Method of Generating Differentials

I-Chiau Huang
Institute of Mathematics, Academia Sinica

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Method of Generating Functions

For a sequence $a_0, a_1, a_2, \cdots$ of numbers with combinatorial or number theoretic interests, we consider the (ordinary) power series

$$a_0 + a_1 T + a_2 T^2 + \cdots$$

or the (exponential) power series

$$a_0 + a_1 T + a_2 (T^2/2!) + a_3 (T^3/3!) + \cdots$$

in $\mathbb{Q}[[T]]$. For example,

- $\displaystyle\sum_{k=0}^{n} \binom{n}{k} T^k = (1+T)^n$;
- $\displaystyle\sum_{k=0}^{\infty} \binom{n+k}{n} T^k = \frac{1}{(1-T)^{n+1}}$;

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- $\displaystyle\sum_{k=0}^{\infty} \binom{2k}{k} T^k = \frac{1}{\sqrt{1-4T}}$;
- $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k} T^k = -\log(1-T)$;
- Catalan numbers C_i are defined by the power series $\mathbf{C} = \sum C_i T^i$ satisfying $\mathbf{C} = 1 + T\mathbf{C}^2$.
- Bernoulli numbers B_i are defined by

$$\frac{T}{e^T - 1} = \sum_{i=0}^{\infty} B_i \frac{T_i}{i!}.$$

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The power series gives rise to a function defined in certain region of the complex plane. We may perform algebraic operations on these functions. For example,

$$(1+T)^n + T(1+T)^n = (1+T)^{n+1}.$$

To obtain combinatorial information from the functions, there are coefficient functionals. Given

$$f = a_0 + a_1 T + a_2 T^2 + \cdots \in \mathbb{Q}[[T]],$$

we define

$$[T^i]f := a_i.$$

For example,

$$\binom{n+1}{i} = [T^i](1+T)^{n+1} = [T^i](1+T)^n + [T^{i-1}](1+T)^n = \binom{n}{i} + \binom{n}{i-1}.$$

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If the functions are defined in an open set, we can also perform analytic operations. With some mild analytic condition on

$$f = a_0 + a_1 T + a_2 T^2 + \cdots ,$$

we can extract coefficients by integration:

$$\frac{1}{2\pi\sqrt{-1}} \oint \frac{f}{T^{i+1}} dT = a_i$$

For example,

$$\begin{aligned}
\binom{n+1}{i} &= \frac{1}{2\pi\sqrt{-1}} \oint \frac{(1+T)^{n+1}}{T^{i+1}} dT \\
&= \frac{1}{2\pi\sqrt{-1}} \oint \frac{(1+T)^{n}}{T^{i+1}} dT + \frac{1}{2\pi\sqrt{-1}} \oint \frac{(1+T)^{n}}{T^{i}} dT \\
&= \binom{n}{i} + \binom{n}{i-1}.
\end{aligned}$$

Method of Generating Functions

The method of generating functions is enhanced by the Lagrange inversion formula. Let w be a power series in $\kappa[[T]]$ defined by $w = T\phi$ for an invertible power series $\phi \in \kappa[[T]]$. The Lagrange inversion formula asserts

$$[T^n]w(T)^k = \frac{k}{n}[T^{n-k}]\phi(T)^n.$$

In the book "Analytic combinatorics in several variables" by R. Pemantle and M. C. Wilson, a proof of Lagrange inversion is supplied, "*because of the danger that the reader will stumble upon the more common and less illuminating formal power series proof*".

We would like to provide a viewpoint from commutative algebra to the method of generating functions.

Power Series Rings

- Let κ be a field. We consider the ring $\kappa[[X_1, \cdots, X_n]]$. Look at vector spaces first...
- There is no canonical choice of variables for a power series ring over a field κ. For example, with $Y = X/(1 - X)$ or with $X = Y/(1 + Y)$, we have $\kappa[[X]] = \kappa[[Y]]$.
- A power series ring R of n variables over a field κ is a complete regular local ring of Krull dimension n with the coefficient field κ.
- If $X_1, \cdots, X_n$ generate the maximal ideal of R, then $R = \kappa[[X_1, \cdots, X_n]]$.
- The notation $\kappa[[X_1, \cdots, X_n]]$ means a power series ring over κ with variables $X_1, \cdots, X_n$ specified.

