

# Algebraic Structure of Convolution Identities

## (A Commutative Algebraist's Journey to Identities)

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# Introduction

- 1 Identities of Bernoulli numbers are obtained using local cohomology residues (1999).
  - Cohomology residues are used for concrete realization of Grothendieck duality and used to prove binomial identities.
- 2 Identities of Bernoulli numbers are realizations of algebraic structures of a module over a Weyl algebra (2010).
  - Local cohomology modules usually are not finitely generated.
  - Some local cohomology modules are finitely generated as modules over a Weyl algebra.
- 3 Identities of Euler numbers are realizations of algebraic structures of a module over the universal enveloping algebra of a Lie algebra (2012).
- 4 For certain sequences of numbers, convolution identities come from a parametrization of a variety equipped with a vector field (2014).

# Geometry and Algebra

Geometry	varieties (vanishing sets of polynomials)	vector fields
Algebra	commutative rings	derivations

## Recursive definitions

### Fibonacci numbers

$F_0 = 0$ ,  $F_1 = 1$  and, for  $n \geq 2$ ,

$$F_{n-1} + F_{n-2} = F_n.$$

### Bernoulli numbers

$B_0 = 1$  and, for  $n \geq 2$ ,

$$\sum_{i=0}^n \binom{n}{i} B_i = B_n.$$

## Analytic definitions

### Euler numbers

$$\frac{2e^z}{e^{2z} + 1} = \sum_{i=0}^{\infty} \frac{E_i}{i!} z^i, \quad |z| < \pi/2$$

### Bernoulli numbers

$$\frac{z}{e^z - 1} = \sum_{i=0}^{\infty} \frac{B_i}{i!} z^i, \quad |z| < 2\pi.$$

Bernoulli numbers appear in values of Riemann zeta function:

$$\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}, \quad n \geq 1.$$

## Algebraic definitions

### Bernoulli numbers

$$\mathbf{B} := \frac{T}{e^T - 1} = \sum_{i=0}^{\infty} \frac{B_i}{i!} T^i$$

### Euler numbers

$$\mathbf{E} := \frac{2e^T}{e^{2T} + 1} = \sum_{i=0}^{\infty} \frac{E_i}{i!} T^i$$

### Genocchi numbers

$$\mathbf{G} := \frac{2T}{e^T + 1} = \sum_{n=0}^{\infty} \frac{G_n}{n!} T^n$$

## Algebraic definitions

### Fibonacci numbers

$$\mathbf{F} := \frac{T}{1 - T - T^2} = \sum_{n=0}^{\infty} F_n z^n$$

### Lucas numbers

$$\mathbf{L} := \frac{2 - T}{1 - T - T^2} = \sum_{n=0}^{\infty} L_n T^n$$

### Pell numbers

$$\mathbf{P} := \frac{T}{1 - 2T - T^2} = \sum_{n=0}^{\infty} P_n T^n$$

# Identities

For the sequences of Fibonacci numbers  $F_n$ , Lucas numbers  $L_n$ , and Pell numbers  $P_n$  etc., there are identities

$$\begin{aligned}\sum_{i+j=n} F_i F_j &= \frac{n-1}{5} F_n + \frac{2n}{5} F_{n-1}, \\ \sum_{i+j=n} L_i L_j &= \frac{5n+11}{5} L_n + \frac{2}{5} L_{n-1}, \\ \sum_{i+j=n} P_i P_j &= \frac{n-1}{4} P_n + \frac{n}{4} P_{n-1}.\end{aligned}$$



# Identities

For the sequences of Bernoulli numbers  $B_n$ , Euler numbers  $E_n$ , Genocchi numbers  $G_n$  etc., there are identities

$$\sum_{i+j=n} \binom{n}{i} B_i B_j = (1-n)B_n - nB_{n-1},$$

$$\sum_{i+j=n} \binom{n}{i} G_i G_j = 2(n-1)G_n + 2nG_{n-1},$$

$$\sum_{i+j+k=n} \binom{n}{i, j, k} E_i E_j E_k = \frac{1}{2}E_n - \frac{1}{2}E_{n+2}.$$

Note that  $B_1 = -1/2$  and  $B_3 = B_5 = B_7 = \dots = 0$ .

