Algebraic Structure of Convolution Identities

(A Commutative Algebraist's Journey to Identities)

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September 29, 2016

Introduction

1. Identities of Bernoulli numbers are obtained using local cohomology residues (1999).
    - Cohomology residues are used for concrete realization of Grothendieck duality and used to prove binomial identities.
2. Identities of Bernoulli numbers are realizations of algebraic structures of a module over a Weyl algebra (2010).
    - Local cohomology modules usually are not finitely generated.
    - Some local cohomology modules are finitely generated as modules over a Weyl algebra.
3. Identities of Euler numbers are realizations of algebraic structures of a module over the universal enveloping algebra of a Lie algebra (2012).
4. For certain sequences of numbers, convolution identities come from a parametrization of a variety equipped with a vector field (2014).

Geometry and Algebra

Geometry	varieties (vanishing sets of polynomials)	vector fields
Algebra	commutative rings	derivations

Recursive definitions

Fibonacci numbers

$F_0 = 0$, $F_1 = 1$ and, for $n \geq 2$,

$$F_{n-1} + F_{n-2} = F_n.$$

Bernoulli numbers

$B_0 = 1$ and, for $n \geq 2$,

$$\sum_{i=0}^{n} \binom{n}{i} B_i = B_n.$$

Analytic definitions

Euler numbers

$$\frac{2e^z}{e^{2z}+1} = \sum_{i=0}^{\infty} \frac{E_i}{i!} z^i, \qquad |z| < \pi/2$$

Bernoulli numbers

$$\frac{z}{e^z - 1} = \sum_{i=0}^{\infty} \frac{B_i}{i!} z^i, \qquad |z| < 2\pi.$$

Bernoulli numbers appear in values of Riemann zeta function:

$$\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}, \qquad n \geq 1.$$

Algebraic definitions

Bernoulli numbers

$$\mathbf{B} := \frac{T}{e^T - 1} = \sum_{i=0}^{\infty} \frac{B_i}{i!} T^i$$

Euler numbers

$$\mathbf{E} := \frac{2e^T}{e^{2T} + 1} = \sum_{i=0}^{\infty} \frac{E_i}{i!} T^i$$

Genocchi numbers

$$\mathbf{G} := \frac{2T}{e^T + 1} = \sum_{n=0}^{\infty} \frac{G_n}{n!} T^n$$

Algebraic definitions

Fibonacci numbers

$$\mathbf{F} := \frac{T}{1 - T - T^2} = \sum_{n=0}^{\infty} F_n z^n$$

Lucas numbers

$$\mathbf{L} := \frac{2 - T}{1 - T - T^2} = \sum_{n=0}^{\infty} L_n T^n$$

Pell numbers

$$\mathbf{P} := \frac{T}{1 - 2T - T^2} = \sum_{n=0}^{\infty} P_n T^n$$

Identities

For the sequences of Fibonacci numbers F_n, Lucas numbers L_n, and Pell numbers P_n *etc.*, there are identities

$$\begin{aligned}
\sum_{i+j=n} F_i F_j &= \frac{n-1}{5} F_n + \frac{2n}{5} F_{n-1}, \\
\sum_{i+j=n} L_i L_j &= \frac{5n+11}{5} L_n + \frac{2}{5} L_{n-1}, \\
\sum_{i+j=n} P_i P_j &= \frac{n-1}{4} P_n + \frac{n}{4} P_{n-1}.
\end{aligned}$$

Identities

For the sequences of Bernoulli numbers B_n, Euler numbers E_n, Genocchi numbers G_n *etc.*, there are identities

$$\begin{aligned}
\sum_{i+j=n} \binom{n}{i} B_i B_j &= (1-n)B_n - nB_{n-1}, \\
\sum_{i+j=n} \binom{n}{i} G_i G_j &= 2(n-1)G_n + 2nG_{n-1}, \\
\sum_{i+j+k=n} \binom{n}{i,j,k} E_i E_j E_k &= \frac{1}{2}E_n - \frac{1}{2}E_{n+2}.
\end{aligned}$$

Note that $B_1 = -1/2$ and $B_3 = B_5 = B_7 = \cdots = 0$.

