Model category structures arising from Drinfeld vector bundles
based on a joint work with P. Guil, M. Prest and J. Trlifaj

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Outline

1 Motivation: Drinfeld’s question
   Infinite-dimensional vector bundles on a scheme

2 The classes of modules involved
   Closure properties
   Quasi-coherent sheaves via compatible systems
   Particular instances

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   Cotorsion pairs
   Distinguished classes of complexes
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   Deconstructibility, and the case of Drinfeld vector bundles
Definition of infinite-dimensional vector bundle

**Theorem (Serre’1955)**

*Finite dimensional vector bundles on the affine scheme Spec(\(R\)) correspond 1-1 to finitely generated projective R-modules.*
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**Drinfeld’2006**


What is the correct definition of an (infinite-dimensional) vector bundle on a scheme *X*?
Proposed answers

Drinfeld’2006

Let $X$ be a scheme. A quasi-coherent sheaf of $O_X$-modules $\mathcal{F}$ is a vector bundle on $X$ if for each open affine subset $\text{Spec}(R) \subseteq X$, the $R$-module of sections $\Gamma(\text{Spec}(R), \mathcal{F})$ is projective.

'Slightly different definition' [Drinfeld’06, E.-Guil-Prest-Trlifaj]

Replace 'projective' by 'flat Mittag-Leffler' above.

Then $\mathcal{F}$ is called the Drinfeld vector bundle.

Gillespie’2007

Replace 'projective' by 'flat'.

Then $\mathcal{F}$ is the flat quasi-coherent sheaf.
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Let $X$ be a scheme. A quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is a **vector bundle** on $X$ if for each open affine subset $\text{Spec}(R) \subseteq X$, the $R$-module of sections $\Gamma(\text{Spec}(R), \mathcal{F})$ is projective.

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The classes of modules involved

- $\mathcal{P}$: the class of all projective modules ($= \text{direct summands of free modules}$ = the modules $P$ such that the functor $\text{Hom}_R(P,-)$ is exact).

- $\mathcal{F}$: the class of all flat modules ($= \text{direct limits of free modules}$ = the modules $F$ such that the functor $F \otimes_R -$ is exact).

- $\mathcal{L}$: the class of all Mittag-Leffler modules, i.e., the modules $M$ such that the canonical map $M \otimes_R \prod_{i \in I} M_i \to \prod_{i \in I} M \otimes_R M_i$ is monic, for each family of $R$-modules $(M_i | i \in I)$.

- $\mathcal{D} = \mathcal{F} \cap \mathcal{L}$: the class of all flat Mittag-Leffler modules. ("projective modules with a human face")
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Drinfeld’s main advantages for considering Flat Mittag-Leffler modules

- “Unlike projectivity, the property of $M$ being a flat Mittag–Leffler module is a first-order property (in the sense of mathematical logic) of $R^{(\mathbb{N})} \otimes_R M$ viewed as a module over $\text{End}_R R^{(\mathbb{N})}$.”
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without AC, projectivity is not equivalent to being a direct summand of a free module.
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- $\mathcal{P} \subseteq \mathcal{D} \subseteq \mathcal{F}$.

- If any two of these classes coincide, then all coincide. This happens iff $R$ is a right perfect ring (i.e., $R$ has dcc on principal left ideals).

- [Kaplansky’1958] The class $\mathcal{P}$ is classifiable: Each module in $\mathcal{P}$ is a direct sum of countably generated modules.

- The class $\mathcal{F}$ is not classifiable in case $R$ is not right perfect (e.g., $\mathcal{F} = \text{all torsion-free groups in case } R = \mathbb{Z}$).
The ’sandwich’ class $\mathcal{D}$

**Theorem (Raynaud-Gruson’1971)**

Let $R$ be a ring and $M$ a module. Then the following are equivalent:

- $M$ is a flat Mittag-Leffler module (i.e., $M \in \mathcal{D}$).
- Every finite (or countable) subset of $M$ is contained in a countably generated projective submodule which is pure in $M$. 

Example (Azumaya-Facchini’1989)

$\mathcal{D}$ is the class of $\aleph_1$-free groups for $R = \mathbb{Z}$.

(A group is $\aleph_1$-free if each of its countable subgroups is free).
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**Definition**

Let $\mathcal{C}$ be a class in a Grothendieck category. $M$ is a **transfinite extension** of objects in $\mathcal{C}$.
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Moreover, $\mathcal{D}$ and $\mathcal{F}$ are closed under pure submodules.
Quasi-coherent sheaves via compatible systems

Theorem (Enochs, E.’2005)

There is an equivalence between the category $\mathcal{Qco}(X)$, and the category of certain representations of the quiver $Q_X = (V, E)$, where

1. $R(v) \otimes R(u) M(u) \rightarrow M(v)$ is an isomorphism ($u \rightarrow v \in E$) and
2. if $w \subseteq v \subseteq u$ in $V$, then $M(u) \rightarrow M(v) \rightarrow M(w)$ is commutative.
Quasi-coherent sheaves via compatible systems