Derivations and Differentials

Let R be an algebra over a field κ. A κ-derivation from R to an R-module M is a κ-linear map $\delta\colon R \to M$ satisfying the Leibniz rule:

$$\delta(r_1 r_2) = r_1\delta(r_2) + r_2\delta(r_1), \qquad r_1, r_2 \in R.$$

The universal object among all κ-derivation from R is called the module of differentials of R over κ and is denoted by $\Omega_{R/\kappa}$.

$$\begin{tikzcd}
R \arrow[r, "d"] \arrow[d, "\delta"'] & \Omega_{R/\kappa} \arrow[dl, dashed] \\
M &
\end{tikzcd}$$

Derivations and Differentials

- The module of differentials of R over κ always exists.
- If $R = \kappa[X_1, \cdots, X_n]$, then $\Omega_{R/\kappa}$ is free of rank n.
  Indeed, $\Omega_{R/\kappa} = RdX_1 + \cdots + RdX_n$.
- $\Omega_{\kappa[[X_1,\cdots,X_n]]/\kappa}$ is not finite.

A κ-derivation $\delta\colon R \to M$ is finite, if M is a finite R-module. The universal object among all finite κ-derivation from R is called the module of finite differentials of R over κ and is denoted by $\tilde{\Omega}_{R/\kappa}$.

- The module of finite differentials of $\kappa[[X_1, \cdots, X_n]]$ over κ exists.
- $\tilde{\Omega}_{\kappa[[X_1,\cdots,X_n]]/\kappa}$ is free of rank n with basis $dX_1, \cdots, dX_n$.
- $\wedge^n \tilde{\Omega}_{\kappa[[X_1,\cdots,X_n]]/\kappa} = \kappa[[X_1, \cdots, X_n]] dX_1 \wedge \cdots \wedge dX_n$.

Local Cohomology

Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. We consider the functor from the category of R-modules to itself given by

$$\Gamma_{\mathfrak{a}}(M) := \{m \in M \colon \mathfrak{a}^i m = 0 \text{ for some } i\}.$$

The n-th right derived functor of $\Gamma_{\mathfrak{a}}(-)$ is denoted by $H^n_{\mathfrak{a}}(-)$. If $\mathfrak{a}$ is generate up to radical by $f_1, \cdots, f_n$, we have an exact sequence

$$\oplus_{i=1}^n M_{f_1 \cdots \hat{f}_i \cdots f_n} \to M_{f_1 \cdots f_n} \to H^n_{\mathfrak{a}}(M) \to 0.$$

$$\frac{f_1^{i-i_1} \cdots f_n^{i-i_n} \omega}{(f_1 \cdots f_n)^i} \mapsto \begin{bmatrix} \omega \\ f_1^{i_1}, \cdots, f_n^{i_n} \end{bmatrix}, \qquad \omega \in M \text{ and } i >> 0$$

Local Cohomology

linearity law For $\omega_1, \omega_2 \in M$, $i_1, \cdots, i_n > 0$, and $g_1, g_2 \in R$,

$$\left[\begin{array}{c} g_1\omega_1 + g_2\omega_2 \\ f_1^{i_1}, \cdots, f_n^{i_n} \end{array} \right] = g_1 \left[\begin{array}{c} \omega_1 \\ f_1^{i_1}, \cdots, f_n^{i_n} \end{array} \right] + g_2 \left[\begin{array}{c} \omega_2 \\ f_1^{i_1}, \cdots, f_n^{i_n} \end{array} \right].$$

transformation law Assume that $\mathfrak{a}$ is also generated up to radical by $f_1', \cdots, f_n'$. For $\omega \in M$,

$$\left[\begin{array}{c} \omega \\ f_1, \cdots, f_\ell \end{array} \right] = \left[\begin{array}{c} \det(r_{ij})\,\omega \\ f_1', \cdots, f_\ell' \end{array} \right],$$

if $f_i' = \sum_{j=1}^n r_{ij} f_j$ for $i = 1, \cdots, n$.

vanishing law For $\omega \in M$,

$$\left[\begin{array}{c} \omega \\ f_1^{i_1}, \cdots, f_n^{i_n} \end{array} \right] = 0$$

if and only if $(f_1^{i_1} \cdots f_n^{i_n})^s \omega \in (f_1^{i_1(s+1)}, \cdots, f_\ell^{i_n(s+1)})M$ for some $s \geq 0$.

