/Q}(-1) = h^1_{R/Q}(-1).$$

Note that $\deg R/Q = \operatorname{adeg} I - \operatorname{adeg} J \le \operatorname{adeg} I - e$. Since R/Q is an one-dimensional ring, by the Grothendieck-Serre formula, we have

$$h^1_{R/Q}(-1) = \deg R/Q \le \operatorname{adeg} I - e.$$

□

The proof of the main theorem is proceeded by induction. The next two lemmas allow us to do induction. The first one is concerning the behavior of the arithmetic degree by hyperplane section. It is more subtle than the usual degree, see [34]. However we have

Lemma 3.4. *Assume that $S = R/I$ is of dimension at least two and positive depth. Assume that x_n is chosen generically. Let $T = R/((I, x_n) : \mathfrak{m}^\infty)$ and $r = \operatorname{reg} T$. Then:*

(i) $\operatorname{reg} T \le \operatorname{reg} S$.
(ii) $\operatorname{adeg} T \le \operatorname{adeg} S$.

Proof. (i) Since x_n is generic, it is a regular element on S. We have

$$\operatorname{reg} T = \operatorname{reg}_1 S/x_nS \le \operatorname{reg} S/x_nS = \operatorname{reg} S.$$

(ii) For an R-module M and $r \ge -1$, let

$$\operatorname{adeg}_r(M) = \sum_{\mathfrak{p} \in \operatorname{Ass}(M),\ \dim R/\mathfrak{p} = r+1} \ell(H^0_{\mathfrak{m}_\mathfrak{p}}(M_\mathfrak{p})) \deg(R/\mathfrak{p})$$

(see [1, Definition 3.4]). Since x_n is generic, by the prime avoidance lemma, we may assume

$$x_n \notin \cup\{\mathfrak{p};\ \mathfrak{m} \ne \mathfrak{p} \in \operatorname{Ass}(S) \cup_{j\ge 1} \operatorname{Ass}(\operatorname{Ext}^{n-j}_R(S, R))\}.$$

By [34, Corollary 2.5] it follows that

$$\operatorname{adeg}_{r-1}(T) = \operatorname{adeg}_r(S) \quad \text{for all } r \ge 1.$$

Since S and T have no zero-dimensional component, we get

$$\begin{aligned} \operatorname{adeg} T &= \operatorname{adeg}_0(T) + \cdots + \operatorname{adeg}_{d-1}(T) \\ &= \operatorname{adeg}_1(S) + \cdots + \operatorname{adeg}_d(S) \le \operatorname{adeg} S. \end{aligned}$$

□

The first three statements of the next lemma are contained in the proof of Mumford's theorem on page 101 of the book [35] (cf. also [26, Proposition 1.4] and [40, Theorem 1.4]). In order to make the paper more self-contained, we give here a sketch of the proof. The proof of (iii) here is also simpler.

Lemma 3.5. *Assume that $S = R/I$ a reduced ring of dimension at least two. Assume that x_n is chosen generically. Let $T = R/((I, x_n) : \mathfrak{m}^\infty)$ and $r = \operatorname{reg} T$. Then T is also a reduced ring and we have*

(i) $\operatorname{reg}_2(S) \le r$.
(ii) $h^1_S(t) \ge h^1_S(t+1)$ *for all* $t \ge r-1$.
(iii) $\operatorname{reg} S \le r + h^1_S(r-1)$.
(iv) $h^1_S(t) \le h^1_T(0) + \cdots + h^1_T(t) + \operatorname{adeg} I - e$, *for all* $t \ge 0$.

Proof. Note that T can be considered as the homogeneous coordinate ring of a generic hyperplane section of the scheme $\operatorname{Proj}(S)$. Since k is an infinite ring and x_n is generic, by Bertini's theorem [18, Corollary 3.4.14], it follows that T is reduced.

The long exact sequence

$$\begin{array}{rl} 0 \to H^0_{\mathfrak m}(S/x_nS)_t & \to \; H^1_{\mathfrak m}(S)_{t-1} \to H^1_{\mathfrak m}(S)_t \xrightarrow{\varphi_t} H^1_{\mathfrak m}(S/x_nS)_t = H^1_{\mathfrak m}(T)_t \\ & \to \; H^2_{\mathfrak m}(S)_{t-1} \to H^2_{\mathfrak m}(S)_t \to \cdots \end{array} \tag{4}$$

implies (i) and the short exact sequence

$$0 \to H^0_{\mathfrak m}(S/x_nS)_t \to H^1_{\mathfrak m}(S)_{t-1} \to H^1_{\mathfrak m}(S)_t \to 0$$

for all $t \geq r$. This yields (ii). If $h^1_S(t_0-1) \geq h^1_S(t_0)$ for some $t_0 \geq r+1$, we would have $h^0_{S/x_nS}(t_0) = 0$. Since $\operatorname{reg}_1(S/x_nS) = \operatorname{reg} T = r$, it then implies that $\operatorname{reg}(S/x_nS) \leq t_0$. Hence $h^1_S(t_0) = h^1_S(t_0+1) = \cdots = 0$. Therefore $h^1_S(t)$ is strictly decreasing to zero when $t \geq r$, which implies (iii).

It remains to show (iv). From the exact sequence (4) we have

$$h^1_S(u) - h^1_S(u-1) = \ell(\operatorname{Im}(\varphi_u)) - h^0_{S/x_nS}(u) \leq h^1_T(u)$$

for all $u \in \mathbb{Z}$. Adding these inequalities and using Lemma 3.3 we get

$$\begin{array}{rl} h^1_S(t) & \leq h^1_T(0) + \cdots + h^1_T(t) + h^1_S(-1) \\ & \leq h^1_T(0) + \cdots + h^1_T(t) + \operatorname{adeg} I - e \end{array}$$

for all $t \geq 0$. □

Since $e = \deg S \leq \operatorname{adeg} S$, Theorem 3.1 is a part of the following result. If $a \in \mathbb{R}$, we denote by $[a]$ the largest integer not exceeding a.

Theorem 3.6. *(*[21, Theorem 1.5]*). Assume that $S = R/I$ is a reduced ring of dimension at least two. Let*

$$m = \frac{e(e-1)}{2} + \operatorname{adeg} I.$$

Then

(i) $\operatorname{reg} S \leq m^{2^{d-2}} - 1$.

(ii) *For all $t \geq 0$, we have $h^1_S(t) \leq m^{2^{d-2}} - e \cdot m^{[2^{d-3}]}$.*

Proof. We may assume that x_n is generic and choose T as in the previous lemma. Hence T is a reduced ring. Set $r = \operatorname{reg} T$.

