Relative homological algebra in categories of representations of quivers
A general overview of the Theory of Covers and Envelopes

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• R. Baer, Abelian groups which are direct summands of every containing group, *Bull. Amer. Math. Soc.* **46** (1940), 800-806.
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General definition of covers and envelopes with respect to a certain class $\mathcal{F}$:
A abelian category and $\mathcal{F}$ a class of objects closed under isomorphisms.

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\downarrow \varphi' & & \downarrow \\
F & & M
\end{array}
\]

can be completed commutatively, with \( F' \in \mathcal{F} \).

If \( \varphi \circ f = \varphi \Rightarrow f \) is an automorphism, \( \varphi \) is an \( \mathcal{F} \)-cover.

\( \mathcal{F} \)-(pre)envelopes are defined in a dual manner.

\( \mathcal{F} \): the class of flat modules, \( \mathcal{F} \)-cover = flat cover.

\( \mathcal{E} \): the class of injective modules, \( \mathcal{E} \)-envelope = injective envelope.
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\( \Rightarrow \)

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$\mathcal{F}$: the class of flat modules, $\mathcal{F}$-cover=flat cover.

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**Flat cover conjecture:** “Every module over an associative ring has a flat cover”.

It was known to be true:

- For modules over a left perfect ring.
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Before of the later resolution to the conjecture, the most significant advance was obtained by:

Positive solution to the conjecture:


Enochs’ proof:

L. Salce, Cotorsion theories for abelian groups, Symposia Mathematica, Vol. 23 (1979), 11-32.


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Quasi-coherent sheaves?

- Existence theorems of covers in categories without enough projectives.

- $\mathcal{Qco}(X)$ is locally $\kappa$-presentable.
For a given $\mathcal{F}$ in $\mathcal{A}$,

$$\mathcal{F}^\perp = \{ C \in \text{Ob}(\mathcal{A}) : \text{Ext}^1(F, C) = 0, \forall F \in \mathcal{F} \}.$$

Analogously $\perp \mathcal{F}$ will denote

$$\perp \mathcal{F} = \{ G \in \text{Ob}(\mathcal{A}) : \text{Ext}^1(G, D) = 0, \forall D \in \mathcal{F} \}.$$

The pair $(\mathcal{F}, \mathcal{F}^\perp)$ is cogenerated by a set $T$ if:

$$C \in \mathcal{F}^\perp \iff \text{Ext}^1(F, C) = 0 \ \forall F \in T.$$

**Definition**

$(\mathcal{F}, C)$ is a cotorsion theory if

$\mathcal{F}^\perp = C$ and $\perp C = \mathcal{F}$. 
1. The pair of classes

$$(\mathcal{Proj}, R\text{-Mod}) \text{ and } (R\text{-Mod}, \text{Inj})$$

are cotorsion theories, $\mathcal{Proj}$ the class of projective modules and $\text{Inj}$ the class of injective modules. $(R\text{-Mod}, \text{Inj})$ is cogenerated by the set $\{R/I: I \leq_R R\}$.

2. The pair $(\mathcal{F}, \mathcal{C})$ composed by flat modules and cotorsion modules (flat cotorsion theory).
Theorem

Let $\mathcal{A}$ be a Grothendieck category and $\mathcal{F}$ a class of objects of $\mathcal{A}$ closed under direct sums, extensions and well ordered direct limits. $(\mathcal{F}, \mathcal{F}^\perp)$ cogenerated by a set $\Rightarrow \forall M \in \text{Ob}(\mathcal{A})$ there exists an $\mathcal{F}$-cover and an $\mathcal{F}^\perp$-envelope.
Proof: Every object in a Grothendieck category is small.

Lemma

There exists an ordinal number $\lambda$ such that $\forall \lambda' \geq \lambda$ and for every well ordered inductive system $(E_\alpha, f_{\beta\alpha})$, $\alpha < \lambda'$ of injective objects of $C$, $\lim_{\alpha < \lambda'} E_\alpha$ is also injective $\forall \lambda' \geq \lambda$.