Identities

Euler (18th century)

For $n \geq 2$,

$$\sum_{i=1}^{n-1} \binom{2n}{2i} B_{2i} B_{2n-2i} = -(2n+1) B_{2n}$$

R. Sitaramachandrarao and B. Davis (1986)

For $n \geq 3$,

$$\sum_{\substack{a,b,c>0 \\ a+b+c=n}} (2n)! \frac{B_{2a}}{(2a)!} \frac{B_{2b}}{(2b)!} \frac{B_{2c}}{(2c)!} = (n+1)(2n+1) B_{2n} + \frac{1}{2} n(2n-1) B_{2n-2}$$

Identities

R. Sitaramachandrarao and B. Davis (1986)

For $n \geq 4$,

$$\sum_{\substack{a,b,c,d>0 \\ a+b+c+d=n}} (2n)! \frac{B_{2a}}{(2a)!} \frac{B_{2b}}{(2b)!} \frac{B_{2c}}{(2c)!} \frac{B_{2d}}{(2d)!} = -\binom{2n+3}{3} B_{2n} - \frac{4}{3} n^2 (2n-1) B_{2n-2}$$

- A. Sankarayanan (1987) has evaluated the sum of products of 5 Bernoulli numbers.
- W.-P. Zhang (1992) has evaluated the sums of products of 6 and 7 Bernoulli numbers.
- K. Dilcher (1996) has a formula for the sum of products of n Bernoulli numbers.
- I-C. Huang & S.-Y. Huang (1999) explain the phenomenon using "complete sums".
- I-C. Huang explains the phenomenon using a D-module.

Identities

Identity

For $n \geq 4$,

$$\sum_{p=2}^{n-2} \frac{(2n-2)!}{(2p-2)!(2n-2p-2)!} \frac{B_{2p}}{2p} \frac{B_{2n-2p}}{2n-2p} = -\frac{(2n+1)(n-3)}{6n} B_{2n}.$$

- by H. Rademacher using Eisenstein series,
- by M.-K. Eie using zeta functions,
- by I-C. Huang and S.-Y. Huang using cohomology residues.

Identities

T. Agoh and K. Dilcher (2007)

$$\sum_{i=0}^{n} \binom{n}{i} B_{i+1} B_{n-i+1} = \frac{1}{6}(n-1)B_n - B_{n+1} - \frac{1}{6}(n+3)B_{n+2},$$

$$\sum_{i=0}^{n} \binom{n}{i} B_{i+2} B_{n-i+3} = \frac{1}{60} n B_{n+1} + \frac{1}{6} B_{n+3} - \frac{1}{60}(n+6)B_{n+5}.$$

- T. Agoh and K. Dilcher (2007) give formulas for the sum of products of two Bernoulli numbers with shifted indices.

Identities

W.-P. Zhang (1992)

$$\sum_{i+j+k=n} \binom{n}{i,j,k} E_i E_j E_k = \frac{1}{2} E_n - \frac{1}{2} E_{n+2}$$

- W.-P. Zhang (1992) has evaluated the sum of products of 5, 7, 9 Euler numbers.

A.-F. Hsu and H.-L. Lee (2011)

$$\sum_{i_1+i_2+\cdots+i_{11}=n} \binom{n}{i_1, i_2, \cdots, i_{11}} E_{i_1} E_{i_2} \cdots E_{i_{11}} = \frac{63}{256} E_n - \frac{117469}{403200} E_{n+2} + \frac{17281}{362880} E_{n+4} - \frac{209}{86400} E_{n+6} + \frac{11}{241920} E_{n+8} - \frac{1}{3628800} E_{n+10}$$

Identities

- K. Dilcher (1996) has a formula for the sum of products of n Euler numbers in terms of Euler polynomial.
- Are there formulas for the sum of product of $2n$ Euler numbers in terms of only Euler numbers? with shifted indices?

