Building integral domains inside power series rings

Sylvia Wiegand
(work of W. Heinzer, C. Rotthaus, S. Wiegand)

Department of Mathematics
University of Nebraska–Lincoln

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Questions

Question1: [Judy Sally, 1990] What rings lie between a ring and its field of fractions?

Question 2: [inspired by S. Abhyankar's work] What rings lie between a ring and a power series ring containing that ring? Such as, between $\mathbb{Q}[x, y]_{(x,y)}$ and $\mathbb{Q}[y]_{(y)}[[x]]$, for $\mathbb{Q}$ the field of rational numbers and x, y indeterminates over k.

•$\exists$ uncountably many $\tau_i \in \mathbb{Q}[y]_{(y)}[[x]]$, that are algebraically independent over $\mathbb{Q}(x, y)$ [Abhyankar, PAMS, 1956]. (Even analytically indep.!)

• In our work we show $\exists$ a wide variety of integral domains fitting Questions 1 and 2. See book at:

[Reference: http://www.math.unl.edu/ swiegand1/2016Aprpower.pdf]

(April 2016 version—May be updated September 2016.)

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  $k[x, y]_{(x,y)}$, k a field, x, y indeterminates over k,
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- $S =$ a homomorphic image of a power series ring over R,
- $L =$ a field with $R \subseteq L \subseteq \mathcal{Q}(S)$, the total quotient ring of S.

Basic Construction 3 yields many unusual Noetherian or non-Noetherian extension rings A of R.

This talk: Classical examples of Nagata and Rotthaus simplified, streamlined by Basic Construction 1 & techniques of the book.

Universality

Basic Construction 3 is universal in the sense that

For EVERY $(A, \mathfrak{n})$ = Noetherian local domain with field of fractions L, if

- $\exists k$ = a "coefficient field" for A ($k \cong A/\mathfrak{n}, k \subseteq A$) and
- L finitely generated over k,

then $\exists (R, \mathfrak{m})$ such that $\bullet A = L \cap S$, as in Basic Construction 1, where

- $S = \widehat{R}/I$, I = ideal of the $\mathfrak{m}$-adic completion $\widehat{R}$,
- $(R, \mathfrak{m})$ a Noetherian local domain
- k = a coefficient field for R,
- L = is the field of fractions of R and
- R is "essentially finitely generated over k"

  (R = (a fin. gen k-algebra), localized).

Note: If time, we may show this universality property.

Two goals of book:

Goal 1: Construct new non-Noetherian integral domains to illustrate recent advances in ideal theory, such as examples featured in Ardibil.

Goal 2: Construct new examples of Noetherian rings,
• Continue tradition of [1930s] Akizuki, Schmidt, (on integral closure)

[1950s] Nagata * (2-dim RLR, not Nagata, not excellent);

[1970-1990s]. Brodmann & Rotthaus, Ferrand & Raynaud, Heitmann, Lequain, Nishimura, Ogoma (normal Noetherian local domain but not universally catenary), Weston and others.
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• New Noetherian examples in book include (1) 3-dim Noetherian RLR $(A, \mathfrak{n})$ with a prime ideal P and $\mathfrak{n}$-adic completion $\widehat{\mathfrak{n}}$, such that $P\widehat{A}$ is not integrally closed. [answers question of C. Huneke]

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(2) $\forall n \geq 2$, a catenary Noetherian local domain with geometrically regular formal fibers that is not universally catenary.

Today: Nagata & Christel examples

Today: Discuss the Nagata and Christel examples:

Nagata Example: A 2-dim RLR, not Nagata.

Christel Example: A Nagata domain that is not excellent.

In the process, • Discuss/define “Nagata ring" & “excellent ring".

- Give simpler form of Basic Construction 3, techniques & theorems used.

General Setting, Simpler Construction

(A recent version, R need not be Noetherian, but "close".)

Setting 4: Let R be an integral domain, $x \in R, x \neq 0$, x a nonunit. Let $R^* = x$-adic completion= inverse limit $(R/(x^n R))$ as $n \to \infty$.

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(2) For $(R, \mathfrak{m})$ local Noetherian, $\widehat{R} = \mathfrak{m}$-adic completion= inverse limit $(R/(\mathfrak{m}^n R))$ as $n \to \infty \sim$ "power series" in more elements.)