## Identities

## Euler (18th century)

For  $n \geq 2$ ,

$$\sum_{i=1}^{n-1} \binom{2n}{2i} B_{2i} B_{2n-2i} = -(2n+1)B_{2n}$$

## R. Sitaramachandrarao and B. Davis (1986)

For  $n \geq 3$ ,

$$\sum_{\substack{a,b,c>0 \\ a+b+c=n}} (2n)! \frac{B_{2a}}{(2a)!} \frac{B_{2b}}{(2b)!} \frac{B_{2c}}{(2c)!} = (n+1)(2n+1)B_{2n} + \frac{1}{2}n(2n-1)B_{2n-2}$$

# Identities

## R. Sitaramachandrarao and B. Davis (1986)

For  $n \geq 4$ ,

$$\sum_{\substack{a,b,c,d>0 \\ a+b+c+d=n}} (2n)! \frac{B_{2a}}{(2a)!} \frac{B_{2b}}{(2b)!} \frac{B_{2c}}{(2c)!} \frac{B_{2d}}{(2d)!} = -\binom{2n+3}{3} B_{2n} - \frac{4}{3} n^2 (2n-1) B_{2n-2}$$

- A. Sankarayanan (1987) has evaluated the sum of products of 5 Bernoulli numbers.
- W.-P. Zhang (1992) has evaluated the sums of products of 6 and 7 Bernoulli numbers.
- K. Dilcher (1996) has a formula for the sum of products of  $n$  Bernoulli numbers.
- I-C. Huang & S.-Y. Huang (1999) explain the phenomenon using “complete sums”.
- I-C. Huang explains the phenomenon using a  $D$ -module.

# Identities

## Identity

For  $n \geq 4$ ,

$$\sum_{p=2}^{n-2} \frac{(2n-2)!}{(2p-2)!(2n-2p-2)!} \frac{B_{2p}}{2p} \frac{B_{2n-2p}}{2n-2p} = -\frac{(2n+1)(n-3)}{6n} B_{2n}.$$

- by H. Rademacher using Eisenstein series,
- by M.-K. Eie using zeta functions,
- by I-C. Huang and S.-Y. Huang using cohomology residues.

# Identities

T. Agoh and K. Dilcher (2007)

$$\sum_{i=0}^n \binom{n}{i} B_{i+1} B_{n-i+1} = \frac{1}{6}(n-1)B_n - B_{n+1} - \frac{1}{6}(n+3)B_{n+2},$$

$$\sum_{i=0}^n \binom{n}{i} B_{i+2} B_{n-i+3} = \frac{1}{60}nB_{n+1} + \frac{1}{6}B_{n+3} - \frac{1}{60}(n+6)B_{n+5}.$$

- T. Agoh and K. Dilcher (2007) give formulas for the sum of products of two Bernoulli numbers with shifted indices.

## Identities

W.-P. Zhang (1992)

$$\sum_{i+j+k=n} \binom{n}{i, j, k} E_i E_j E_k = \frac{1}{2} E_n - \frac{1}{2} E_{n+2}$$

- W.-P. Zhang (1992) has evaluated the sum of products of 5, 7, 9 Euler numbers.

A.-F. Hsu and H.-L. Lee (2011)

$$\begin{aligned} \sum_{i_1+i_2+\dots+i_{11}=n} \binom{n}{i_1, i_2, \dots, i_{11}} E_{i_1} E_{i_2} \cdots E_{i_{11}} &= \frac{63}{256} E_n - \frac{117469}{403200} E_{n+2} \\ &+ \frac{17281}{362880} E_{n+4} - \frac{209}{86400} E_{n+6} + \frac{11}{241920} E_{n+8} - \frac{1}{3628800} E_{n+10} \end{aligned}$$

# Identities

- K. Dilcher (1996) has a formula for the sum of products of  $n$  Euler numbers in terms of Euler polynomial.
- Are there formulas for the sum of product of  $2n$  Euler numbers in terms of only Euler numbers? with shifted indices?