Bernoulli numbers

Bernoulli numbers are defined by $\mathbf{B} := \dfrac{T}{e^T - 1} = \sum \dfrac{B_i}{i!} T^i$.

$$\frac{d\mathbf{B}}{dT} = \frac{1}{e^T - 1} - \frac{Te^T}{(e^T - 1)^2} = \frac{1}{e^T - 1} - \frac{T}{e^T - 1} - \frac{T}{(e^T - 1)^2}$$

Let δ be the derivation $T(d/dT)$. Then

$$\delta \cdot \mathbf{B} = \mathbf{B} - T\mathbf{B} - \mathbf{B}^2,$$

or

$$\mathbf{B}^2 = (1 - T - \delta) \cdot \mathbf{B}.$$

Comparing coefficients, we recover

$$\sum_{i+j=n} \frac{B_i}{i!}\frac{B_j}{j!} = \frac{B_n}{n!} - \frac{B_n}{(n-1)!} - \frac{B_{n-1}}{(n-1)!}.$$

Bernoulli numbers

Formulas of the sum of products of n Bernoulli numbers are obtained from the following relation.

M. É. Lucas (1878)

$$\mathbf{B}^{n+1} = \left(1 - T - \frac{T}{n}\frac{d}{dT}\right) \cdot \mathbf{B}^n$$

For example,

$$\begin{aligned} \mathbf{B}^3 &= \left(1 - T - \frac{T}{2}\frac{d}{dT}\right) \cdot \mathbf{B}^2 \\ &= \left(1 - T - \frac{T}{2}\frac{d}{dT}\right)\left(1 - T - T\frac{d}{dT}\right) \cdot \mathbf{B}, \end{aligned}$$

which can be simplified using the relation

$$[\frac{d}{dT}, T] := \frac{d}{dT}T - T\frac{d}{dT} = 1.$$

Bernoulli numbers and Euler numbers

$\mathbb{Q}[T, \mathbf{B}]_T$ is a module over the Weyl algebra $\mathbb{Q}\langle T, \frac{d}{dT}\rangle$.
The polynomial ring $\mathbb{Q}[T, \mathbf{B}]$ is a module over $\mathbb{Q}\langle T, T\frac{d}{dT}\rangle$.
Note that $(T\frac{d}{dT}) \cdot T = T$.

For Euler numbers $\mathbf{E} = \frac{2e^T}{e^{2T}+1}$, we have $\frac{d}{dT}\mathbf{E} = \mathbf{E} - e^T\mathbf{E}^2$.
In other words, $\mathbb{Q}[e^T, \mathbf{E}]$ is a module over $\mathbb{Q}\langle e^T, \frac{d}{dT}\rangle$.
$\mathbb{Q}[e^T, \mathbf{E}]$ is isomorphic to $\mathbb{Q}[X, Y]/(X^2Y - 2X + Y)$.
Note that $\frac{d}{dT} \cdot e^T = e^T$.

Structure

For each sequence of numbers, we consider a subvariety of the affine plane.

numbers	vanishing set of	$\mathbb{Q}$-algebra
Bernoulli	0	$\mathbb{Q}[X, Y]$
Euler	$X^2Y - 2X + Y$	$\mathbb{Q}[X, Y]/(X^2Y - 2X + Y)$
Genocchi	0	$\cdot$
Fibonacci	$X^2Y + XY + X - Y$	$\cdot$
Lucas	$X^2Y - 5XY + X + 5Y$	$\cdot$
Pell	$X^2Y + 2XY + X - Y$	$\cdot$

Structure

Parametrization

A *parametrization* of a $\mathbb{Q}$-algebra $\mathbb{Q}[x, y]$ is a $\mathbb{Q}$-algebra embedding $\mathbb{Q}[x, y] \to \mathbb{Q}[[T]]$.

For Bernoulli numbers, Euler numbers, Genocchi numbers, Fibonacci numbers, Lucas numbers, and Pell numbers, we will show that identities are obtained from a parametrization sending y to **B**, **E**, **G**, **F**, **L** and **P** together with a derivation δ stabilizing x (that is $\delta \cdot x = x$).