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Construction 5: Let $\underline{\tau} = \tau_1, \tau_2, \ldots, \tau_s \in xR^*$ be algebraically independent over R.

Assume
- The elements of $R[\underline{\tau}]$ are regular in R^*.

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Define the Intersection Domain:

$$A = \mathcal{Q}(R[\underline{\tau}]) \cap R^*.$$

Prototype examples:

Example 6: k a field $R = k[x,y]_{(x,y)}$; $R^* = k[y]_{(y)}[[x]]$.
Let $\tau \in xk[[x]]$ be transcendental over $k(x)$; eg $\tau = e^x - 1$ for $k = \mathbb{Q}$.

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Define $\boxed{D = k(x,y,\tau) \cap (k[y]_{(y)}[[x]])}$.

Fact: $D = V[y]_{(x,y)}$, where $V = k(x,\tau) \cap k[[x]]$.

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Example 7: k a field $\quad R = k[x,y,z]_{(x,y,z)}$; $\quad R^* = k[y,z]_{(y,z,)}[[x]]$.

Similarly define $\boxed{D' = k(x,y,z,\tau,\sigma) \cap (k[y,z]_{(y,z)}[[x]])}$, where $\tau, \sigma \in xk[[x]]$ are alg. indep. over $k(x)$, such as for $k = \mathbb{Q}$, $\sigma = e^x - 1$, $\tau = e^{x^2} - 1$.

Fact: $D' = V'[y,z]_{(x,y,z)}$, where $V' = k(x,\tau,\sigma) \cap k[[x]]$.

So V' is a DVR, D' is an RLR.

Approximation Domains:

Definitions/notes 8: (with Setting 4, Construction 5 above).

For $\tau \in xR^*$, write $\tau = \sum_{i=1}^{\infty} a_i x^i$. (non-unique)

For every $n \in \mathbb{N}$, the n^{th} endpiece τ_n of τ is:

$$\tau_n := \sum_{i=n+1}^{\infty} a_i x^{i-n}.$$

Note: $\tau_n := a_{n+1}x + \sum_{i=n+2}^{\infty} a_i x^{i-n-1} = a_{n+1}x + x\tau_{n+1}$.

For several elements, set $\underline{\tau} = \tau_1, \dots, \tau_s$, and $\underline{\tau}_n = \tau_{1n}, \tau_{2n}, \dots, \tau_{sn}$.

Define $U_n := R[\underline{\tau}_n]$, $\quad B_n := (U_n)_{(\mathfrak{m}_R, \underline{\tau}_n)}$.

By Note, $U_n \subseteq U_{n+1} \subseteq U_n[1/x]$, $\quad B_n \subseteq B_{n+1}$, and

$B_n[1/x]$ is a localization of $U_0 = R[\underline{\tau}]$.

Define $U := \bigcup U_n$, and the Approximation Domain B

$$B := \bigcup B_n.$$

Construction Properties & Flatness Theorems:

New versions: R not necessarily Noetherian.
Use Setting 4, Construction 5 above.

Construction properties theorem 9:

- $A^* = B^* = R^*$ (x-adic completions).
- $R/xR = B/xB = A/xA = R^*/xR^*$.
- $B[1/x]$ is a localization of $U_0 = R[\underline{\tau}]$.
- If R is a UFD, so is B.
- If R is a regular Noetherian UFD, so is $B[1/x]$.
- If R is Noetherian, so is $B[1/x]$.
- If $(R, \mathfrak{m})$ is quasi-local, so are A, B, R^*, with max ideals $\mathfrak{m}_A = \mathfrak{m}A, \mathfrak{m}_B = \mathfrak{m}B$, and $\mathfrak{m}^*_R = \mathfrak{m}R^*$.

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Noetherian Flatness Theorem 10 TAE:

1. $\psi : U_0 = R[\underline{\tau}] \hookrightarrow R^*[1/x]$ is flat.
2. B is Noetherian.
3. $B \hookrightarrow R^*$ is faithfully flat.
4. A is Noetherian *and* $A = B$.