Let $d = 2$. In order to show (ii), by Lemma 3.5(ii), we may assume that $t \leq r-1$. Note that T is a Cohen-Macaulay ring and $e(T) = e$. Then Lemma 3.5(iv) and Lemma 3.2 yield:

$$\begin{array}{rl} h^1_S(t) & \leq h^1_T(0) + \cdots + h^1_T(t) + \operatorname{adeg} I - e \\ & \leq h^1_T(0) + \cdots + h^1_T(r-1) + \operatorname{adeg} I - e \\ & \leq \frac{e(e-1)}{2} + \operatorname{adeg} I - e = m - e. \end{array}$$

Using this inequality and the fact that $r \leq e-1$ (since T is a Cohen-Macaulay ring), by Lemma 3.5(iii) we get

$$\operatorname{reg} S \leq e - 1 + m - e = m - 1.$$

Thus the case $d = 2$ is proven.

Let $d \geq 3$. Since $\dim T = d-1,\ e(T) = e$, and $\operatorname{adeg} T \leq \operatorname{adeg} S$ (by Lemma 3.4(ii)), the induction hypothesis gives

$$r \leq m^{2^{d-3}} - 1, \tag{5}$$

and for all $t \geq 0$

$$h^1_T(t) \leq m^{2^{d-3}} - e \cdot m^{[2^{d-4}]} \leq m^{2^{d-3}} - e. \tag{6}$$

In order to prove (ii), again by Lemma 3.5(ii), we may assume that $t \leq r-1$. Then, by Lemma 3.5(iv), for all $t \geq 0$ we have

$$\begin{aligned} h^1_S(t) &\leq h^1_T(0) + \cdots + h^1_T(t) + \operatorname{adeg} I - e \\ &\leq r(m^{2^{d-3}} - e) + \operatorname{adeg} I - e \quad (\text{by} \quad (6)) \\ &\leq (m^{2^{d-3}} - 1)(m^{2^{d-3}} - e) + \operatorname{adeg} I - e \quad (\text{by} \quad (5)) \\ &= m^{2^{d-2}} - e \cdot m^{2^{d-3}} - m^{2^{d-3}} + \operatorname{adeg} I \\ &\leq m^{2^{d-2}} - e \cdot m^{2^{d-3}}. \end{aligned}$$

To prove (i) we use (ii) and Lemma 3.5(iii) :

$$\operatorname{reg} S \leq r + h^1_S(r-1) \leq m^{2^{d-3}} - 1 + m^{2^{d-2}} - e \cdot m^{2^{d-3}} \leq m^{2^{d-2}} - 1.$$

□

Note that Theorem 3.6 does not hold if the ring R/I is not reduced.

Example 3.7. (see [51, Example 9.3.1]) Let $S = K[x,y,u,v]/((x,y)^2, xu^t + yv^t),\ t \geq 1$. Then $\operatorname{adeg} S = e = 2$, while $\operatorname{reg} S = t$ can be arbitrarily large.

Another direction is to bound the Castelnuovo-Mumford regularity of a homogenous ideal in terms of its defining degree. The best known bound is given in [21, Theorem 2.1]. The bound is almost sharp and free of characteristic of the base field. We just recall here the result without giving a proof. Let $R = k[X_1, ..., X_n]$ be a polynomial over a filed k and $I \subset R$ a homogeneous ideal. Let $d = \dim R/I$ and $c = \operatorname{codim}(I) = n - d$. We always write the degrees of polynomials in a minimal homogeneous basis of I in a decreasing sequence

$$\Delta := \delta_1 \geq \delta_2 \geq \cdots$$

and assume $\Delta \geq 2$.

Theorem 3.8. *(See* [21, Theorem 2.1]*). Let k be an arbitrary field and I be an arbitrary homogeneous ideal of dimension $d \geq 1$ and codimension c. Then*

$$\operatorname{reg} I \leq (\delta_1 \cdots \delta_c + \Delta - 1)^{2^{d-1}} \leq (\Delta^c + \Delta - 1)^{2^{d-1}}.$$

This theorem is a slight improvement of a result by G. Caviglia and E. Sbarra in [8] and gives a positive answer to a question posed by Bayer and Mumford in [1]. Note that using the counterexamples of J. McCullough and I. Peeva given at the beginning of the section one can show that the bound in Theorem 3.8 is also almost optimal for prime ideals!

In the next section we give another kind of bounds (in terms of Hilbert coefficients).

Exercise 3.9. 1. Given examples of prime ideals $\mathfrak{p} \subset R$ containing no linear form such that $\operatorname{reg}(\mathfrak{p}) = \deg(\mathfrak{p}) - \operatorname{codim}(\mathfrak{p}) + 1$.

2. Assume that S is a Cohen-Macaulay ring. Prove that $\operatorname{reg}(S) \leq \deg(S) - 1$. Does it still hold if S is a Buchsbaum ring?
3. (Cf. [46, Proposition 3.2].) Let $\dim S = d$ and $x_1, ..., x_d \in S_1$ be any filter-regular sequence of S. Prove that
$$a_d(S) + d \leq a_0(S/(x_1, ..., x_d)S) \leq \operatorname{reg}(S).$$
4. Let M be a graded S-module. Denote by $M^{(m)} = \oplus_{j\in\mathbb{Z}} M_{jm}$ the m-th Veronesean of M. Show that
$$H^i_{S^{(m)}_+}(M^{(m)}) \cong (H^i_{S_+}(M))^{(m)}.$$
Deduce from this a relation between $\operatorname{reg}(M)$ and $\operatorname{reg}(M^{(m)})$, where $M^{(m)}$ is considered as a graded module over $S^{(m)}$.

4. Castelnuovo-Mumford regularity and Hilbert coefficients

The Castelnuovo-Mumford regularity can be bounded in terms of Hilbert coefficients and vice verse. Write the Hilbert polynomial of M in the form:
$$P_M(t) = e_0(M)\binom{t+d-1}{d-1} - e_1(M)\binom{t+d-2}{d-2} + \cdots + (-1)^{d-1}e_{d-1}(M).$$
Then $e_0(M), e_1(M), ..., e_{d-1}(M)$ are called *Hilbert coefficients* of M. Note that $e_0(M) = \deg(M)$. First we show that Hilbert coefficients can be bounded in terms of the Castelnuovo-Mumford regularity.