Proposition

For every object $M \in \text{Ob}(C)$ there exists an ordinal number $\lambda$ such that $\forall \lambda' \geq \lambda$

$$\text{Ext}^n(M, \lim_{\alpha < \lambda'} M_\alpha) \cong \lim_{\alpha < \lambda'} \text{Ext}^n(M, M_\alpha),$$

for every w. o. inductive system $(M_\alpha, f_{\beta\alpha})$, $\alpha < \lambda'$ in $C$. 
There is another proof without using homological methods, by means of “The Small Object Argument”

by using a more general version which appears in


So in any case everything reduces to prove that the pair \((\mathcal{F}, \mathcal{F}^\perp)\) is cogenerated by a set.
Lemma (Eklof)

Let $\mathcal{A}$ be an abelian category with direct limits and $A, C$ objects of $\mathcal{A}$. Let us suppose that

1. $A = \bigcup_{\alpha<\lambda} A_\alpha$ for an ordinal number $\lambda$
2. $\text{Ext}^1(A_0, C) = 0$ and $\text{Ext}^1(A_{\alpha+1}/A_\alpha, C) = 0$ for every $\alpha < \lambda$.

Then $\text{Ext}^1(A, C) = 0$.

Proposition

Let $F \in \mathcal{F}$ and $x \in F$. Suppose there exists a cardinal $\aleph$ and $S \subseteq F$ such that

1. $x \in S$, $|S| \leq \aleph$
2. $S, F/S \in \mathcal{F}$.

Then the pair $(\mathcal{F}, \mathcal{F}^\perp)$ is cogenerated by a set.
Representations of quivers

A quiver $Q$ is a directed graph.

A path $p$ of $Q$ is a sequence of arrows. If $t(p) = i(q)$ we get the path $qp$.

$P(Q)_v$, (left) path tree associated to $Q$ with root in $v$: is a quiver whose vertices are the paths $p$ of $Q$ beginning in $v$ and the arrows the pairs $(p, ap)$.

A representation by modules of $Q$ is a functor $X : Q \to R$-Mod. A morphism between $X$ and $Y$ is a natural transformation.

$(Q, R$-Mod$)$ is the family of representations by modules of a quiver $Q$, it is a Grothendieck category with enough projectives.
$RQ$ (path algebra of $Q$) is the free $R$-module whose base are the paths $p$ of $Q$, and

$$q \cdot p = \begin{cases} 
q p & \text{if } t(p) = i(q) \\
0 & \text{in other case}
\end{cases}$$

If $Q$ has a finite number of vertices, $RQ$ has identity

$$1 = v_1 + \cdots + v_n,$$

if not is a ring with local units. 

The categories $(Q, R\text{-Mod})$ and $RQ\text{-Mod}$ are equivalent.
Quasi-coherent $\mathcal{R}$-modules

$Q = (V, E)$ is a quiver.
$\mathcal{R}$ is a representation of $Q$ in the category of rings, that is, for each $v \in V$ we have a ring $\mathcal{R}(v)$ and for an arrow $a : v \to w \in E$ a homomorphism of rings

$$\mathcal{R}(a) : \mathcal{R}(v) \to \mathcal{R}(w).$$

An $\mathcal{R}$-module $M$ is an $\mathcal{R}(v)$-module $M(v)$ for $v \in V$ and an $\mathcal{R}(v)$-morphism $M(a) : M(v) \to M(w)$ for an arrow $a : v \to w$. 
$M$ is quasi-coherent if for each edge $a$ the morphism

$$\mathcal{R}(w) \otimes_{\mathcal{R}(v)} M(v) \to M(w)$$

is an $\mathcal{R}(w)$-isomorphism.

The category of quasi-coherent $\mathcal{R}$-modules is cocomplete with exact direct limits and abelian if $\mathcal{R}(w)$ is a flat $\mathcal{R}(v)$-module for $v \to w$. 
Consider quasi-coherent sheaves over \((X, \mathcal{O}_X)\).

If \(\mathcal{U}\) are the affine opens \(U \subseteq X\), a quasi-coherent sheaf over \((X, \mathcal{O}_X)\)
(or a quasi-coherent \(\mathcal{O}_X\)-module) is uniquely determined by
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- An \(O(U)\)-module \(M_U\) for each \(U\)

\[\text{Maps } M_U \to M_V \text{ if } V \subseteq U, V, U \in U\]

\[\text{The compatibility condition, if } W \subseteq V \subseteq U, (W, V, U) \in U, \text{ then } M_U \circ @ R M_V - M_W \text{ is commutative.}\]
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- An \(O(U)\)-module \(M_U\) for each \(U\)
- Maps \(M_U \rightarrow M_V\) if \(V \subseteq U\), \(V, U \in \mathcal{U}\) satisfying