Prototypes are Approximation Domains

Proof for Prototype Example 6 above $R = k[x,y]_{(x,y)}$, where k, a field characteristic $k \neq 2$, $R^* = k[y]_{(y)}[[x]]$, and $\tau \in xk[[x]]$
$D = k(x,y,\tau) \cap R^*$.

$k[x][\tau] \hookrightarrow k[[x]][1/x]$ is flat (always true for an inclusion into a field).
$\implies k[y] \otimes_k k[x][\tau] \hookrightarrow k[y] \otimes_k k[[x]][1/x]$ is flat.
$\implies k[x,y][\tau] \hookrightarrow k[y][[x]][1/x]$ is flat, and
$\implies U_0 = R[\tau] \hookrightarrow R^*[1/x]$ is flat.
$\implies D =$ the associated Approximation Domain and D is Noetherian, by Noetherian Flatness Theorem 10 (overkill). □

Similarly D' in Prototype Example 7 equals its Approximation Domain and is Noetherian.

Nagata Example set-up

(Nagata constructs A as a nested union of localized polynomial rings; here A is an intersection.)

(Inside Prototype Example 6 above) $R = k[x,y]_{(x,y)}$, with characteristic $k \neq 2$, $R^* = k[y]_{(y)}[[x]]$, and $\tau \in xk[[x]]$. Then
$\bullet D = k(x,y,\tau) \cap R^* =$ its Approximation Domain, $\bullet D$ is Noetherian
$\bullet U_0 = R[\tau] \hookrightarrow R^*[1/x]$ is flat.

Nagata Example 11: Let $f = (y+\tau)^2 \in xR^*$, z an indeterminate.

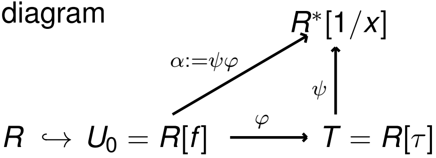
Define $A := \ k(x,y,f) \cap k[y]_{(y)}[[x]]$ $\qquad E := \dfrac{A[z]}{(z^2 - f)A[z]}$,

Then: $\bullet A$ has unique max ideal $\mathfrak{m}_A = (x,y)A$, $\bullet$ The element f is prime in A, $\bullet \widehat{A} = k[[x,y]]$.

The Nagata Example A is a 2-dim RLR

Here $R = k[x,y]_{(x,y)}$, characteristic $k \neq 2$, $R^* = k[y]_{(y)}[[x]]$,
$A := \ k(x,y,f) \cap k[y]_{(y)}[[x]]$.

Proof: Consider the diagram



The ring $T = R[\tau]$ is a free $R[f]$-module with free basis $\langle 1, y + \tau \rangle$.
$\therefore \varphi$ is flat.

Since τ defines the Prototype D, $\psi : R[\tau] \hookrightarrow R^*[1/x]$ is flat.

Now $\alpha = \psi \circ \varphi \implies \alpha$ is flat.
Noetherian Flatness Theorem 10 $\implies$ A is Noetherian (and $=$ its Approximation Domain). Since $A = B \hookrightarrow k[[x,y]]$ is a flat local homomorphism, A is a RLR [Matsumura, "Comm Rings", Thm 23.7].

dim $A = 2$, since $\mathfrak{m}_A = (x,y)A$. □

Properties of Nagata example

Here $R = k[x,y]_{(x,y)}$, char $k \neq 2$, $R^* = k[y]_{(y)}[[x]]$,
$A := k(x,y,f) \cap k[y]_{(y)}[[x]]$, $fA = (y+\tau)^2$.

Definition: A Noetherian ring R is Nagata if • $\forall P \in \operatorname{Spec} R$ and $\forall$ finite extension field L of $\mathcal{Q}(R/P)$, $\overline{R/P}^L$ (integral closure of R/P in L) is finitely generated as a module over R/P.

Claim: A is not a Nagata ring.

Proof: $fA = (y+\tau)^2 \implies \widehat{A/fA} = \widehat{A}/f\widehat{A}$ has a nonzero nilpotent element, and $\dim(A/fA) = 1$. $\therefore$ the integral closure $\overline{A/fA}$ is not finitely generated over A/fA by

Theorem: [Nagata "Local rings", Ex. 1, p. 22] For R a 1-dim Noeth. local domain, $\overline{R}$ (integral closure of R) is a finitely generated R-module $\iff$ R is "analytically unramified" ($\widehat{R}$ has no nilpotent elements).