Lemma 4.1. (See [13, Claim on p. 245]) *Let M be a graded S-module of dimension d and let $s = \operatorname{reg}(M)$. Assume that $y_1, ..., y_d \in S_1$ is an S_+-filter-regular sequence w.r.t M. Put $h^i_M(t) := \ell_{S_0}(H^i_{S_+}(M)_t)$. Then for all $i \geq 1$ and $s' \geq s$ we have*
$$h^i_M(t) \leq \binom{s'-1-t}{i-1} h_{M/(y_1,...,y_{i-1})M}(s'). \tag{7}$$

Proof. Since $h^i_M(t) = 0$ for all $t \geq s$, we may assume that $t \leq s-1 \leq s'-1$. We proceed by induction on i. For $i = 1$, let $M' := \oplus_{p \geq s'} M_p$. Then $\operatorname{reg}(M') = s'$ and y_1 is regular on M'. The exact sequence
$$H^0_{S_+}(M'/y_1M')_t \to H^1_{S_+}(M')_{t-1} \to H^1_{S_+}(M')_t \to H^1_{S_+}(M'/y_1M')_t$$
implies
$$h^1_{M'}(t-1) - h^1_{M'}(t) \leq h^0_{M'/y_1M'}(t) \leq h_{M'/y_1M'}(t).$$
Hence
$$h^1_M(t) \leq h^1_{M'}(t) = \sum_{i=t+1}^{s'} (h^1_{M'}(i-1) - h^1_{M'}(i)) \leq \sum_{i=t+1}^{s'} h_{M'/y_1M'}(i) = h_{M'}(s') = h_M(s').$$

The case $i \geq 2$ follows from the induction hypothesis and the inequality

$$h^i_M(t-1) - h^i_M(t) \leq h^{i-1}_{M/y_{i-1}M}(t).$$

So, the proof of the claim (7) in completed. Now, taking $s' = s$ and using [9, Lemma 4.4(i)], we obtain

$$h^i_M(t) \leq \ell(M/(y_1, ..., y_d)M)\binom{\operatorname{reg}(M) - 1 - t}{i-1}\binom{\operatorname{reg}(M) + d - i}{d-i}.$$

□

Now we can show

Theorem 4.2. ([9, Theorem 4.6]). *Let M be a graded S-module of dimension d. Let $x_1, \ldots, x_d \in S_1$ be an S_+-filter regular sequence w.r.t M and $B = \ell(M/(x_1, ..., x_d)M)$. Then for all $1 \leq i \leq d-1$ we have*

$$|e_i(M)| \leq B(\operatorname{reg}^1(M) + 1)^i.$$

Proof. As usual we set $\overline{r} = \operatorname{reg}(\overline{M}) = \operatorname{reg}^1(M)$, where $\overline{M} = M/H^0_{S_+}(M)$. We do induction on d. Note that $0 \leq e_0(M) \leq B$. Hence the inequality holds true for $i = 0$. In particular the statement holds for $d = 1$. Assume that the statement holds for all modules of dimension $d - 1 \geq 1$. Let M be a module of dimension d and $M_1 = M/x_1M$. Then $e_i(M) = e_i(M_1)$ for all $i \leq d-2$. Since $\operatorname{reg}(\overline{M_1}) \leq \operatorname{reg}(\overline{M})$ and $\ell(M_1/(x_2, ..., x_d)M_1) = B$, by the induction hypothesis it suffices to show the inequality

$$|e_{d-1}(M)| \leq B(\overline{r} + 1)^{d-1}.$$

Note that we may assume $M = \overline{M}$, i.e. $H^0_{S_+}(M) = 0$. From the Grothendieck-Serre formula (3) we get (setting $t = -1$):

$$(-1)^{d-1}e_{d-1}(M) = C_d - D_d,$$

where

$$C_d = \ell(H^1_{S_+}(M)_{-1}) + \ell(H^3_{S_+}(M)_{-1}) \ \cdots ,$$

and

$$D_d = \ell(H^2_{S_+}(M)_{-1}) + \ell(H^4_{S_+}(M)_{-1}) \ \cdots .$$

By Lemma 4.1 we have

$$C_d \leq B \sum_{1 \leq 2j+1 \leq d} \binom{\overline{r}}{2j}\binom{\overline{r} + d - 2j - 1}{d - 2j - 1} =: B.\tilde{C}_d.$$

We show by induction on d that $\tilde{C}_d \leq (\overline{r} + 1)^{d-1}$. We have $\tilde{C}_2 = \overline{r} + 1$ and $\tilde{C}_3 = \overline{r}^2 + \overline{r} + 1 < (\overline{r} + 1)^2$. Let $d \geq 4$. Assume that

$$\tilde{C}_{d-1} \leq (\overline{r} + 1)^{d-2}.$$

If d is even, then $d - 2j - 1 \geq 1$ and

$$\binom{\overline{r}+d-2j-1}{d-2j-1} = \frac{\overline{r}+d-2j-1}{d-2j-1}\binom{\overline{r}+(d-1)-2j-1}{(d-1)-2j-1} \leq (\overline{r} + 1)\binom{\overline{r}+(d-1)-2j-1}{(d-1)-2j-1}.$$

Hence, by the induction hypothesis on $\tilde{C}_{d-1}$ we get

$$\tilde{C}_d \leq (\overline{r}+1)\sum_{1\leq 2j+1\leq d}\binom{\overline{r}}{2j}\binom{\overline{r}+(d-1)-2j-1}{(d-1)-2j-1} = (\overline{r}+1)\tilde{C}_{d-1} \leq (\overline{r}+1)^{d-1}.$$

If d is odd, say $d = 2\delta+1$, then for $j < \delta$ we have $d-2j-1 \geq 2$ and

$$\tbinom{\overline{r}+d-2j-1}{d-2j-1} = (\tfrac{\overline{r}}{d-2j-1}+1)\tbinom{\overline{r}+(d-1)-2j-1}{(d-1)-2j-1} \leq (\tfrac{\overline{r}}{2}+1)\tbinom{\overline{r}+(d-1)-2j-1}{(d-1)-2j-1}.$$

Therefore

$$\begin{aligned}\tilde{C}_d &\leq (\tfrac{\overline{r}}{2}+1)\sum_{1\leq 2j+1\leq d-1}\binom{\overline{r}}{2j}\binom{\overline{r}+(d-1)-2j-1}{(d-1)-2j-1} + \binom{\overline{r}}{d-1}\\ &< (\frac{\overline{r}}{2}+1)\tilde{C}_{d-1} + \frac{(\overline{r}+1)^{d-2}\overline{r}}{2}\\ &\leq (\frac{\overline{r}}{2}+1)(\overline{r}+1)^{d-2} + (\overline{r}+1)^{d-2}\frac{\overline{r}}{2}\\ &= (\overline{r}+1)^{d-1}.\end{aligned}$$

Thus we have proved $\tilde{C}_d \leq (\overline{r}+1)^{d-1}$, and so $C_d \leq B(\overline{r}+1)^{d-1}$. Similarly, $D_d \leq B(\overline{r}+1)^{d-1}$. Hence

$$|e_{d-1}(M)| \leq \max\{C_d,\ D_d\} \leq B(\overline{r}+1)^{d-1},$$

as required. □

Remark. (i) If M is a Cohen-Macaulay module, then $B = \deg(M)$.

(ii) Considering $M/(x_1,...,x_d)M$ as a module over $S/(x_1,...,x_d)S$, it is easy to show that

$$B \leq \mu(M)\binom{\mathrm{reg}^1(M)+n-d}{n-d},$$

where $\mu(M)$ is the number of generators of M.