## Bernoulli numbers

Bernoulli numbers are defined by  $\mathbf{B} := \frac{T}{e^T - 1} = \sum \frac{B_i}{i!} T^i$ .

$$\frac{d\mathbf{B}}{dT} = \frac{1}{e^T - 1} - \frac{T e^T}{(e^T - 1)^2} = \frac{1}{e^T - 1} - \frac{T}{e^T - 1} - \frac{T}{(e^T - 1)^2}$$

Let  $\delta$  be the derivation  $T(d/dT)$ . Then

$$\delta \cdot \mathbf{B} = \mathbf{B} - T\mathbf{B} - \mathbf{B}^2,$$

or

$$\mathbf{B}^2 = (1 - T - \delta) \cdot \mathbf{B}.$$

Comparing coefficients, we recover

$$\sum_{i+j=n} \frac{B_i}{i!} \frac{B_j}{j!} = \frac{B_n}{n!} - \frac{B_n}{(n-1)!} - \frac{B_{n-1}}{(n-1)!}.$$



## Bernoulli numbers

Formulas of the sum of products of  $n$  Bernoulli numbers are obtained from the following relation.

M. É. Lucas (1878)

$$\mathbf{B}^{n+1} = \left( 1 - T - \frac{T}{n} \frac{d}{dT} \right) \cdot \mathbf{B}^n$$

For example,

$$\begin{aligned} \mathbf{B}^3 &= \left( 1 - T - \frac{T}{2} \frac{d}{dT} \right) \cdot \mathbf{B}^2 \\ &= \left( 1 - T - \frac{T}{2} \frac{d}{dT} \right) \left( 1 - T - T \frac{d}{dT} \right) \cdot \mathbf{B}, \end{aligned}$$

which can be simplified using the relation

$$\left[ \frac{d}{dT}, T \right] := \frac{d}{dT} T - T \frac{d}{dT} = 1.$$

## Bernoulli numbers and Euler numbers

$\mathbb{Q}[T, \mathbf{B}]_T$  is a module over the Weyl algebra  $\mathbb{Q}\langle T, \frac{d}{dT} \rangle$ .

The polynomial ring  $\mathbb{Q}[T, \mathbf{B}]$  is a module over  $\mathbb{Q}\langle T, T \frac{d}{dT} \rangle$ .

Note that  $(T \frac{d}{dT}) \cdot T = T$ .

For Euler numbers  $\mathbf{E} = \frac{2e^T}{e^{2T} + 1}$ , we have  $\frac{d}{dT} \mathbf{E} = \mathbf{E} - e^T \mathbf{E}^2$ .

In other words,  $\mathbb{Q}[e^T, \mathbf{E}]$  is a module over  $\mathbb{Q}\langle e^T, \frac{d}{dT} \rangle$ .

$\mathbb{Q}[e^T, \mathbf{E}]$  is isomorphic to  $\mathbb{Q}[X, Y]/(X^2Y - 2X + Y)$ .

Note that  $\frac{d}{dT} \cdot e^T = e^T$ .

# Structure

For each sequence of numbers, we consider a subvariety of the affine plane.

numbers	vanishing set of	$\mathbb{Q}$ -algebra
Bernoulli	0	$\mathbb{Q}[X, Y]$
Euler	$X^2Y - 2X + Y$	$\mathbb{Q}[X, Y]/(X^2Y - 2X + Y)$
Genocchi	0	.
Fibonacci	$X^2Y + XY + X - Y$	.
Lucas	$X^2Y - 5XY + X + 5Y$	.
Pell	$X^2Y + 2XY + X - Y$	.

# Structure

## Parametrization

A *parametrization* of a  $\mathbb{Q}$ -algebra  $\mathbb{Q}[x, y]$  is a  $\mathbb{Q}$ -algebra embedding  $\mathbb{Q}[x, y] \rightarrow \mathbb{Q}[[T]]$ .

For Bernoulli numbers, Euler numbers, Genocchi numbers, Fibonacci numbers, Lucas numbers, and Pell numbers, we will show that identities are obtained from a parametrization sending  $y$  to **B**, **E**, **G**, **F**, **L** and **P** together with a derivation  $\delta$  stabilizing  $x$  (that is  $\delta \cdot x = x$ ).