Residues

Let $R = \kappa[[X_1, \cdots, X_n]]$ and $\mathfrak{a}$ be its maximal ideal. We define the residue map

$$\mathrm{res}_{X_1,\cdots,X_n} : H^n_{\mathfrak{a}}(\wedge^n \Omega_{\tilde{R}/\kappa}) \to \kappa$$

with respect to $X_1, \cdots, X_n$ by

$$\mathrm{res}_{X_1,\cdots,X_n}\left[\begin{array}{c} \sum b_{i_1\cdots i_n} X_1^{i_1}\cdots X_n^{i_n} dX_1 \wedge \cdots \wedge dX_n \\ X_1^{i_1+1}, \cdots, X_n^{i_n+1} \end{array}\right] = b_{i_1\cdots i_n}.$$

Theorem

If $R = \kappa[[X_1, \cdots, X_n]] = \kappa[[Y_1, \cdots, Y_n]]$, then $\mathrm{res}_{X_1,\cdots,X_n} = \mathrm{res}_{Y_1,\cdots,Y_n}$.

The residue map is a pairing for differentials and system of parameters.

Residues

Saalschützs Theorem

Let a and b be positive integers. Let m and n be non-negative integers. Then

$$\sum_{k\geq 0}\binom{a}{m-k}\binom{b}{n-k}\binom{a+b+k}{k}=\binom{a+n}{m}\binom{b+m}{n}.$$

The identity is from a change of variables $\kappa[[X_1, X_2]] = \kappa[[Y_1, Y_2]]$, where

$$\begin{cases} X_1 = Y_1/(1+Y_2), \\ X_2 = Y_2/(1+Y_1). \end{cases}$$

Residues

From the relation

$$\begin{cases} X_1 = Y_1/(1+Y_2) \\ X_2 = Y_2/(1+Y_1), \end{cases}$$

we have

$$\begin{cases} 1+Y_1 = (1+X_1)/(1-X_1X_2) \\ 1+Y_2 = (1+X_2)/(1-X_1X_2). \end{cases}$$

Furthermore,

$$dY_1 \wedge dY_2 = \frac{\partial(Y_1, Y_2)}{\partial(X_1, X_2)} dX_1 \wedge dX_2 = \frac{(1+X_1)(1+X_2)}{(1-X_1X_2)^3} dX_1 \wedge dX_2.$$

Residues

The coefficient of $X_1^m X_2^n \partial(X_1, X_2)/\partial(Y_1, Y_2)$ in the power series $(1+Y_1)^{a-1}(1+Y_2)^{b-1}$ is given by

$$
\begin{aligned}
& \operatorname{res}\left[\begin{array}{c} (1+Y_1)^{a-1}(1+Y_2)^{b-1} dY_1 dY_2 \\ X_1^{m+1}, X_2^{n+1} \end{array}\right] \\
= \; & \operatorname{res}\left[\begin{array}{c} (1+Y_1)^{a+n}(1+Y_2)^{b+m} dY_1 dY_2 \\ Y_1^{m+1}, Y_2^{n+1} \end{array}\right] = \binom{a+n}{m}\binom{b+m}{n}.
\end{aligned}
$$

The residue can be also computed in terms of x. The Saalschützs theorem is recovered from the computation

$$
\operatorname{res}\left[\begin{array}{c} \frac{(1+X_1)^a(1+X_2)^b}{(1-X_1X_2)^{a+b+1}} dX_1 dX_2 \\ X_1^{m+1}, X_2^{n+1} \end{array}\right] = \sum_{k\geq 0}\binom{a}{m-k}\binom{b}{n-k}\binom{a+b+k}{k}.
$$

Lagrange Inversion

Theorem

Let w be a power series in $\kappa[[T]]$ defined by $w = T\phi$ for an invertible power series $\phi \in \kappa[[T]]$. Then

$$[T^n]w(T)^k = \frac{k}{n}[T^{n-k}]\phi(T)^n.$$

The above formula is built into the framework of residue calculus. Note that $\kappa[[T]] = \kappa[[w]]$.

$$\operatorname{res}\left[\begin{array}{c} w^k dT \\ T^{n+1} \end{array}\right] = \frac{1}{n}\operatorname{res}\left[\begin{array}{c} dw^k \\ T^n \end{array}\right] = \frac{1}{n}\operatorname{res}\left[\begin{array}{c} \phi^n dw^k \\ w^n \end{array}\right] = \frac{k}{n}\operatorname{res}\left[\begin{array}{c} \phi^n dw \\ w^{n-k+1} \end{array}\right]$$