Now we want to solve a reverse problem, namely to bound $\mathrm{reg}^1(M)$ in terms of Hilbert coefficients of M. We need

Lemma 4.3. (see [40, Theorem 1.4], [31, Theorem 2.7]). *Let* $\dim M \geq 1$ *be a module generated by elements of degree at most* m. *Let* $x \in S_1$ *an* M*-filter regular. Assume that* $\mathrm{reg}^1(M/xM) \leq m$. *Then*

$$\mathrm{reg}^1(M) \leq m + P_M(m) - 1.$$

Proof. Since $\mathrm{reg}^1(M) = \mathrm{reg}^1(\overline{M})$ and $P_M(t) = P_{\overline{M}}(t)$, we may assume that $\mathrm{depth}(M) > 0$. By Lemma 1.3(ii), $\mathrm{reg}^2(M) \leq m$. Hence, we have short exact sequences

$$0 \longrightarrow H^0_{S_+}(M/xM)_p \longrightarrow H^1_{S_+}(M)_{p-1} \longrightarrow H^1_{S_+}(M)_p \longrightarrow 0,$$

for all $p \geq m$. Moreover, by the Grothendieck-Serre formula (3),

$$0 \leq \ell(H^1_{S_+}(M)_p) = P_M(p) - H_M(p) \leq P_M(p) - 1.$$

Let $p_0 \geq m$ be the smallest in integer such that $H^0_{S_+}(M/xM)_{p_0} = 0$. Then, by Castelnuovo's Lemma 1.4, $\operatorname{reg}(M/xM) \leq p_0$. Then we have $H^1_{S_+}(M)_{p-1} \cong H^1_{S_+}(M)_p$ for all $p \geq p_0$ which implies $\operatorname{reg}^1(M) \leq p_0$. If $p_0 = m$, then the lemma is proved, because $P_M(m) - 1 \geq 0$. Assume $p_0 > m$. Then $H^0_{S_+}(M/xM)_p \neq 0$ for all $m \leq p \leq p_0 - 1$. In this case $\ell(H^1_{S_+}(M)_{p-1}) < \ell(H^1_{S_+}(M)_p)$ for all $m \leq p \leq p_0 - 1$, which yields $\ell(H^1_{S_+}(M)_m) \geq p_0 - m$. Therefore

$$\operatorname{reg}^1(M) \leq p_0 \leq m + \ell(H^1_{S_+}(M)_m) \leq m + P_M(m) - 1.$$

□

One can bound $\operatorname{reg}^1(M)$ in terms of $\Delta(M), e_0(M), ..., e_{d-1}(M)$, as shown in [6, Theorem 17.2.7] and [45, Theorem 2]. Below we present the bound by Trivedi which does not depend on the number of generators of M as the one in [6]. The result was originally formulated in [45] for associated graded ring of an ideal in a local ring, which corresponds to the case M being generated by elements of degree zero. But this assumption is not essential. The proof was eventually given in [44, Lemma 4]. The proof presented below is more algebraic. Let

$$\Delta'(M) = \max\{\Delta(M),\ 0\}.$$

We inductively define a sequence of integers as follows: $m_1 = e_0(M) + \Delta'(M)$, and for all $i \geq 2$,

$$m_i = m_{i-1} + \sum_{k=0}^{i-1} (-1)^k e_k(M) \binom{m_{i-1} + i - 2 - k}{i - 1 - k}. \tag{8}$$

Then

Theorem 4.4. ([45, Theorem 2]) *Assume that $d \geq 1$. Then $\operatorname{reg}^1(M) \leq m_d - 1$.*

Proof. Induction by d. One may assume $\operatorname{depth} M = 0$. Let $d = 1$ and x_1 be a M-filter-regular element. Then $\operatorname{reg}^1(M) = \operatorname{reg}(M/xM) = a(M/xM) =: a$. Note that $[M/xM]_p \neq 0$ for all $\Delta(M) \leq p \leq a$. Hence $e_0(M) = \ell(M/xM) \geq a - \Delta(M) + 1$ whence $a \leq m_1 - 1$. Let $d > 1$. Let x_1 be a M-filter-regular element. By Lemma 4.3 and induction hypothesis,

$$\operatorname{reg}^1(M) \leq m_{d-1} + P_M(m_{d-1}) - 1 = m_d.$$

□

The following bound is weaker, but it is explicit.

Theorem 4.5. ([13, Lemma 1.2]) *Let M be a finitely generated graded S-module of dimension $d \geq 1$. Put*

$$\xi_{d-1}(M) = \max\{e_0(M), |e_1(M)|, ..., |e_{d-1}(M)|\}.$$

Then we have

$$\operatorname{reg}^1(M) \leq (\xi_{d-1}(M) + \Delta'(M) + 1)^{d!} - 2.$$

Proof. For short, we put $e_i := e_i(M)$, $\xi := \xi_{d-1}(M)$ and $\Delta' := \Delta'(M)$. By Theorem 4.4 it suffices to show that $m_d \le (\xi + \Delta' + 1)^{d!} - 1$. This is a purely arithmetic issue, which is trivial for $d = 1$. By the induction hypothesis we may assume

$$m_{d-1} \le (\xi + \Delta' + 1)^{(d-1)!} - 1 =: \alpha.$$

Note that

$$\sum_{i=0}^{d-1} (-1)^i e_i \binom{\alpha + d - 2 - i}{d - 1 - i} \le \xi \sum_{i=0}^{d-1} \binom{\alpha + d - 2 - i}{d - 1 - i} = \xi \binom{\alpha + d - 1}{d - 1}.$$

Hence, by the recurrence formula (8) applied to $i = d$, we get

$$m_d \le \alpha + \xi \binom{\alpha + d - 1}{d - 1}.$$

If $d = 2$, then $\alpha = \xi + \Delta'$, and

$$m_2 \le \xi + \Delta' + \xi(\xi + \Delta' + 1) = (\xi + \Delta' + 1)(\xi + 1) - 1 \le (\xi + \Delta' + 1)^2 - 1.$$

Assume $d \ge 3$. Observing that $\binom{\alpha+d-1}{d-1} \le (\alpha + 1)^{d-1}$ for all $\alpha \ge 1$ and $\alpha \ge (\xi + 1)^2 > \xi + 1$, we obtain

$$m_d \le \alpha + \xi(\alpha + 1)^{d-1} \le (1 + \xi)(\alpha + 1)^{d-1} - 1 \le (\alpha + 1)^d - 1 = (\xi + \Delta' + 1)^{d!} - 1.$$

□

The above results can be applied to the local case. Let $\mathcal{M}$ be a finite module of dimension d over a local ring $(A, \mathfrak{m})$ and I be an $\mathfrak{m}$-primary ideal of A. The Hilbert-Samuel function $H^1_{I,\mathcal{M}}(t) := \ell(\mathcal{M}/I^{t+1}\mathcal{M})$ agrees with the Hilbert-Samuel polynomial $P^1_{I,\mathcal{M}}(t)$ for all $t \gg 0$. Write the Hilbert-Samuel in the form:

$$P^1_{I,\mathcal{M}}(t) = e_0(I, \mathcal{M})\binom{t + d}{d} - e_1(I, \mathcal{M})\binom{t + d - 1}{d - 1} + \cdots + (-1)^d e_d(I, \mathcal{M}).$$

Then $e_0(I, \mathcal{M}), e_1(I, \mathcal{M}), ..., e_d(I, \mathcal{M})$ are also called *Hilbert coefficients* of $\mathcal{M}$ w.r.t. I. It is clear that $e_i(I, \mathcal{M}) = e_i(G_{\mathcal{M}}(I))$ for all $i \le d - 1$, where

$$G_{\mathcal{M}}(I) := \oplus_{j \ge 0} I^j\mathcal{M}/I^{j+1}\mathcal{M}$$

is a graded module over $G(I) := \oplus_{j \ge 0} I^j/I^{j+1}$.