\[
\begin{align*}
\text{i) } & \quad & O(V) \otimes O(U) M_U & \rightarrow M_V \\
\text{ii) } & \quad & \text{The compatibility condition, if } W \subseteq V \subseteq U, \quad (W, V, U \in \mathcal{U}), \text{ then } M_U \otimes O(W) & \rightarrow M_V - M_W \text{ is commutative.}
\end{align*}
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i) \(O(V) \otimes_{O(U)} M_U \rightarrow M_V\) is an isomorphism \(\forall V \subseteq U\)
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  ii) The compatibility condition,
if \(W \subseteq V \subseteq U\), \((W, V, U \in \mathcal{U})\), then

\[
\begin{array}{ccc}
M_U & \longrightarrow & M_V \\
\downarrow & & \downarrow \\
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\end{array}
\]

is commutative.
Proposition

For $M$ a right $R$-module and $N$ a left $R$-module, the tensor product $M \otimes_R N$ is the $\mathbb{Z}$-module

$$(\mathbb{Z}(v) = \mathbb{Z}, \forall v \in V \text{ and } \mathbb{Z}(a) = id_{\mathbb{Z}} \text{ for all } a \in E)$$

such that

$$M \otimes_R N(v) = M(v) \otimes_R N(v),$$

with $M \otimes_R N(a)$ the obvious map.

$\diamond R F$ is flat $\overset{\text{def}}{\iff} \quad - \otimes_R F$ is exact.

$\diamond F$ will denote the class of all flat quasi-coherent $R$-modules.

**Proposition**

\( Q = (V, E) \) is a quiver, and \( M \) a quasi-coherent \( R \)-module over \( Q \).
Let \( \kappa \) be an infinite cardinal such that \( \kappa \geq |\mathcal{R}(v)| \) \( \forall v \in V \) and \( \kappa \geq |E|, |V| \).
Let \( X_v \subseteq M(v) \) be subsets with \( |X_v| \leq \kappa \) \( \forall v \in V \).
There exits a quasi-coherent submodule \( M' \subseteq M \) with
\begin{enumerate}
  \item \( M'(v) \subseteq M(v) \) pure, \( \forall v \in V \)
  \item \( X_v \subseteq M'(v), \forall v \in V \) and
  \item \( |M'| \leq \kappa \).
\end{enumerate}
Corollary (Gabber)

The category of quasi-coherent \( \mathcal{R} \)-modules is locally \( \kappa \)-presentable.

Theorem

Every quasi-coherent \( \mathcal{R} \)-module has a flat cover and a cotorsion envelope.

Corollary

For a given scheme \( (X, O_X) \), every quasi-coherent sheaf on \( O_X \) admits a flat cover and a cotorsion envelope.
Gorenstein categories

\( \mathcal{A} \) Grothendieck category.

**Definition**

\[ X \in Ob(\mathcal{A}) \]

\[ pd \ X \leq n \Leftrightarrow Ext^i(X, -) = 0 \text{ for } i \geq n + 1. \]

\[ FPD(\mathcal{A}) = \sup\{pd \ X : \forall X, \ pd \ X < \infty\} \]

\[ FID(\mathcal{A}) = \sup\{id \ X : \forall X, \ id \ X < \infty\}. \]

**Definition**

We will say that \( \mathcal{A} \) is a Gorenstein category if the following hold:

1) For any object \( L \) of \( \mathcal{A} \), \( pd \ L < \infty \) if and only if \( id \ L < \infty \).

2) \( FPD(\mathcal{A}) < \infty \) and \( FID(\mathcal{A}) < \infty \).

3) \( \mathcal{A} \) has a generator \( L \) such that \( pd \ L < \infty \).
Definition

*E is Gorenstein injective if there exists an exact complex*

\[ \cdots \rightarrow E_{-1} \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \]

*such that* \( E = \text{Ker} \ (E_0 \rightarrow E_1) \) *and it is*

\( \text{Hom}(U, -) \)-exact for all injective *U.*

*Dually we define Gorenstein projective objects.*

Definition

*Gpd*(X) = *n* *if the first syzygy of* X *that is Gorenstein projective is the* *n*-th one *and* Gpd(X) = \( \infty \) *if there is no such syzygy.*

\[ \text{glGpd}(\mathcal{A}) = \sup \{ \text{Gpd}(X) : X \in \text{Ob}(\mathcal{A}) \} \]

*Then also define Gid(Y) and glGid(\mathcal{A}).*
\[ \mathcal{L} = \{ X \in \text{Ob}(\mathcal{A}) : \text{id} X < \infty \} \]

**Theorem**

*If \((\mathcal{A}, \mathcal{L})\) is a Gorenstein category then*

1. \((\mathcal{L}, \mathcal{L}^\perp)\) is a cotorsion theory.
2. \(\mathcal{L}^\perp\) is the class of Gorenstein injective objects of \(\mathcal{A}\).
3. *For Each* \(M \in \text{Ob}(\mathcal{A})\) *has a special* \(\mathcal{L}\)-*precover and a special* \(\mathcal{L}^\perp\)-*preenvelope (so \((\mathcal{L}, \mathcal{L}^\perp)\) *is a complete and hereditary cotorsion theory*).