Structure

Given $0 \neq f \in \mathbb{Q}[[T]]$, we denote

$$\frac{d \log f}{dT} := \frac{1}{f} \frac{df}{dT} \in \mathbb{Q}((T)).$$

Given futhermore $g \in \mathbb{Q}[[T]] \setminus \mathbb{Q}$, we denote

$$\frac{d \log f}{d \log g} := \frac{d \log f}{dT} \left(\frac{d \log g}{dT} \right)^{-1}.$$

The map

$$\frac{d}{d \log g} := \left(\frac{d \log g}{dT} \right)^{-1} \frac{d}{dT}$$

is a $\mathbb{Q}$-derivation stabilizing g:

$$\frac{dg}{d \log g} = \left(\frac{d \log g}{dT} \right)^{-1} \frac{dg}{dT} = \left(\frac{1}{g} \frac{dg}{dT} \right)^{-1} \frac{dg}{dT} = g.$$

Structure

Let $\mathbb{Q}[x, y]$ be a $\mathbb{Q}$-subalgebra of $\mathbb{Q}[[T]]$, where $x \notin \mathbb{Q}$. If

$$\frac{d \log y}{d \log x} \in \mathbb{Q}[x, y],$$

the restriction of

$$\delta := \frac{d}{d \log x} := \left(\frac{d \log x}{dT}\right)^{-1} \frac{d}{dT}$$

to $\mathbb{Q}[x, y]$ is a $\mathbb{Q}$-derivation stabilizing x. Note that

$$\delta \cdot y = \left(\frac{d \log x}{dT}\right)^{-1} \frac{dy}{dT} = y \left(\frac{d \log x}{dT}\right)^{-1} \frac{d \log y}{dT} = y \frac{d \log y}{d \log x}.$$

Structure

Examples:

- Bernoulli numbers: Let $x = T$, $y = \mathbf{B}$ and $\delta = d/d \log x$.

$$\frac{d \log y}{d \log x} = 1 - x - y \quad \text{and} \quad \begin{cases} \delta \cdot x = x \\ \delta \cdot y = y - xy - y^2 \end{cases}$$

- Euler numbers: Let $x = e^T$, $y = \mathbf{E}$ and $\delta = d/d \log x$.

$$\frac{d \log y}{d \log x} = 1 - xy \quad \text{and} \quad \begin{cases} \delta \cdot x = x \\ \delta \cdot y = y - xy^2 \end{cases}$$

- Genocchi numbers, Fibonacci numbers, Lucas numbers, Pell numbers, *etc.* fit in the framework.

Structure (universal enveloping algebra)

Let δ be a $\mathbb{Q}$-derivation on $\mathbb{Q}[x, y]$ stabilizing x. We denote by $\mathbb{Q}\langle x, \delta\rangle$ the $\mathbb{Q}$-algebra of $\mathbb{Q}$-linear operators on $\mathbb{Q}[x, y]$ generated by δ and the multiplication map x.

The Leibniz rule gives the relation $\delta x - x\delta = x$ in $\mathbb{Q}\langle x, \delta\rangle$.

Proposition

Let δ be a $\mathbb{Q}$-derivation on $\mathbb{Q}[x, y]$ stabilizing x. Assume that x is not a zero-divisor of $\mathbb{Q}[x, y]$. The $\mathbb{Q}$-algebra $\mathbb{Q}\langle x, \delta\rangle$ is isomorphic to the universal enveloping algebra of the two-dimensional Lie algebra with basis $\{\delta, x\}$ and Lie bracket $[\delta, x] := \delta x - x\delta = x$.

Structure (Weyl algebra)

Let δ be a $\mathbb{Q}$-derivation on $\mathbb{Q}[x, y]$ stabilizing x. The commutative ring $\mathbb{Q}[x, x^{-1}, y]$ has a module structure over $\mathbb{Q}\langle x, x^{-1}\delta\rangle$, where $x^{-1}\delta$ is the $\mathbb{Q}$-derivation on $\mathbb{Q}[x, x^{-1}, y]$ defined by

$$(x^{-1}\delta) \cdot \frac{f}{x^n} := \frac{1}{x}\left(\frac{\delta \cdot f}{x^n} - n\frac{f}{x^{n+1}}\delta \cdot x\right) = \frac{(\delta - n) \cdot f}{x^{n+1}}.$$

Proposition

Let δ be a $\mathbb{Q}$-derivation on $\mathbb{Q}[x, y]$ stabilizing x. Assume that x is not a zero-divisor. The $\mathbb{Q}$-algebra $\mathbb{Q}\langle x, x^{-1}\delta\rangle$ is isomorphic to the first Weyl algebra over $\mathbb{Q}$.