An element $x \in I$ is called $\mathcal{M}$-*superficial element* for I if there exists a non-negative integer c such that $(I^{m+1}\mathcal{M} :_{\mathcal{M}} x) \cap I^c\mathcal{M} = I^m\mathcal{M}$ for every $m \ge c$ and we say that a sequence of elements $x_1, ..., x_r$ is an $\mathcal{M}$-*superficial sequence* for I if, for $i = 1, 2, ..., r$, x_i is an $\mathcal{M}/(x_1, ..., x_{i-1})$-superficial sequence for I.

Using Theorem 4.5 we can already bound $\operatorname{reg}^1(G_{\mathcal{M}}(I))$ in terms of $e_0(I, \mathcal{M}), ..., e_{d-1}(I, \mathcal{M})$. If $\operatorname{depth}(\mathcal{M}) > 0$, by [12, Lemma 1.8], $\operatorname{reg}(G_{\mathcal{M}}(I)) = \operatorname{reg}^1(G_{\mathcal{M}}(I))$, and so it is bounded in terms of $e_i(I, \mathcal{M})$, $i < d$. The following example shows that this is not true if $\operatorname{depth}(\mathcal{M}) = 0$.

Example 4.6. Let $A = K[[x,y]]/(x^2, xy^s),\ s \geq 1$. Then $G_A(\mathfrak{m}) \cong k[x,y]/(x^2, xy^s)$. Since (x^2, xy^s) is a so-called stable ideal, $\operatorname{reg}(G_A(\mathfrak{m})) = s$ can be arbitrarily large, while $e_0(A) = 1$.

Our first goal is to show that also using $e_d(I, \mathcal{M})$ we can bound $\operatorname{reg}(G_{\mathcal{M}}(I))$. In the sequel we will often use the following notation:

$$\xi_s(I, \mathcal{M}) = \max\{e_0(I, \mathcal{M}), |e_1(I, \mathcal{M})|, ..., |e_s(I, \mathcal{M})|\},$$

where $0 \leq s \leq d$.

Theorem 4.7. *Let $\mathcal{M}$ be a A-module of dimension $d \geq 1$. Then*

$$\operatorname{reg}(G_{\mathcal{M}}(I)) < (\xi_d(I, \mathcal{M}) + 1)^{(d+1)!} - 2.$$

Proof. Let x be an indeterminate of $\deg(x) = 1$, $G' = G(I)[x] \cong G(I) \otimes_{A/\mathfrak{m}} A/\mathfrak{m}[x]$ be a standard graded ring, and $\mathcal{G}' = G_{\mathcal{M}}(I) \otimes_{A/\mathfrak{m}} A/\mathfrak{m}[x]$ be graded G'-module of dimension $(d+1)$ over $A/\mathfrak{m}$. Then

$$H_{\mathcal{G}'}(m) = \sum_{i=0}^{m} \ell_A(I^i M/I^{i+1}M) = \ell(M/I^{m+1}M).$$

Hence $e_i(I, \mathcal{M}) = e_i(\mathcal{G}')$ for all $i \leq d$ and $\xi_d(\mathcal{G}') = \xi_d(I, \mathcal{M})$. Since x is regular on $\mathcal{G}'$,

$$\operatorname{reg}(\mathcal{G}') = \operatorname{reg}^1(\mathcal{G}') = \operatorname{reg}(\mathcal{G}'/x\mathcal{G}') = \operatorname{reg}(G_{\mathcal{M}}(I)).$$

Let $e_i = e_i(I, \mathcal{M})$ and $\xi := \xi_d(I, \mathcal{M})$. By Theorem 4.5 we have

$$\operatorname{reg}(G_{\mathcal{M}}(I)) = \operatorname{reg}^1(G(\mathcal{G}')) \leq (\xi_d(I, \mathcal{M}) + 1)^{(d+1)!} - 2.$$

□

A little better bound is given in [13, Theorem1.8], but the proof is more complicate. Refining the above technique one can bound $\operatorname{reg}(G_{\mathcal{M}}(I))$ in terms of $\xi_{d-t}(I, \mathcal{M})$, where $t = \operatorname{depth} \mathcal{M}$.

Theorem 4.8. ([13, Theorem 1.12]) *Let* $\dim(\mathcal{M}) = d \geq 1$ *and* $\operatorname{depth}(\mathcal{M}) = t$. *Then*

$$\operatorname{reg}(G_{\mathcal{M}}(I)) \leq (\xi_{d-t}(I, \mathcal{M}) + 1)^{2(d-t+1)d!} - 2.$$

Conversely, as in the graded case, one can bound Hilbert coefficients in terms of the Castelnuovo-Mumford regularity of the associated graded module.

Proposition 4.9. (See [14, Proposition B]). *Let $x_1, \ldots, x_d \in I$ be an $\mathcal{M}$-superficial sequence for I and $B = \ell(\mathcal{M}/(x_1, ..., x_d)\mathcal{M})$. Then $|e_i(\mathcal{M})| < B(2\operatorname{reg}(G_{\mathcal{M}}(I)) + 2)^i$ for all $1 \leq i \leq d$.*

Proof. We do induction on d. Let $a = \operatorname{reg}(G_{\mathcal{M}}(I))$ and $e_i = e_i(\mathcal{M})$. By [13, Lemma 1.5],

$$H^1_{I,\mathcal{M}}(a) = P^1_{I,\mathcal{M}}(a) = \sum_{i=0}^{d} (-1)^i e_i \binom{a+d-i}{d-i}.$$

By [12, Lemma 1.7],

$$H^1_{I,\mathcal{M}}(a) = \ell(\mathcal{M}/\mathcal{M}_{a+1}) \leq \ell(\mathcal{M}/I^{a+1}\mathcal{M}) \leq B\binom{a+d}{d}.$$

Note that $\binom{a+j}{j} \leq (a+1)^j$ and $e_0 = e_0(I,\mathcal{M}) \leq B$.