$\therefore$ A is not a Nagata ring.

What about E?

Here $R = k[x, y]_{(x,y)}$, where k, a field characteristic $k \neq 2$, $R^* = k[y]_{(y)}[[x]]$, and $\tau \in xk[[x]]$. Let $f = (y + \tau)^2 \in xR^*$, and let z be an indeterminate.

$A := k(x, y, f) \cap k[y]_{(y)}[[x]] \quad E := \frac{A[z]}{(z^2 - f)A[z]}$,

•$E =$ integrally closed Noetherian local domain.

•E is “analytically reducible" ($\widehat{E}$ is not an integral domain), because

$f\widehat{A} = (y + \tau)^2\widehat{A} \implies \widehat{E} = \frac{k[[x,y,z]]}{(z-(y+\tau))(z+(y+\tau))} \implies$ not integral domain.

This was important —the first such example (about 1956).

Christel's Example

(Originally A was constructed as a direct limit.)

Christel's Example: (Inside D' of Prototype Example 7)
$R = k[x, y, z]_{(x,y,z)}$, $x, y, z =$ indeterminates over $k =$ field of characteristic 0, $\sigma, \tau \in xk[[x]]$ alg. indep. over $k(x)$.

Let $f := (y + \sigma)(z + \tau) \in xk[[x]])$. Define

$$\boxed{A \ := \ k(x, y, z, f) \ \cap \ (k[y, z]_{(y,z)]}[[x]]).}$$

• The completion $\widehat{A}$ of A is $k[[x, y, z]]$. If A is Noetherian, then A is a 3-dimensional regular local domain.

We show A is Noetherian on the next slide.

Christel's Example A is Noetherian

Here $R = k[x, y, z]_{(x,y,z)}$, characteristic $k = 0$, $R^* = k[y, z]_{(y,z)}[[x]]$, $\sigma, \tau \in xk[[x]]$, $f := (y + \sigma)(z + \tau)$.$A := \ k(x, y, f) \cap k[y, z]_{(y,z)}[[x]]$.

Consider the diagram

$$
\begin{array}{ccccc}
 & & & & R^*[1/x] \\
 & & & \overset{\alpha:=\psi\varphi}{\nearrow} & \uparrow \psi \\
R \hookrightarrow & U_0 = R[f] & & \xrightarrow{\varphi} & T = R[\sigma, \tau]
\end{array}
$$

Fact: The ring $T = R[\sigma, \tau]$ is flat over $R[f]$, since the coefficients of $f = yz + \sigma z + \tau y + \sigma\tau$ as a polynomial in σ, τ generate R.

$\therefore \varphi$ is flat.

Since σ, τ defines the Prototype D', $\psi : R[\sigma, \tau] \hookrightarrow R^*[1/x]$ is flat.

Now $\alpha = \psi \circ \varphi \implies \alpha$ is flat.
Noetherian Flatness Theorem 10 $\implies$ A is Noetherian (and $=$ its Approximation Domain). □

Excellence

Definitions: • For $f : A \to B$ a ring homomorphism of Noetherian rings, and $P \in \operatorname{Spec} A$. The fiber over P with respect to f is geometrically regular if $F =$ finite extension field of $\mathcal{Q}(A/P) \implies B \otimes_A F$ is a Noetherian regular ring (every localization is an RLR).

• Let $(R, \mathfrak{m}) =$ Noetherian local ring, $\widehat{R} = \mathfrak{m}$-adic completion of R.

• A Noetherian ring A is excellent if

(i) A is universally catenary,

(ii) For every prime ideal P of A, the map from A_P to its PA_P-adic completion is regular (has geometrically regular fibers).

(iii) For every finitely generated A-algebra B, the set $\operatorname{Reg}(B) = \{P \in \operatorname{Spec} B \mid B_P \text{ is an RLR}\}$ is an open subset of $\operatorname{Spec} B$.

Remarks: •$\mathbb{Z}$, all fields and all complete Noetherian local rings are excellent. • All Dedekind domains of characteristic zero are excellent. • Every excellent ring is a Nagata ring by [Matsumura “Comm. Alg", Thm. 78, p. 257].

