2011 Institute for Research in Fundamental Sciences – IPM

Model category structures arising from Drinfeld vector bundles

based on a joint work with P. Guil, M. Prest and J. Trlifaj

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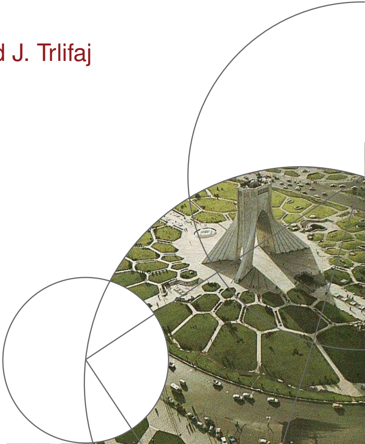

Outline

1. Motivation: Drinfeld's question
   - Infinite-dimensional vector bundles on a scheme
2. The classes of modules involved
   - Closure properties
   - Quasi-coherent sheaves via compatible systems
   - Particular instances
3. Model category structures on $\mathrm{Ch}(\mathfrak{Q}co(X))$
   - Model Categories
   - Cotorsion pairs
   - Distinguished classes of complexes
   - General Theorem
   - Main applications
   - Deconstructibility, and the case of Drinfeld vector bundles

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Finite dimensional vector bundles on the affine scheme $\mathrm{Spec}(R)$ *correspond 1-1 to finitely generated projective* R*-modules.*

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What is the correct definition of an (infinite-dimensional) vector bundle on a scheme X?

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Replace 'projective' by 'flat'.
Then $\mathfrak{F}$ is the flat quasi-coherent sheaf.

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$$M \otimes_R \prod_{i \in I} M_i \to \prod_{i \in I} M \otimes_R M_i$$

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- $\mathcal{D} = \mathcal{F} \cap \mathcal{L}$ the class of all flat Mittag-Leffler modules. ("projective modules with a human face")

Drinfeld's main adventages for considering Flat Mittag-Leffler modules

- "Unlike projectivity, the property of M being a flat Mittag–Leffler module is a first-order property (in the sense of mathematical logic) of $R^{(\mathbb{N})} \otimes_R M$ viewed as a module over $\mathrm{End}_R R^{(\mathbb{N})}$".

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  without AC, projectivity is not equivalent to being a direct summand of a free module.

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- [Kaplansky'1958] The class $\mathcal{P}$ is classifiable: Each module in $\mathcal{P}$ is a direct sum of countably generated modules.
- The class $\mathcal{F}$ is not classifiable in case R is not right perfect (e.g., $\mathcal{F}$ = all torsion-free groups in case $R = \mathbb{Z}$).

The 'sandwich' class $\mathcal{D}$

Theorem (Raynaud-Gruson'1971)

Let R be a ring and M a module. Then the following are equivalent:

- *M is a flat Mittag-Leffler module (i.e., $M \in \mathcal{D}$).*
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Example (Azumaya-Facchini'1989)

$\mathcal{D}$ is the class of $\aleph_1$-free groups for $R = \mathbb{Z}$.
(A group is $\aleph_1$-free if each of its countable subgroups is free).

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Moreover, $\mathcal{D}$ and $\mathcal{F}$ are closed under pure submodules.

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We identify qc sheaves on X with compatible systems of $R(v)$-modules, $\mathcal{M} = (M(v) |\ v \in V)$, where $R(v) = \mathcal{O}_X(v)$, and systems of $R(u)$-morphisms $M(u) \to M(v)$ for $u \to v \in E$, satisfying

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3. ***Lifting Properties**: each trivial cofibration (resp. cofibration) has the left lifting property with respect to fibrations (resp. trivial fibrations).*
4. ***Factorizations**:*

$$f = \beta(f)\alpha(f) \begin{cases} \beta(f) \text{ triv. fib.} \\ \alpha(f) \text{ cof.} \end{cases} \qquad f = \delta(f)\gamma(f) \begin{cases} \delta(f) \text{ fib.} \\ \gamma(f) \text{ triv. cof.} \end{cases}$$

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Then we can identify $\mathcal{D}(\mathcal{A})$ and $\mathrm{Ho}(\mathcal{M})$.

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- Many results constructing model structures in $\mathrm{Ch}(\mathcal{A})$.

Model structures $\Leftrightarrow$ Cotorsion pairs

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- **Cotorsion pair cogenerated by** $\mathcal{X}$: $({}^{\perp}(\mathcal{X}^{\perp}), \mathcal{X}^{\perp})$.
- **Complete cotorsion pairs**: $(\mathcal{C}, \mathcal{C}^{\perp})$ is complete if for each $M \in \mathcal{A}$ there exist short exact sequences

$$0 \longrightarrow L \longrightarrow D \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow M \longrightarrow L' \longrightarrow D' \longrightarrow 0$$

with $D, D' \in \mathcal{C}$ and $L, L' \in \mathcal{C}^{\perp}$.

Hovey's Theorem

Theorem (Hovey, 2002)

Let $\mathcal{F}$, $\mathcal{W}$ and $\mathcal{C}$ be classes in $\mathcal{A}$ such that:

1. *$\mathcal{W}$ is thick, that is, it is closed under retracts, and whenever two out of three entries in a short exact sequence are in $\mathcal{W}$, so is the third.*
2. *$(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are complete cotorsion pairs (e.g. if they are cogenerated by a set).*

The there is a unique model structure on $\mathcal{A}$ such that $\mathcal{C}$ is the class of cofibrant objects, $\mathcal{F}$ is the class of fibrant objects and $\mathcal{W}$ is the class of trivial objects.

Complexes in $\mathcal{A}$

- Given a complex (X, d)

$$\cdots \xrightarrow{d_{n+2}} X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots$$

we shall denote

- $Z_nX = \mathrm{Ker}\ d_n$ ($n+1$*th*-syzygy);
- $B_nX = \mathrm{Im}\ d_{n+1}$;
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- **The** ***Hom*****-complex**: $Hom(X, Y)$
  - $Hom(X, Y)_n = \prod_{k\in\mathbb{Z}} Hom(X_k, Y_{k+n})$
  - $(fd_n^H)_k = f_k d_{k+n}^Y - (-1)^n d_k^X f_{k-1}$.

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- **Exact complexes**: $\mathcal{E}$.

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- Dually we can define the classes $\widetilde{\mathcal{C}}$ and $dg\,\widetilde{\mathcal{C}^{\perp}}$ of $\mathcal{C}$-complexes and dg-$\mathcal{C}^{\perp}$ complexes of objects in $\mathcal{A}$.

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(The Hill Lemma was primarily used as a preparatory tool for applications of Shelah's Singular Compactness Theorem).

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Further cases

[E.-Guil-Prest-Trlifaj] 'Restricted' Drinfeld vector bundles, etc.

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The homotopy theory tools above apply to vector bundles and flat qc sheaves, but **not** *to Drinfeld vector bundles.*

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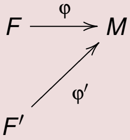

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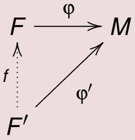

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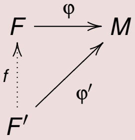

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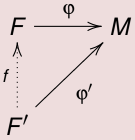

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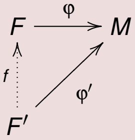

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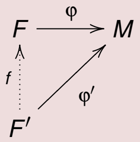

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Theorem (Eklof-Shelah'03,E.-Guil-Prest-Trlifaj)

The class $\mathcal{D}$ (= all $\aleph_1$-free abelian groups) is not precovering for $R = \mathbb{Z}$.