If $d = 1$, then

$$|e_1| = |H^1_{I,\mathcal{M}}(a) - e_0(a+1)| \leq \max\{B(a+1), e_0(a+1)\} = B(a+1).$$

Let $d \geq 2$. First we prove the statement for $0 < i \leq d-1$. Assume that $\operatorname{depth}(\mathcal{M}) > 0$. Then $\dim(\mathcal{M}/x_1\mathcal{M}) = d-1$ and by [39, Proposition 1.2], $e_i(\mathcal{M}) = e_i(\mathcal{M}/x_1\mathcal{M})$ for all $i \leq d-1$. By [13, Lemma 1.9], $\operatorname{reg}(G_{\mathcal{M}/x_1\mathcal{M}}(I)) \leq a$. Hence, by the induction hypothesis applied to $\mathcal{M}/x_1\mathcal{M}$ and the sequence $x_2, ..., x_d$, we get

$$|e_i(\mathcal{M})| < B(2\operatorname{reg}(G_{\mathcal{M}/x_1M}(I)) + 2)^i \leq B(2a+2)^i.$$

We now assume that $\operatorname{depth}(\mathcal{M}) = 0$. Let $\overline{\mathcal{M}} = \mathcal{M}/H^0_{\mathfrak{m}}(\mathcal{M})$ and $\overline{\mathcal{M}} = \mathcal{M}/H^0_{\mathfrak{m}}(\mathcal{M})$. Note that $e_i\mathcal{M}) = e_i(\overline{\mathcal{M}})$ for all $i \leq d-1$ and $\ell(\overline{\mathcal{M}}/(x_1, ..., x_d)\overline{\mathcal{M}}) \leq B$. In the proof of [12, Lemma 1.9], it was shown that there is an exact sequence

$$0 \to L \to G_{\mathcal{M}}(I) \to G_{\overline{\mathcal{M}}}(I) \to 0,$$

where L has a finite length. Hence $\operatorname{reg}(G_{\overline{\mathcal{M}}}(I)) \leq \operatorname{reg} G_{\mathcal{M}}(I) = a$, and

$$|e_i(\mathcal{M})| = e_i(\overline{\mathcal{M}}) < \ell(\overline{\mathcal{M}}/(x_1, ..., x_d)\overline{\mathcal{M}})(2\operatorname{reg}(G_{\overline{\mathcal{M}}}(I)) + 2)^i \leq B(2a+2)^i.$$

Finally, we have

$$\begin{aligned} |e_d| &\leq H^1_{I,\mathcal{M}}(a) + \textstyle\sum_{i=0}^{d-1} |e_i|\binom{a+d-i}{d-i} \\ &< B\binom{a+d}{d} + B\textstyle\sum_{i=0}^{d-1} 2^i(a+1)^i\binom{a+d-i}{d-i} \\ &\leq B(a+1)^d + B\textstyle\sum_{i=0}^{d-1} 2^i(a+1)^i(a+1)^{d-i} \\ &= B2^d(a+1)^d. \end{aligned}$$

□

From this result Theorem 4.8, we get the following extension of a very nice result obtained by Srinivas and Trivedi ([44, Theorem 1]) for all Cohen-Macaulay modules to the non-Cohen-Macaulay case.

Theorem 4.10. ([13, Corollary 2.5]) *Assume that* $\dim(\mathcal{M}) = d \geq 1$ *and* $\operatorname{depth}(\mathcal{M}) = t \geq 1$. *Then for all* $d-t+1 \leq j \leq d$ *we have*

$$|e_j(I,\mathcal{M})| < (\xi_{d-t} + 1)^{3j(d-t+1)j!},$$

where

$$\xi_{d-t} = \max\{e_0(I,\mathcal{M}), |e_1(I,\mathcal{M})|, ..., |e_{d-t}(I,\mathcal{M})|\}.$$

In other words, if $d-t+1 \leq j \leq d$, $|e_j(I,\mathcal{M})|$ *is bounded in terms of* $d, e_0(I,\mathcal{M})$, $e_1(I,\mathcal{M}), ..., e_{d-t}(I,\mathcal{M})$.

The proof of the above theorem and of Theorem 4.8 involve many computation, so we omit them here. It is easy to construct examples to show that one cannot reduce the number of "independent" coefficients in Theorem 4.10 (see Exercise 4.11(4) below).

Even under certain additional assumption on A we cannot reduce the number of "independent" coefficients. For example, in [43] there were constructed a complete regular local ring R and an infinite sequence of prime ideals $\mathfrak{p}_n$ of R such that $\dim(R/\mathfrak{p}_n) = 2$, $e_0(R/\mathfrak{p}_n) = 4$, but $e_1(R/\mathfrak{p}_n) = 8 - n$.

Exercise 4.11.

1. Show that $x \in I \setminus \mathfrak{m}I$ is a superficial $\mathcal{M}$-element of I if and only if its form $x^* \in G(I)$ is a $G_{\mathcal{M}}(I)$-filter regular element.
2. Let $x \in I \setminus \mathfrak{m}I$ be a superficial $\mathcal{M}$-element of I. Show that
$$x\mathcal{M} \cap I^p\mathcal{M} = xI^{p-1}\mathcal{M},$$
for all $p \geq \operatorname{reg}(G_{\mathcal{M}}(I)) + 1$.
3. Let $x \in I \setminus \mathfrak{m}I$ be a superficial $\mathcal{M}$-element of I. Show that for all $p \gg 0$,
$$I^{p+1}\mathcal{M} : x = I^p\mathcal{M} + (0 :_{\mathcal{M}} x),$$
and
$$(0 :_{\mathcal{M}} x) \cap I^p\mathcal{M} = 0.$$
4. Let $A = K[[x_1, ..., x_{d+1}]]/(x_1^2, x_1x_2, ..., x_1x_d, x_1x_{d+1}^s)$, where $s \geq 1$, and $I = \mathfrak{m} = (\bar{x_1}, ..., \bar{x}_{d+1})$. Then $\dim A = d$, $\operatorname{depth} A = 0$, $e_0 = 1$, $e_1 = \cdots = e_{d-1} = 0$, while $e_d = (-1)^d s$. This shows one cannot reduce the number of "independent" coefficients in Theorem 4.10.
5. Let $\dim \mathcal{M} \geq 1$ and $x \in \mathfrak{m}$ be a superficial $\mathcal{M}$-element for I. Show that
$$\operatorname{reg}(G_{\mathcal{M}/x\mathcal{M}}(I/Ax)) \leq \operatorname{reg}(G_{\mathcal{M}}(I)).$$

5. Some problems

There are many open problems left in the theory of the Castelnuovo-Mumford regularity.

After the Eisenbud-Goto Conjecture has been disproved, the most important and interesting problem is

Eisenbud-Goto Conjecture Does the Eisenbud-Goto Conjecture holds for smooth projective varieties?