Therefore

$$[T^n]w^k = \frac{k}{n}[w^{n-k}]\phi^n.$$

Lagrange Inversion

Catalan numbers C_n are defined by the power series $\mathbf{C} = \sum C_i X^i$ satisfying $\mathbf{C} = 1 + X\mathbf{C}^2$. Let $Y := \mathbf{C} - 1$. Then $\kappa[[X]] = \kappa[[Y]]$. Indeed,

$$X = \frac{\mathbf{C} - 1}{\mathbf{C}^2} = \frac{Y}{(1+Y)^2}.$$

For $n > 0$,

$$\begin{aligned} C_n = \operatorname{res}\left[\begin{array}{c} YdX \\ X^{n+1} \end{array}\right] &= \frac{1}{n}\operatorname{res}\left[\begin{array}{c} dY \\ X^n \end{array}\right] \\ &= \frac{1}{n}\operatorname{res}\left[\begin{array}{c} (1+Y)^{2n}dY \\ Y^n \end{array}\right] = \frac{1}{n}\binom{2n}{n-1}. \end{aligned}$$

Lagrange Inversion

Lagrange-Good formula

Let $\kappa[[X_1, \cdots, X_n]] = \kappa[[Y_1, \cdots, Y_n]]$, where $Y_i = X_i\varphi_i$ for an invertible φ_i. Then

$$\begin{aligned} & \operatorname{res}\left[\begin{array}{c} G dX_1 \cdots dX_n \\ X_1^{i_1+1}, \cdots, X_n^{i_n+1} \end{array}\right] \\ = \ & \operatorname{res}\left[\begin{array}{c} G\varphi_1^{i_1}\cdots\varphi_n^{i_n}\det(\delta_{ij} - \frac{Y_i}{\varphi_i}\frac{\partial\varphi_i}{\partial Y_j}) dY_1 \cdots dY_n \\ Y_1^{i_1+1}, \cdots, Y_n^{i_n+1} \end{array}\right] \end{aligned}$$

$$dX_i = d\frac{Y_i}{\varphi_i} = \sum_{j=1}^{n} \frac{\partial(Y_i/\varphi_i)}{\partial Y_j} dY_j = \frac{1}{\varphi_i}\sum_{j=1}^{n}(\delta_{ij} - \frac{Y_i}{\varphi_i}\frac{\partial\varphi_i}{\partial Y_j}) dY_j.$$

Lagrange Inversion is indeed a phenomenon of changes of variables.

Schauder Bases

A power series ring R over a field κ is a complete metric space.

Definition

A sequence $f_0, f_1, f_2, \cdots \in R$ is a Schauder basis if every element in R can be represented uniquely as $a_0 f_0 + a_1 f_a + a_2 f_2 + \cdots$ for $a_0, a_1, a_2, \cdots \in \kappa$

- Ordinary Schauder basis: $(X^k)_{k \geq 0}$
- Exponential Schauder basis: $(X^k/k!)_{k \geq 0}$, if char $\kappa = 0$

Schauder Bases

Let $\kappa[[X]] = \kappa[[Y]]$.

- Gould-Schauder basis: $(Y^k(1+X)^p)_{k\geq 0}$, where $p \in \mathbb{Z}$
- Abel-Schauder basis: $(Y^k e^{pX})_{k\geq 0}$, where $p \in \kappa$ and char $\kappa = 0$
- Bernoulli-Schauder basis: $(Y^k(X/(e^X - 1))^p)_{k\geq 0}$, where $p \in \mathbb{Z}$ and char $\kappa = 0$

- Interplay of representations of a power series by two Schauder bases is exactly an inverse relation.
- The theory of Riordan arrays can be explained using Schauder bases.

Comparison

- Non-canonical vs. Fixed Choice variables
- Commutative vs. Non-commutative operations
- *Relations vs. Transformations*. In linear algebra, a matrix may be interpreted as a linear transformation of vector spaces. It may be also regarded as relations between two sets of vectors. In the literature, a Riordan array is treated as a map for power series. From the viewpoint of Schauder bases, the array is regarded as a relation between two power series.
- Differentials vs. Functions
  There is a pairing given by local cohomology residues for differentials and systems of parameters. The pairing is an algebraic analogue of the integration of a differential form on a manifold. It has an effect of equating coefficients in a way independent of choices of a set of variables.

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