IPM, December 29, at 10:00

Let $S=\bigoplus_{n=0}^{\infty} S_n$ be a graded ring and

$S_+ = \bigoplus_{n \gneq 0} S_n$ and consider to

$$X := \mathrm{Proj}(S) := \{ \text{homo. prime ideal } p \mid p \not\supseteq S_+ \}$$

For any homo. ideal I of S, set

$$V_+(I) = \{ p \in X \mid p \supseteq I \}$$

<u>Ex</u>. • $V_+(I) \cup V_+(J) = V_+(IJ)$

• $\bigcap V_+(I_\alpha) = V_+(\sum I_\alpha)$

Thus X is topological space.

Let $U \subseteq X$ be open set and $p \in U$. Set

$S_{(p)} := \{ f/g \mid f, g \in S_d,\ g \notin p \}$, and

define

$$\mathcal{O}(U) := \left\{ s : U \longrightarrow \bigsqcup_{p \in U} S_{(p)} \;\middle\|\; \begin{array}{l} \bullet\ s(p) \in S_{(p)} \\ \bullet\ \forall p \in U\ \exists V \underset{\text{open}}{\subseteq} U,\ p \in V \\ \text{and } f, g \in S \text{ s.t. } \forall q \in V \\ s(q) = f/g \text{ (of course with same degree)} \end{array} \right\}$$

Call the second condition by "locally cte"

$\mathcal{O}(U)$ is a ring with usual $+, \times$ of function.

Let us to recall def. of sheaf:

Def Let X be topological space (e.g. $Proj(S)$).

Consider to a category $\mathcal{C}$ where

$$Ob(\mathcal{C}) = \left\{ \begin{array}{c} \text{open subsets} \\ \text{of } X \end{array} \right\}$$

$$Mor(U, V) = \begin{cases} U \hookrightarrow V & \text{if } U \subset V \\ \phi & \text{if } U \not\subset V \end{cases}$$

* By a presheaf $\mathcal{F} : \mathcal{C} \longrightarrow$ Abelian group we mean a contravariant functor s.t. $\mathcal{F}(\phi) = \{0\}$

* A presheaf is called sheaf if $\{U_i\}$ is an open covering for $U \subseteq X$ open and $s_i \in F(U_i)$ are s.t.

image (s_i under $F(U_i) \longrightarrow F(U_i \cap U_j)$

is the same as $F(U_j) \longrightarrow F(U_i \cap U_j)$

Then $\exists !\ s \in F(U)$ s.t. the map

$F(U) \longrightarrow F(U_i)$ send s to s_i.

Remark Let F be any presheaf. Then $\exists !$ sheaf F^+ and a natural trasformation $F^+ \longrightarrow F$ s.t. if G is any sheaf with a natural transformation $F \longrightarrow G$ then $\exists !\ \phi$ s.t.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & G \\ F^+ \arrow[ur, dashed] & \end{tikzcd}$$

Let us comeback to our setting

$X := \mathrm{Proj}(S)$ and $\forall\, U \subseteq X$ open

$\mathcal{O}(U)$ was defined.

<u>Lemma</u> $\mathcal{O}(-)$ is a sheaf of rings on $\mathrm{Proj}(S)$

* Let $f \in S$ be homogen. and define

$D(f) = \{ p \in X \mid f \notin p \}$

Also, for $p \in X$, define

$$\mathcal{O}_{X,p} := \varinjlim_{p \in U} \mathcal{O}(U)$$

<u>FACT A</u> $\mathrm{Proj}(S) = \bigcup_{f \text{ is hom}} \underbrace{D(f)}_{\text{open}}$

<u>FACT B</u> $\mathcal{O}(D(f)) = S_{(f)} := \left\{ g/f^n \;\middle|\; \deg g = n \cdot \deg f \right\}$

<u>FACT C</u> $\mathcal{O}_{X,p} = S_{(p)}$

<u>Def</u>: A morphism $\left(X := \mathrm{Proj}(S), \mathcal{O}_X\right) \longrightarrow \left(Y := \mathrm{Proj}(R), \mathcal{O}_Y\right)$ is a pair $(f, f^\#)$ s.t.

$f : \mathrm{Proj}(S) \longrightarrow \mathrm{Proj}(R)$ is cont. map of top. spaces

$f^\#$ defined as follows: $\forall\, U' \subseteq U \subseteq Y$ open,

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{f^\#} & \mathcal{O}_X(f^{-1}(U)) \\ \downarrow & & \downarrow \\ \mathcal{O}_Y(U') & \xrightarrow{f^\#} & \mathcal{O}_X(f^{-1}(U')) \end{array}$$

RHS is called "direct image of $\mathcal{O}_X$"

Now let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a graded S-module and $U \subseteq X$ be open

$$\widetilde{M}(U) := \left\{ s : U \longrightarrow \bigsqcup M_{(p)} \;\middle|\; \begin{array}{l} s(p) \in M_{(p)} \\ \text{and } s \text{ is locally cte} \end{array} \right\}$$

The usual scalar mult. shows that $\widetilde{M}(U)$ is $\mathcal{O}(U)$-module. Also, $\widetilde{M}$ is a sheaf, called such a thing quasi-coherent.

• Ex: How can define stalk of a ~~t~~coherent sheaf?