In some cases, there are known good bounds, but people are trying to find better bounds. For an example, let I be a square-free monomial ideal generated of dimension d. Then from Hochster's formula of computing local cohomology modules of R/I, one can easily show that $\operatorname{reg}(R/I) \leq d$, and it is easy to construct examples to show that this bound is sharp (cf. Exercise 7 in 1.10). However, when one restricts to a special class, there are still many interesting problem. Recently the following problem attracts interest of many researchers

Castelnuovo-Mumford regularity of edge ideals. Assume that I is a square-free monomial ideal whose generators are of degree 2. In this case I is associated to a simple undirected graph G on the vertex set $V = \{1, ..., n\}$, such that $I = I(G) := (X_iX_j|\ \{i,j\} \in E)$, where E is the set of edges of G. Give good lower and upper bounds on $\text{reg}(I(G)$ in terms of G.

Asymptotic behavior

The asymptotic behavior of the Castelnuovo-Mumford regularity is a hot topic during last 15 years. It was first discovered by A. Betram, L. Ein and R. Lazarsfeld in [4] that if I is the defining ideal of a smooth complex projective variety, then $\text{reg}(I^s)$ is bounded by a linear function of s. Later on, D. Cutkosky, J. Herzog and N.V. Trung [11] and independently Kodiyalam [27] proved that for an arbitrary ideal I of a polynomial ring over k, $\text{reg}(I^s)$ is actually a linear function for $s \gg 0$. Then this result is extended to a rather general context in [50]. Although the techniques used in these three papers are quite different, the common point is based on the finite generation of the Rees algebra of I. Here we will present the result of [50].

Theorem 5.1. (See [50, Theorem 3.2].) *Let S be a Noetherian standard graded ring over a local ring S_0*[3]*. Let M be a finite graded S-module. Then there is $a \in \mathbb{N}$ and $b \in \mathbb{Z}$ such that*

$$\text{reg}(I^sM) = as + b \quad \textit{for all} \quad s \gg 0.$$

The number a is called the slope of this function.

Even more generally,

Theorem 5.2. (Cf. [11, Theorem 3.1].) *Let S be a Noetherian standard graded ring over a local ring S_0. Let M be a finite graded S-module and $s \geq 0$. Then there is $c_1, c_2 \in \mathbb{N}$ and $d_1, d_2 \in \mathbb{Z}$ such that*

$$\text{reg}_l(I^sM) = c_1s + d_1 \quad \textit{and} \quad a_l^*(I^sM) = c_2s + d_2 \quad \textit{for all} \quad s \gg 0.$$

If we consider other filtrations of ideals, then the corresponding Rees algebra is not necessarily finitely generated, and a similar result of Theorem 5.1 maybe does not hold. For example $\text{reg}((I^s)^{sat})$ of the saturation of I^s is not necessarily an asymptotically linear function (see [11, Examples 4.2-4.4]). Since $\text{reg}^2(I^s) = \text{reg}((I^s)^{sat})$, this example also shows that Theorem 5.2 does not necessarily hold for $\text{reg}^l(I^s)$.

On the other hand, Cutkosky, Ein and Lazarsfeld [10] showed

Theorem 5.3. *The limit $\lim_{s\to\infty} \frac{\text{reg}^1(R/I^s)}{s}$ always exists.*

Note that the above limit can be an irrational number. Therefore the following problems are of interest:

Let I be a homogenous ideal in $R = k[X_1, ..., X_n]$. Assume that $\text{reg}(I^s) = as + b$ for all $s \geq s_0$. The invariant a can be computed: it is related to reduction of I, see [27].

[3] In fact it is sufficient to assume that S_0 is a Noetherian ring

Problem 1. Determine/characterize b.

Problem 2. Give an upper bound on s_0.

Problem 3. Let $(A, \mathfrak{m})$ be a local ring and I an ideal in A. Let $G(A/I^s)$ denote the associated graded ring of A/I^s with respect to $\mathfrak{m}$. Does $\lim_{s\to\infty} \frac{\operatorname{reg}(G(A/I^s))}{s}$ exist?

Problem 4. Let R be a polynomial ring and I a homogeneous ideal in R. Let $I^{(s)}$ denote the sth symbolic power of I. Does $\lim_{s\to\infty} \frac{\operatorname{reg}(I^{(s)})}{s}$ exist?

Problem 5. Let R be a polynomial ring and I a homogeneous ideal in R. Let $\operatorname{in}(I)$ denote the initial ideal of I with respect to an arbitrary term order. Does $\lim_{s\to\infty} \frac{\operatorname{reg}(\operatorname{in}(I^s))}{s}$ exist?

Note that there are some partial answers to Problems 3-5 for small dimensions in [20].

One can also ask about the asymptotic behavior of each partial Castelnuovo-Mumford regularity $a_i(R/I^s)$, namely

Problem 6. Let R be a polynomial ring and I a homogeneous ideal in R. Let $i \le \dim R/I$. Does $\lim_{s\to\infty} \frac{a_i(R/I^s))}{s}$ exist?

There are some partial answers to Problem 6 for monomial ideas given in [25].

References

[1] D. Bayer, D. Mumford, *What can be computed in algebraic geometry?*, in: "Computational algebraic geometry and commutative algebra (Cortona, 1991)", 1–48, Sympos. Math., XXXIV, Cambridge Univ. Press, Cambridge, 1993.

[2] D. Bayer and M. Stillman, *A criterion for detecting m-regularity*, Invent. Math. **87**(1987), 1-11.

[3] D. Bayer and M. Stillman, *On the complexity of computing syzygies.* In: Computational aspects of commutative algebra, J. Symbolic Comput. **6** (1988), 135147.

[4] A. Betram, L. Ein and R. Lazarsfeld, *Vanishing theorems, a theorem of Severi, and the equations defining projective varieties*, J. Amer. Math. Soc. **4**(1991), 587-602.

[5] M. Brodmann and T. Götsch, *Bounds for the Castelnuovo-Mumford regularity*, J. Comm. Algebra **1**(2009), 197-225.

[6] M. Brodmann and R. Sharp, Local cohomology. An algebraic introduction with geometric applications, Cambridge Univ. Press, Cambridge, 1998.

[7] W. Bruns and J. Herzog, *On multigraded resolutions*, Math. Proc. Camb. Phil. Soc. **118**(1995), 245-257.

[8] G. Caviglia and E. Sbarra, *Characteristic-free bounds for Castelnuovo-Mumford regularity*, Compos. Math. 141 (2005), no. 6, 1365–1373

[9] M. Chardin, D.T. Ha and L. T. Hoa, *Castelnuovo-Mumford regularity of Ext modules and homological degree*, Trans. Amer. Math. Soc. **363**, 3439 - 3456 (2011).

[10] D. Cutkosky, L. Ein and R. Lazarsfeld, *Positivity and complexity of ideal sheaves*, Math. Ann. **321**(2001), no. 2, 213–234.

[11] D. Cutkosky, J. Herzog and N.V. Trung, *Asymptotic behaviour of the Castelnuovo-Mumford regularity*, Compositio Math. **118**(1999), 243–261.