*If \(\text{FID}(\mathcal{A}) = n\) then \(\text{Gid}(Y) \leq n\) for all objects \(Y\) of \(\mathcal{A}\).*
**Theorem**

If \((\mathcal{A}, \mathcal{L})\) be a Gorenstein category of dimension at most \(n\) having enough projectives. Then for an object \(C\) of \(\mathcal{A}\) the following are equivalent:

1) \(C\) is an \(n\)-th syzygy.
2) \(C \in \perp \mathcal{L}\).
3) \(C\) is Gorenstein projective.

As a consequence we get that \(\text{gl}\text{Gpd}(\mathcal{A}) \leq n\) and that \((\perp \mathcal{L}, \mathcal{L})\) is a complete hereditary cotorsion pair.
A Grothendieck category with enough projectives.

**Theorem**

Let $\mathcal{A}$ be a Grothendieck category with enough projectives. Then the following are equivalent:

1) $\mathcal{A}$ is Gorenstein.
2) $\text{glGpd}(\mathcal{A}) < \infty$ and $\text{glGid}(\mathcal{A}) < \infty$.

Moreover, if (1) (or (2)) holds we have

$$\text{FID}(\mathcal{A}) = \text{FPD}(\mathcal{A}) = \text{glGpd}(\mathcal{A}) = \text{glGid}(\mathcal{A}).$$
$\text{Gext}^i$ functors

A Gorenstein category.

- For a given $Y$ there exists a \textit{Gorenstein injective resolution}, that is an exact sequence

$$0 \to Y \to G_0 \to G_1 \to \cdots$$

such that $\text{Hom}(-, G)$ leaves the sequence exact, for all Gorenstein injective $G$.

- We define right derived functors $\text{Gext}^i(X, Y), i \geq 0$ of $\text{Hom}$ by using Gorenstein injective resolutions of $Y$. 
A Gorenstein category with $gl\text{Gid}(\mathcal{A}) = n$.

**Tate cohomology functors** $\hat{\text{Ext}}^i(X, Y), i \in \mathbb{Z}$

- The $n$-th cosyzygy of $Y, G$, is Gorenstein injective, so there exists

  $$E : \cdots \to E^{-1} \to E^0 \to E^1 \to \cdots$$

  with $G = \text{Ker}(E^0 \to E^1)$, such that

  $$\text{Hom}(U, E)$$

  is exact, $U$ injective.

- $E$ is unique up to homotopy

  $$\hat{\text{Ext}}^i(X, Y) \overset{\text{def}}{=} i\text{-th cohomology groups of } \text{Hom}(X, E)$$


**Proposition**

If \( \mathcal{A} \) is a Gorenstein category of dimension at most \( n \) then for all objects \( X \) and \( Y \) of \( \mathcal{A} \) there exist natural exact sequences

\[
0 \to \text{Gext}^1(X, Y) \to \text{Ext}^1(X, Y) \to \text{Ext}^1(X, Y) \to \text{Gext}^2(X, Y) \to \cdots \to \text{Gext}^n(X, Y) \to \text{Ext}^n(X, Y) \to \text{Ext}^n(X, Y) \to 0.
\]
$X \subseteq \mathbb{P}^n(A)$ closed subscheme.
$X$ locally Gorenstein scheme (so $\mathcal{R}(\nu)$ is Gorenstein ring, i.e.,
commutative noetherian and $\text{id } \mathcal{R}(\nu) < \infty, \forall \nu$).

**Theorem**

$\mathcal{Qco}(X)$ is a Gorenstein category.
Projective and injective model structure on Gorenstein categories

**Theorem**

(Hovey). If \((\mathcal{A}, \mathcal{L})\) is a Gorenstein category then there is a cofibrantly generated model structure on \(\mathcal{A}\) with \(\mathcal{L}\) the full subcategory of trivial objects and such that the fibrant objects are the Gorenstein injective objects.

If \(\mathcal{A}\) has enough projectives then there is a cofibrantly generated model structure on \(\mathcal{A}\) with \(\mathcal{L}\) the trivial objects and such that the cofibrant objects are the Gorenstein projective objects.