## Structure

Given  $0 \neq f \in \mathbb{Q}[[T]]$ , we denote

$$\frac{d \log f}{dT} := \frac{1}{f} \frac{df}{dT} \in \mathbb{Q}((T)).$$

Given furthermore  $g \in \mathbb{Q}[[T]] \setminus \mathbb{Q}$ , we denote

$$\frac{d \log f}{d \log g} := \frac{d \log f}{dT} \left( \frac{d \log g}{dT} \right)^{-1}.$$

The map

$$\frac{d}{d \log g} := \left( \frac{d \log g}{dT} \right)^{-1} \frac{d}{dT}$$

is a  $\mathbb{Q}$ -derivation stabilizing  $g$ :

$$\frac{dg}{d \log g} = \left( \frac{d \log g}{dT} \right)^{-1} \frac{dg}{dT} = \left( \frac{1}{g} \frac{dg}{dT} \right)^{-1} \frac{dg}{dT} = g.$$

## Structure

Let  $\mathbb{Q}[x, y]$  be a  $\mathbb{Q}$ -subalgebra of  $\mathbb{Q}[[T]]$ , where  $x \notin \mathbb{Q}$ . If

$$\frac{d \log y}{d \log x} \in \mathbb{Q}[x, y],$$

the restriction of

$$\delta := \frac{d}{d \log x} := \left( \frac{d \log x}{dT} \right)^{-1} \frac{d}{dT}$$

to  $\mathbb{Q}[x, y]$  is a  $\mathbb{Q}$ -derivation stabilizing  $x$ . Note that

$$\delta \cdot y = \left( \frac{d \log x}{dT} \right)^{-1} \frac{dy}{dT} = y \left( \frac{d \log x}{dT} \right)^{-1} \frac{d \log y}{dT} = y \frac{d \log y}{d \log x}.$$

# Structure

Examples:

- Bernoulli numbers: Let  $x = T$ ,  $y = \mathbf{B}$  and  $\delta = d/d \log x$ .

$$\frac{d \log y}{d \log x} = 1 - x - y \quad \text{and} \quad \begin{cases} \delta \cdot x = x \\ \delta \cdot y = y - xy - y^2 \end{cases}$$

- Euler numbers: Let  $x = e^T$ ,  $y = \mathbf{E}$  and  $\delta = d/d \log x$ .

$$\frac{d \log y}{d \log x} = 1 - xy \quad \text{and} \quad \begin{cases} \delta \cdot x = x \\ \delta \cdot y = y - xy^2 \end{cases}$$

- Genocchi numbers, Fibonacci numbers, Lucas numbers, Pell numbers, *etc.* fit in the framework.

## Structure (universal enveloping algebra)

Let  $\delta$  be a  $\mathbb{Q}$ -derivation on  $\mathbb{Q}[x, y]$  stabilizing  $x$ . We denote by  $\mathbb{Q}\langle x, \delta \rangle$  the  $\mathbb{Q}$ -algebra of  $\mathbb{Q}$ -linear operators on  $\mathbb{Q}[x, y]$  generated by  $\delta$  and the multiplication map  $x$ .

The Leibniz rule gives the relation  $\delta x - x\delta = x$  in  $\mathbb{Q}\langle x, \delta \rangle$ .

### Proposition

Let  $\delta$  be a  $\mathbb{Q}$ -derivation on  $\mathbb{Q}[x, y]$  stabilizing  $x$ . Assume that  $x$  is not a zero-divisor of  $\mathbb{Q}[x, y]$ . The  $\mathbb{Q}$ -algebra  $\mathbb{Q}\langle x, \delta \rangle$  is isomorphic to the universal enveloping algebra of the two-dimensional Lie algebra with basis  $\{\delta, x\}$  and Lie bracket  $[\delta, x] := \delta x - x\delta = x$ .



## Structure (Weyl algebra)

Let  $\delta$  be a  $\mathbb{Q}$ -derivation on  $\mathbb{Q}[x, y]$  stabilizing  $x$ . The commutative ring  $\mathbb{Q}[x, x^{-1}, y]$  has a module structure over  $\mathbb{Q}\langle x, x^{-1}\delta \rangle$ , where  $x^{-1}\delta$  is the  $\mathbb{Q}$ -derivation on  $\mathbb{Q}[x, x^{-1}, y]$  defined by

$$(x^{-1}\delta) \cdot \frac{f}{x^n} := \frac{1}{x} \left( \frac{\delta \cdot f}{x^n} - n \frac{f}{x^{n+1}} \delta \cdot x \right) = \frac{(\delta - n) \cdot f}{x^{n+1}}.$$

### Proposition

Let  $\delta$  be a  $\mathbb{Q}$ -derivation on  $\mathbb{Q}[x, y]$  stabilizing  $x$ . Assume that  $x$  is not a zero-divisor. The  $\mathbb{Q}$ -algebra  $\mathbb{Q}\langle x, x^{-1}\delta \rangle$  is isomorphic to the first Weyl algebra over  $\mathbb{Q}$ .