[12] L. X. Dung and L. T. Hoa, *Castelnuovo-Mumford regularity of associated graded modules and fiber cones of filtered modules*, Comm. Algebra **40**, 404 - 422 (2012).
[13] L. X. Dung and L. T. Hoa, *Dependence of Hilbert coefficients*, Manu. Math. **149**(2016), 235-249.
[14] L. X. Dung and L. T. Hoa, *Erratum to: Dependence of Hilbert coefficients*, Manu. Math. **154**(2017), 551-552.
[15] D. Eisenbud, Commutative Algebra with a View toward Algebraic Geometry, Springer-Verlag, 1995.
[16] D. Eisenbud and S. Goto, *Linear free resolutions and minimal multiplicities*, J. Algebra **88**(1984), 89-133.
[17] S. Eliahou and M. Kervaire, *Minimal resolutions of some monomial ideals*, J. Algebra **129**(1990), 1-25.
[18] H. Flenner, L. O'Carroll and W. Vogel, Joins and intersections. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1999.
[19] L. Gruson, R. Lazarsfeld and C. Peskine, *On a theorem of Castelnuovo, and the equations defining space curves*, Invent. Math. **72**(1983), 491–506.
[20] J. Herzog, L.T. Hoa and N. V. Trung, *Asymptotic linear bounds for the Castelnuovo-Mumford regularity*, Trans. Amer. Math. Soc. **354**(2002), 1793–1809.
[21] L.T. Hoa, *Finiteness of Hilbert functions and bounds for Castelnuovo-Mumford regularity of initial ideals*, Trans. Amer. Math. Soc. **360**(2008), 4519-4540.
[22] L. T. Hoa and J. Stückrad, *Castelnuovo-Mumford regularity of simplicial toric rings*, J. Algebra **259** (2003), 127-146.
[23] L. T. Hoa, J. Stückrad and W. Vogel, *Towards a structure theory for projective varieties of degree =codimension +2*, J. Pure Appl. Algebra, **71**, 203-231(1991).
[24] L. T. Hoa and N. D. Tam, *On some invariants of a mixed product of ideals*, Arch. Math. **94**(2010), 327-337.
[25] L. T. Hoa and T.N. Trung, *Partial Castelnuovo-Mumford regularities of sums and intersections of powers of monomial ideals*, Math. Proc. Cambridge Phil. Soc. **149**(2010), 229-246.
[26] S. L. Kleiman, *Les théorèmes de finitude pour le foncteur de Picard* , in: "Théorie des intersections et théorème de Riemann-Roch" (French) Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), pp. 616–666. Lecture Notes in Mathematics, Vol. **225**. Springer-Verlag, Berlin-New York, 1971
[27] V. Kodiyalam, *Asymptotic behaviour of Castelnuovo-Mumford regularity*, Proc. Amer. Math. Soc. **128**(2000), 407-411.
[28] J. Koh, *Ideals generated by quadrics exhibiting double exponential degrees*, J. Algebra **200**(1998), 225245.
[29] S. Kwak, *Castelnuovo regularity for smooth subvarieties of dimension 3 and 4*, J. Algebraic Geom. **7**(1998), 195-206.
[30] R. Lazarsfeld, *A sharp Castelnuovo bound for smooth surfaces*, Duke Math. J. **55**(1987), 423–429.
[31] C.H. Linh, *Upper bound for Castelnuovo-Mumford regularity of associated graded modules*, Comm. Algebra. **33** (2005), 1817-1831.
[32] J. McCullough, I. Peeva, *Counterexamples to the Eisenbud-Goto regularity conjecture*, J. Amer. Math. Soc., DOI: 10.1090/jams/891.
[33] E. Mayr and A. Meyer, *The complexity of the word problem for commutative semigroups and polynomial ideals*, Adv. Math. **46** (1982), 305329.
[34] C. Miyazaki, W. Vogel and K. Yanagawa, *Associated primes and arithmetic degrees*, J. Algebra **192**(1997), no. 1, 166–182.
[35] D. Mumford, Lectures on curves on an algebraic surface, Princeton Univ. Press, Priceton, NJ, 1966.

[36] U. Nagel, *Comparing Castelnuovo-Mumford regularity and extended degree: the borderline cases*, Trans. Amer. Math. Soc. **357**(2005), 3585-3603.
[37] I. Peeva and B. Sturmfels, *Syzygies of codimension* 2 *lattice ideals*, Math. Z. **229**(1998), 163–194.
[38] Z. Ran, *Local differential geometry and generic projections of threefolds*, J. Differential Geom. **32**(1990), 131–137.
[39] Rossi, M. E., Valla, G.: Hilbert functions of filtered modules. Lecture Notes of the Unione Matematica Italiana, **9**. Springer, Heidelberg (2010).
[40] M. E. Rossi, N.V. Trung and G. Valla, *Castelnuovo-Mumford regularity and extended degree*, Trans. Amer. Math. Soc. **355**(2003), 1773–1786.
[41] P. Schenzel, *On the use of local cohomology in Algebra and Geometry.* In: Six lectures on commutative algebra (Bellaterra, 1996), pp. 241-292, Progr. Math., 166, Birkhäuser, Basel, 1998.
[42] J. Stückrad and W. Vogel, *Castelnuovo bounds for certain subvarieties in* $\mathbb{P}^n$, Math. Ann. **276**(1987), 341–352.
[43] Srinivas, V., Trivedi, V.: *A finiteness theorem for the Hilbert functions of complete intersection local rings.* Math. Z. **225**, 543 - 558 (1997).
[44] Srinivas, V., Trivedi, V.: *On the Hilbert function of a Cohen-Macaulay local ring.* J. Algebraic Geom. **6**, 733 - 751 (1997).
[45] Trivedi, V.: *Hilbert functions, Castelnuovo-Mumford regularity and uniform Artin-Rees numbers.* Manuscripta Math. **94**, 485 - 499 (1997).
[46] N.V. Trung, *Reduction exponent and degree bound for the defining equations of graded rings*, Proc. Amer. Math. Soc. **101**(1987), 229–236.
[47] N.V. Trung, *The Castelnuovo regularity of the Rees algebra and the associated graded ring*, Trans. Amer. Math. Soc. **350**(1998), 2813–2832.
[48] N.V. Trung, *The largest non-vanishing degree of graded local cohomology modules*, J. Algebra **215** (1999), 481-499.
[49] N. V. Trung, *Gröbner bases, local cohomology and reduction number*, Proc. Amer. Math. Soc. **129**(2001), no. 1, 9–18.
[50] N.V. Trung and H-J. Wang, *On the asymptotic linearity of Castelnuovo-Mumford regularity.* J. Pure Appl. Alg. **201**(2005), 42-48.
[51] W. V. Vasconcelos, Computational methods in commutative algebra and algebraic geometry. With chapters by David Eisenbud, Daniel R. Grayson, Jürgen Herzog and Michael Stillman. Algorithms and Computation in Mathematics, 2. Springer-Verlag, Berlin, 1998.

Institute of Mathematics, VAST, 18 Hoang Quoc Viet, 10307 Hanoi, Viet Nam
E-mail address: lthoa@math.ac.vn