Properties of Christel's Example

Claim: Christel's Example A is a Nagata domain that is not excellent.

Proof: $\bullet(y-\sigma, z-\tau)\widehat{A} =$ height-two prime ideal of A. Fact: $(y-\sigma, z-\tau)\widehat{A} \cap A = (y-\sigma)(z-\tau)A$. $\therefore (y-\sigma)(z-\tau)A \in \operatorname{Spec} A$. But $\widehat{A}_{(y-\sigma),z-\tau)A)\widehat{A}}/(y-\sigma)(z-\tau)\widehat{A}_{(y-\sigma),z-\tau)\widehat{A}}$ is a non-regular formal fiber of A. ("formal"="fiber of the map to $\widehat{A}$".) $\therefore A$ is not excellent.

• Since $k \subseteq A$ & k characteristic zero, A is a Nagata domain if $\forall P \in \operatorname{Spec} A$, $\overline{A/P}$ is a finite A/P-module by

Theorem [Matsumura, "Comm. Rings", p. 262] Let R be an integrally closed Noetherian integral domain with field of fractions K. If L/K is a finite separable algebraic field extension, then the integral closure of R in L is a finite R-module. If R has characteristic zero, then the integral closure of R in a finite algebraic field extension is a finite R-module.

• Since the formal fibers of A are reduced, the integral closure of A/P is a finite A/P-module, by Theorem used for the Nagata example. □

d-coefficient Theorem:

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d-coeff. Theorem: [HRW] Let $d \geq 2$, $R := k[x, y_1, \dots y_d]_{(x, y_1, \dots y_d)}$.
Then $\exists B \mid R \subseteq B \subseteq R^* := k[y_1, \dots, y_d]_{(-)}[[x]] \subseteq \widehat{R} := k[[x, y_1, \dots, y_d]]$, and B has a prime ideal $Q := (y_1 y_2, \dots, y_d)R^* \cap B$ such that:

1. $B =$ a non-Noetherian local UFD, maximal ideal $\mathfrak{m}_B = \mathfrak{m}_R B$, $\dim(B) = d + 2$.
2. The $\mathfrak{m}_B$-adic completion $\widehat{B} = k[[x, \underline{y}]]$, $\dim(\widehat{B}) = d + 1$.
3. $B[1/x] =$ a Noetherian regular UFD; $B/xB =$ an RLR, dim d; $\forall P \in \operatorname{Spec} B$, $P \neq \mathfrak{m}_B \implies B_P =$ an RLR.

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4. $Q = \bigcup_{n=1}^{\infty} Q_n$, $Q_n = (y_1, y_2, \dots, y_d, f_n)B_n$; Q not finitely generated. $\{Q\} = \{P \in \operatorname{Spec} B \mid \operatorname{ht} P = d + 1\}$. ($f_n$ = later.)

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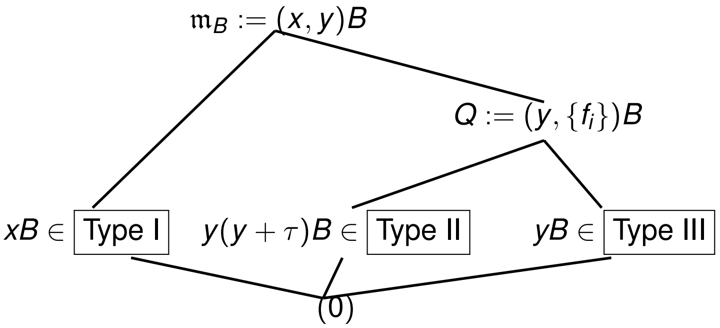
3. $B[1/x] =$ a Noetherian regular UFD; $B/xB =$ an RLR, dim d; $\forall P \in \operatorname{Spec} B$, $P \neq \mathfrak{m}_B \implies B_P =$ an RLR.
4. $Q = \bigcup_{n=1}^{\infty} Q_n$, $Q_n = (y_1, y_2, \dots, y_d, f_n)B_n$; Q not finitely generated. $\{Q\} = \{P \in \operatorname{Spec} B \mid \text{ht } P = d + 1\}$. ($f_n =$ later.)
5. $\mathcal{C} =$ a saturated chain in Spec $B \implies$ length $(\mathcal{C}) = d + 1$ or $d + 2$; $\exists$ such $\mathcal{C}$ with length $(\mathcal{C}) = d + 1, d + 2$; $\therefore B$ not catenary,



Spec B

“Type I" =“B/P is Noetherian"; “Type III"= “P not contracted."