## Bernoulli numbers

For Bernoulli numbers, we consider the polynomial ring  $\mathbb{Q}[x, y]$  with the  $\mathbb{Q}$ -derivation  $\delta$  given by

- $\delta \cdot x = x,$
- $\delta \cdot y = y - xy - y^2.$

Leibniz rule:

- $\delta x - x\delta = x \in \mathbb{Q}\langle x, \delta \rangle.$

Concrete information is obtained from the realization  $x = T,$   
 $y = \mathbf{B}$  and

$$\delta = \frac{d}{d \log x}.$$

## Bernoulli numbers

For Bernoulli numbers,  $x^{-1}\delta$  can be realized as  $D := d/dT$ .

A formula for the sum

$$\sum_{i=0}^{\ell} \binom{\ell}{i} B_{i+m} B_{n-i+n}$$

is given by the membership  $T^{m+n}(D^m \cdot y)(D^n \cdot y) \in \mathbb{Q}\langle T, D \rangle \cdot y$ .

For example,

$$(D \cdot y)^2 = \left( -\frac{1}{6}TD^3 - \frac{1}{2}D^2 + \frac{1}{6}TD - D - \frac{1}{6} \right) \cdot y.$$

gives rise to

$$\sum_{i=0}^n \binom{n}{i} B_{1+i} B_{1+n-i} = \frac{1}{6}(n-1)B_n - B_{n+1} - \frac{1}{6}(n+3)B_{n+2}.$$

## Euler numbers

For Euler numbers, we consider  $x$  and  $y$  with the relation

- $x^2y - 2x + y = 0$ ;

and consider the  $\mathbb{Q}$ -derivation  $\delta$  satisfying

- $\delta \cdot x = x$ ,
- $\delta \cdot y = y - xy^2$ .

Leibniz rule:

- $\delta x - x\delta = x \in \mathbb{Q}\langle x, \delta \rangle$ .

The relation

$$y^{2n+1} = \frac{(1^2 - \delta^2)(3^2 - \delta^2) \cdots ((2n-1)^2 - \delta^2)}{(2n)!} \cdot y$$

gives rise to formulas for the sum of product of  $2n+1$  Euler numbers. Note that  $y^{2n} \notin \mathbb{Q}[\delta] \cdot y$  but  $y^{2n} \in \mathbb{Q}[\delta]x \cdot y$ .

# Euler numbers

## Proposition

If the sequence  $i_1, \dots, i_n$  consists of even number of odd integers and odd number of even integers, then

$$(\delta^{i_1} \cdot y)(\delta^{i_2} \cdot y) \cdots (\delta^{i_n} \cdot y) \in \mathbb{Q}[\delta] \cdot y.$$

## Examples

$$(\delta \cdot y)^2 y = \left( \frac{1}{8} - \frac{1}{12} \delta^2 - \frac{1}{24} \delta^4 \right) \cdot y$$

$$(\delta^2 \cdot y) y^2 = \left( -\frac{1}{4} + \frac{1}{3} \delta^2 - \frac{1}{12} \delta^4 \right) \cdot y$$

## Euler numbers

The relation

$$(\delta \cdot y)^2 y = \left( \frac{1}{8} - \frac{1}{12} \delta^2 - \frac{1}{24} \delta^4 \right) \cdot y$$

gives rise to the identity

$$\sum_{i+j+k=n} \binom{n}{i, j, k} E_{i+1} E_{j+1} E_k = \frac{1}{8} E_n - \frac{1}{12} E_{n+2} - \frac{1}{24} E_{n+4}.$$

## Euler numbers

The relation

$$(\delta^2 \cdot y)y^2 = \left(-\frac{1}{4} + \frac{1}{3}\delta^2 - \frac{1}{12}\delta^4\right) \cdot y$$

gives rise to the identity

$$\sum_{i+j+k=n} \binom{n}{i, j, k} E_{2+i} E_j E_k = -\frac{1}{4} E_n + \frac{1}{3} E_{n+2} - \frac{1}{12} E_{n+4}.$$