Notaion Let $n \in \mathbb{Z}$ and denote shifting degree n-times to the left by $M(n)$, i.e.,

$$\boxed{M(n)_i := M_{i+n}}$$

Notaion • $\mathcal{O}(n) := \widetilde{S(n)}$

• $\mathcal{F}(n)$ the sheaf associated to the presheaf:

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(n)(U)$$

Our next aim is to define Cech Cohomology of a coherent sheaf of $\mathcal{O}_X$-module over $\operatorname{Proj}(S)$

Def Let $\mathcal{U} = \{U_i\}$ be an affine open covering of X, i.e., $X = \bigcup U_i$ and $U_i = \operatorname{Spec}(A_i)$

$$C^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p})$$

Take $\alpha \in C^p(\mathcal{U}, \mathcal{F})$. We want to define $d(\alpha)$ as an element in $C^{p+1}(\mathcal{U}, \mathcal{F})$. Note that $\alpha = (\alpha_{i_0, \dots, i_p})_{i_0 < i_1 < \cdots < i_p}$

Each element of $C^{p+1}(\mathcal{U}, \mathcal{F})$ is a seq. of $(d+2)$-tuple. Set

- $V_k := \bigcap_{\substack{0 \le j \le p+1 \\ \text{and } j \neq k}} U_{i_j}$ (where i_{p+1} is an integer $> i_p$)
- $W = V_k \cap U_{i_k}$
- $\rho_{V_k, W} : \mathcal{F}(V_k) \longrightarrow \mathcal{F}(W)$

Now defin $i_0, \dots, i_{p+1}$ component of $d(\alpha)$ by

$$\sum_{k=0}^{p+1} (-1)^k \rho_{V_k, W} (\alpha_{i_0, \dots, \hat{i}_k, \dots, i_{p+1}})$$

the other component are zero

<u>Lemma</u>
- $d^2 = 0$
- Let $\mathcal{U}' = \{ U_i' \}$ be any affine open covering of X. Then $H^i(C(\mathcal{U}, \mathcal{F})) = H^i(C(\mathcal{U}', \mathcal{F}))$

☆ <u>Def</u> $\boxed{H^i(X, \mathcal{F}) := H^i(C(\mathcal{U}, \mathcal{F}))}$

Thm

- $H^i(X, \widetilde{M}(n)) = H^i_{S_+}(M)_n \quad \forall i \geq 1.$

- There is the following exact seq:

$$0 \longrightarrow H^0_{S_+}(M) \longrightarrow M \longrightarrow \bigoplus_{n \in \mathbb{Z}} H^0(X, \widetilde{M(n)}) \longrightarrow H^1_{S_+}(M) \longrightarrow 0$$

In particular,
- $H^i(X, \mathcal{F})$ is f.g. S_0-module
- $H^i(X, \widetilde{M}(n)) = 0 \quad \forall\ n \gg 0$

this is called "cohomological criterion of ampleness"

- $H^i(X, \mathcal{F}) = 0 \quad \forall i > \dim X$

Note that:
$\dim X = \dim S - 1$

☆ $\chi(F) := \sum_{i=0}^{\dim X} (-1)^i \dim H^i(X, \mathcal{F})$ is well-define.

A little derived functor

Let $\varphi: \mathcal{F} \longrightarrow \mathcal{G}$ be morph. of sheaves (i.e. a natural transformation between two functors)

- $\ker(\varphi)(U) := \ker(\mathcal{F}(U) \longrightarrow \mathcal{G}(U))$
- $\operatorname{Im}(\varphi)$ is the sheaf associated to the following presheaf $U \longmapsto \operatorname{Im} \varphi(U)$

Def $\mathcal{F}_1 \xrightarrow{\varphi_1} \mathcal{F}_2 \xrightarrow{\varphi_2} \mathcal{F}_3$ is called exact if $\operatorname{im} \varphi_1 = \ker \varphi_2$.

FACT Category of sheaves of $\mathcal{O}_X$-module has enough injective.

Let $\mathcal{F}$ be any sheaf of $\mathcal{O}_X$-module and let $0 \to \mathcal{F} \to I^0 \to I^1 \to \cdots$ be the injective resolution of $\mathcal{F}$ and let

$$\mathcal{J} : \left\{ \begin{array}{c} \text{Sheaves of} \\ \mathcal{O}_X\text{-module} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Ab.} \\ \text{groups} \end{array} \right\}$$

be any (left exact) functor.

By $(R^i \mathcal{J})(\mathcal{F})$ we mean

$$H^i(\cdots \to \mathcal{J}(I^{i-1}) \to \mathcal{J}(I^i) \to \mathcal{J}(I^{i+1}) \to \cdots)$$

Ex: Define $\operatorname{Ext}^i(\mathcal{F}, \mathcal{G})$ as derived functor of $\operatorname{Hom}(\mathcal{F}, \mathcal{G})$

Ex: Let P be any local property of ring (or modules) (e.g. Regularity, CMness, freeness, ...). One can exted the property P in the context of scheme (or sheaves) by passing through stalks.

What is locally free sheaf?

Notation $\mathbb{P}^n_A := \operatorname{Proj}(A[x_0, \dots, x_n])$

Ex 1) Forget the gradeding of S and define structure of scheme on $Spec(S)$

2) Let R be any ring. What is $\mathbb{A}^n_R$?

3) Find a relation between $\mathbb{A}^n_{\mathbb{C}}$ and $\mathbb{C}^n$ (with usual Zar. top.).

Ref R. Hartshorne, Algebraic Geometry.