**Theorem (Enochs, E.'2005)**

There is an equivalence between the category $\mathcal{QC}(X)$, and the category of certain representations of the quiver $Q_X = (V, E)$, where $V$ are all open affine subsets of $X$, and there is an arrow $u \to v$ iff $v \subseteq u$. We identify qc sheaves on $X$ with compatible systems of $R(v)$-modules, $M = (M(v) | v \in V)$, where $R(v) = O_X(v)$, and systems of $R(u)$-morphisms $M(u) \to M(v)$ for $u \to v \in E$, satisfying

- $R(v) \otimes R(u) M(u) \to M(v)$ is an isomorphism ($u \to v \in E$) and
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**Flat quasi-coherent sheaves:** $\mathcal{M}$ consists of flat modules.
Model Categories
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Definition (Quillen)

A model category $C$ has three classes of morphisms called \textit{fibrations}, \textit{cofibrations} and \textit{weak equivalences}:

1. \textbf{2-out-of-3}: if two of $f$, $g$, $fg$ are weak equivalences, then so does the third.
2. \textbf{Retracts}: the classes are closed under retracts.
3. \textbf{Lifting Properties}: each trivial cofibration (resp. cofibration) has the left lifting property with respect to fibrations (resp. trivial fibrations).
4. \textbf{Factorizations}: $f = \beta(\alpha(f))$ and $f = \delta(\gamma(f))$. 

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    f = \beta(f)\alpha(f) \left\{ \begin{array}{l} \beta(f) \text{ triv. fib.} \\ \alpha(f) \text{ cof.} \end{array} \right. \\
    f = \delta(f)\gamma(f) \left\{ \begin{array}{l} \delta(f) \text{ fib.} \\ \gamma(f) \text{ triv. cof.} \end{array} \right.
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Application of Model Categories

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**NOTATION**

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\text{Ch}(\mathcal{A}) \quad \text{category of unbounded complexes.}
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Then we can identify \( \mathcal{D}(\mathcal{A}) \) and \( \text{Ho}(\mathcal{M}) \).
Model structures in $\text{Ch}(\mathcal{A})$

Conclusion

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- Many results constructing model structures in $\text{Ch}(\mathcal{A})$.

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Model structures $\iff$ Cotorsion pairs
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- **Cotorsion pair cogenerated by** \( \mathcal{X} \): \((^\perp (\mathcal{X}^\perp), \mathcal{X}^\perp)\).
- **Complete cotorsion pairs**: \( (C, C^\perp) \) is complete if for each \( M \in \mathcal{A} \) there exist short exact sequences
  \[
  0 \longrightarrow L \longrightarrow D \longrightarrow M \longrightarrow 0
  \]
  and
  \[
  0 \longrightarrow M \longrightarrow L' \longrightarrow D' \longrightarrow 0
  \]
  with \( D, D' \in C \) and \( L, L' \in C^\perp \).
Hovey’s Theorem

Theorem (Hovey, 2002)

Let $\mathcal{F}$, $\mathcal{W}$ and $\mathcal{C}$ be classes in $\mathcal{A}$ such that:

1. $\mathcal{W}$ is thick, that is, it is closed under retracts, and whenever two out of three entries in a short exact sequence are in $\mathcal{W}$, so is the third.

2. $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are complete cotorsion pairs (e.g. if they are cogenerated by a set).

The there is a unique model structure on $\mathcal{A}$ such that $\mathcal{C}$ is the class of cofibrant objects, $\mathcal{F}$ is the class of fibrant objects and $\mathcal{W}$ is the class of trivial objects.
Complexes in $\mathcal{A}$

- Given a complex $(X, d)$

$$
\cdots \xrightarrow{d_{n+2}} X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{} \cdots
$$

we shall denote

- $Z_nX = \ker d_n$ ($n+1$th-syzygy);
- $B_nX = \text{im } d_{n+1}$;
- $H_nX = \frac{Z_nX}{B_nX}$ ($n$th-homology).
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- **The Hom-complex**: $\text{Hom}(X, Y)$
  - $\text{Hom}(X, Y)_n = \prod_{k \in \mathbb{Z}} \text{Hom}(X_k, Y_{k+n})$
  - $(fd^H_n)_k = f_kd_Y^{k+n} - (-1)^n d_X^k f_{k-1}$. 

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- **Exact complexes:** $\mathcal{E}$. 
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- A complex \(M = (M^n)\) in \(\mathcal{C}(\mathcal{A})\) is a \(dg\)-\(\mathcal{C}\) complex if each \(M \to E\) is nullhomotopic for any complex \(E \in \mathcal{C}^\perp\) and \(M^n \in \mathcal{C}\), for each \(n \in \mathbb{Z}\).
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- Dually we can define the classes $\mathcal{C}$ and $\text{dg} \mathcal{C}^\perp$ of $\mathcal{C}$-complexes and dg-$\mathcal{C}^\perp$ complexes of objects in $\mathcal{A}$. 
Notation and assumptions

- $Q_X = (V, E)$ a quiver associated to a scheme $X$, 

- $\kappa$ an infinite cardinal such that $\kappa \geq |V|$ for all $v \in V$.