Bernoulli numbers

For Bernoulli numbers, we consider the polynomial ring $\mathbb{Q}[x, y]$ with the $\mathbb{Q}$-derivation δ given by

- $\delta \cdot x = x$,
- $\delta \cdot y = y - xy - y^2$.

Leibniz rule:

- $\delta x - x\delta = x \in \mathbb{Q}\langle x, \delta \rangle$.

Concrete information is obtained from the realization $x = T$, $y = \mathbf{B}$ and

$$\delta = \frac{d}{d \log x}.$$

Bernoulli numbers

For Bernoulli numbers, $x^{-1}\delta$ can be realized as $D := d/dT$.
A formula for the sum

$$\sum_{i=0}^{\ell} \binom{\ell}{i} B_{i+m} B_{n-i+n}$$

is given by the membership $T^{m+n}(D^m \cdot y)(D^n \cdot y) \in \mathbb{Q}\langle T, D\rangle \cdot y$.
For example,

$$(D \cdot y)^2 = \left(-\frac{1}{6}TD^3 - \frac{1}{2}D^2 + \frac{1}{6}TD - D - \frac{1}{6}\right) \cdot y.$$

gives rise to

$$\sum_{i=0}^{n} \binom{n}{i} B_{1+i} B_{1+n-i} = \frac{1}{6}(n-1)B_n - B_{n+1} - \frac{1}{6}(n+3)B_{n+2}.$$

Euler numbers

For Euler numbers, we consider x and y with the relation

- $x^2y - 2x + y = 0$;

and consider the $\mathbb{Q}$-derivation δ satisfying

- $\delta \cdot x = x$,
- $\delta \cdot y = y - xy^2$.

Leibniz rule:

- $\delta x - x\delta = x \in \mathbb{Q}\langle x, \delta \rangle$.

The relation

$$y^{2n+1} = \frac{\left(1^2 - \delta^2\right)\left(3^2 - \delta^2\right) \cdots \left((2n-1)^2 - \delta^2\right)}{(2n)!} \cdot y$$

gives rise to formulas for the sum of product of $2n+1$ Euler numbers. Note that $y^{2n} \notin \mathbb{Q}[\delta] \cdot y$ but $y^{2n} \in \mathbb{Q}[\delta]x \cdot y$.

Euler numbers

Proposition

If the sequence $i_1, \cdots, i_n$ consists of even number of odd integers and odd number of even integers, then $(\delta^{i_1} \cdot y)(\delta^{i_2} \cdot y) \cdots (\delta^{i_n} \cdot y) \in \mathbb{Q}[\delta] \cdot y$.

Examples

$$\begin{aligned} (\delta \cdot y)^2 y &= \left(\frac{1}{8} - \frac{1}{12}\delta^2 - \frac{1}{24}\delta^4\right) \cdot y \\ (\delta^2 \cdot y) y^2 &= \left(-\frac{1}{4} + \frac{1}{3}\delta^2 - \frac{1}{12}\delta^4\right) \cdot y \end{aligned}$$

Euler numbers

The relation

$$(\delta \cdot y)^2 y = \left(\frac{1}{8} - \frac{1}{12}\delta^2 - \frac{1}{24}\delta^4\right) \cdot y$$

gives rise to the identity

$$\sum_{i+j+k=n} \binom{n}{i,j,k} E_{i+1} E_{j+1} E_k = \frac{1}{8} E_n - \frac{1}{12} E_{n+2} - \frac{1}{24} E_{n+4}.$$

Euler numbers

The relation

$$(\delta^2 \cdot y)y^2 = \left(-\frac{1}{4} + \frac{1}{3}\delta^2 - \frac{1}{12}\delta^4\right) \cdot y$$

gives rise to the identity

$$\sum_{i+j+k=n} \binom{n}{i,j,k} E_{2+i} E_j E_k = -\frac{1}{4}E_n + \frac{1}{3}E_{n+2} - \frac{1}{12}E_{n+4}.$$