“Type II" = “$P = P^* \cap B$, $\exists P^* \in \mathrm{Spec}(k[[x, y]]$."

$yz\tau$ Theorem:

"$yz\tau$" Example Theorem: $\exists B \mid k[x,y,z]_{(x,y,z)} \subseteq B \subseteq k[[x,y,z]]$ and:

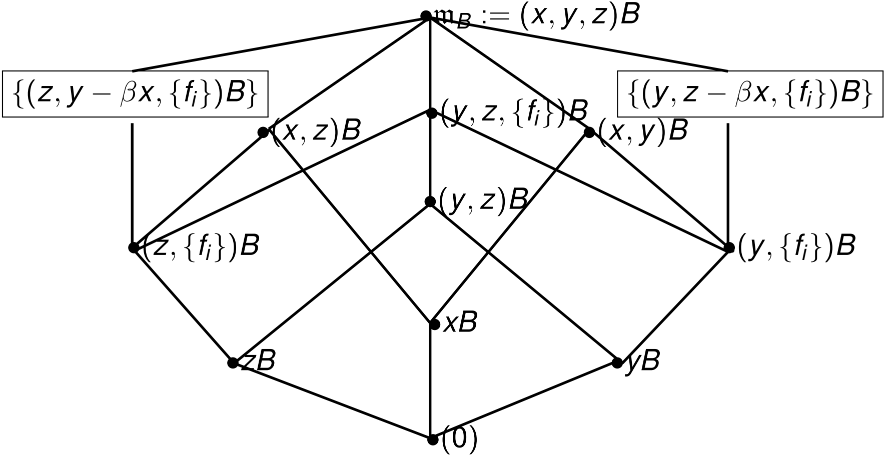
1. $B =$ 4-dim local UFD, max ideal $\mathfrak{m}_B = (x,y,z)B$, $\widehat{B} = k[[x,y,z]]$.
2. $B[1/x] =$ Noetherian regular UFD, $\dim(B/xB) = 2$. If $P \in \operatorname{Spec} B$, B_P an RLR $\iff$ B_P is Noetherian $\iff$ $(yz,x)R^* \cap B \not\subseteq P$. $\therefore$ ht $P \leq 2 \implies B_P$ is an RLR.
3. $B_{(x,y)B}$ and $B_{(x,z)B}$ are 3-dim non-Noetherian local UFDs.

$yz\tau$ Theorem, cont

$yz\tau$ Question: What ideals of B are finite generated? Partial answer:

1. Every height-one prime ideal is principal.
2. $Q_1 := (y, \{f_n\})B = yR^* \cap B$, $\quad Q_2 := (z, \{f_n\})B = zR^* \cap B$, $Q_3 := (y, z, \{f_n\})B = (y, z)R^* \cap B$ are prime ideals, not finitely generated; $\quad$ ht $Q_1 =$ ht $Q_2 = 2$, ht $Q_3 = 3$.
3. The prime ideals $(x, y)B$ and $(x, z)B$ have height three.
4. If P is a height-two prime ideal of B that contains an element of the form $y + g(z, x)$ or $z + h(x, y)$, where $0 \neq g(z, x) \in (x, z)k[x, z]$ and $0 \neq h(x, y) \in (x, y)k[x, y]$, then P is generated by two elements.
5. If $\mathfrak{a}$ is an ideal of B that contains $x + yzg(y, z)$, for some polynomial $g(y, z) \in k[y, z]$, then $\mathfrak{a}$ is finitely generated.
6. $\exists\infty$ many ht-3 non-finitely generated prime ideals, e.g. $Q_{i,\alpha} = (y - \alpha x^i, z, \{f_n\})B$, where $i \in \mathbb{N}$ and $\alpha \in k$.

Part of Spec B, for the $yz\tau$ example, $B \subset k[[x, y, z]]$



THANKS!