- For each $v \in V$, $S_v$ a class of $\leq \kappa$–presented $R(v)$–modules, $F_v = \perp (S_v \perp)$ is resolving, and

- $C$ be the class of all qc sheaves $M$ such that $M(v) \in F_v$ for each $v \in V$.

- Assume that $C$ contains a generator for $Q_X^{co}(X)$. 

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Moreover, if every $M \in S_v$ is a flat $R(v)$-module,
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then **the model category structure is monoidal**.

Comments to the proof:
Key point is to use the Hill Lemma to make compatible all the individual filtrations,
first at the level of qc sheaves and then at the level of complexes of qc.
(The Hill Lemma was primarily used as a preparatory tool for applications of Shelah’s Singular Compactness Theorem).
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First application: Locally projective model structure on $\mathbb{C}(\Omega \mathcal{O}(X))$.

Corollary (Enochs-E.-García Rozas’08; $S_v = \{R(v)\}$)

If $X$ is a scheme having enough vector bundles then there is a monoidal model category structure on $\mathbb{C}(\Omega \mathcal{O}(X))$ such that weak equivalences are homology isomorphisms, and cofibrations are monomorphisms whose cokernels are dg-locally projective complexes of vector bundles.

Corollary (Gillespie’07; $S_v$ = a representative set of $\leq \sup_{v \in V} |R(v)|$-generated flat modules)

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Further cases [E.-Guil-Prest-Trlifaj] ‘Restricted’ Drinfeld vector bundles, etc.
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**Further cases**

[E.-Guil-Prest-Trlifaj ’Restricted’ Drinfeld vector bundles, etc.]

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Why ’restricted’ Drinfeld vector bundles?

**Definition (Eklof)**

A class of modules $\mathcal{C}$ is **deconstructible** in case there is a cardinal $\kappa$ such that each $M \in \mathcal{C}$ is a transfinite extension of $\leq \kappa$-presented modules in $\mathcal{C}$.
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Theorem

The homotopy theory tools above apply to vector bundles and flat qc sheaves, but not to Drinfeld vector bundles.
Precovering classes

**Definition (Enochs)**

A class $\mathcal{F}$ is **precovering** if for each $M$ there is a morphism $\varphi : F \to M$, such that $F \in \mathcal{F}$ and every diagram

\[
F \xrightarrow{\varphi} M
\]

Each class of the form $\bot^E \mathcal{E}$ is precovering provided it is deconstructible. In particular the classes $\mathcal{P}$ and $\mathcal{F}$ are precovering.

**Theorem (Eklof-Shelah'03, E.-Guil-Prest-Trlifaj)**

The class $D(=\text{all } \aleph_1\text{-free abelian groups})$ is not precovering for $R=\mathbb{Z}$. 
Precovering classes

Definition (Enochs)

A class $\mathcal{F}$ is precovering if for each $M$ there is a morphism $\varphi : F \to M$, such that $F \in \mathcal{F}$ and every diagram

\[
\begin{array}{ccc}
F & \xrightarrow{\varphi} & M \\
\downarrow & & \downarrow \\
F' & \xrightarrow{\varphi'} & M
\end{array}
\]

can be completed commutatively, with $F' \in \mathcal{F}$.

Each class of the form $\perp_{\mathcal{E}}$ is precovering provided it is deconstructible. In particular the classes $\mathcal{P}$ and $\mathcal{F}$ are precovering.

Theorem (Eklof-Shelah'03, E.-Guil-Prest-Trlifaj)

The class $\mathcal{D}$ ( = all $\aleph_1$-free abelian groups) is not precovering for $\mathcal{R} = \mathbb{Z}$. 
Precovering classes

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A class $\mathcal{F}$ is precovering if for each $M$ there is a morphism $\phi : F \rightarrow M$, such that $F \in \mathcal{F}$ and every diagram

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F' & \xrightarrow{\phi'} & \ \\
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Theorem (Eklof-Shelah'03, E.-Guil-Prest-Trlifaj)

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Precovering classes

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Precovering classes

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A class \( \mathcal{F} \) is precovering if for each \( M \) there is a morphism \( \varphi : F \to M \), such that \( F \in \mathcal{F} \) and every diagram

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Precovering classes

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A class \( \mathcal{F} \) is precovering if for each \( M \) there is a morphism \( \varphi : F \to M \), such that \( F \in \mathcal{F} \) and every diagram

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\downarrow f & & \downarrow \\
F' & \nearrow \varphi' & \\
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The class \( \mathcal{D} = (\text{all } \aleph_1\text{-free abelian groups}) \) is not precovering for \( R = \mathbb{Z} \).
Precovering classes

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