CLASSIFICATION PROBLEMS OF SUBCATEGORIES

RYO TAKAHASHI

Contents

0. Introduction

Aim. Given a category $\mathcal{C}$, classify the $***$ subcategories of $\mathcal{C}$.

- Ring Theory
- Stable Homotopy Theory
- Modular Representation Theory
- Algebraic Geometry

Strategy. For a category $\mathcal{C}$, find a commutative ring R and make a 1-1 correspondence

$$\begin{array}{c}\{\, *** \text{ subcategories of } \mathcal{C} \,\} \\ f \Big\downarrow \quad \cong \quad \Big\uparrow g \\ \{\, +++ \text{ subsets of } \operatorname{Spec} R \,\}.\end{array}$$

Convention. Throughout, unless otherwise specified,

- R = commutative Noetherian ring
- $\operatorname{mod} R$ = the category of f.g. (finitely generated) R-modules
- subcategory = nonempty full subcategory closed under isomorphisms
- A subcategory $\mathcal{X}$ of a category $\mathcal{C}$ $\rightsquigarrow$ $\boxed{\mathcal{X} \subseteq \mathcal{C}}$

1. Serre subcategories of module categories

Theorem 1.1 (Gabriel (1962))**.**

$$\begin{array}{c}\{\, \text{Serre subcategories of } \operatorname{mod} R \,\} \\ \text{Supp} \Big\downarrow \quad \cong \quad \Big\uparrow \text{Supp}^{-1} \\ \{\, \text{Specialization closed subsets of } \operatorname{Spec} R \,\}\end{array}$$

Definition 1.2. $\mathcal{X} \subseteq \operatorname{mod} R$ is Serre if for every exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{mod} R$, it holds

$$M \in \mathcal{X} \Leftrightarrow L, N \in \mathcal{X}.$$

In other words, $\mathcal{X}$ is closed under $\begin{cases} \text{submodules,} \\ \text{quotient modules,} \\ \text{extensions.} \end{cases}$

Example 1.3. The following are Serre.
(1) $\{ M \in \operatorname{mod} R \mid M \text{ has finite length} \}$.
(2) $\{ M \in \operatorname{mod} R \mid M \text{ is a torsion module} \}$.
(3) $\{ M \in \operatorname{mod} R \mid I^n M = 0 \text{ for some } n \geq 0 \}$ for a fixed ideal $I \subseteq R$.

Definition 1.4. $W \subseteq \operatorname{Spec} R$ is specialization-closed if it satisfies

$$\mathfrak{p} \in W,\ \mathfrak{p} \subseteq \mathfrak{q} \in \operatorname{Spec} R \ \Rightarrow \ \mathfrak{q} \in W.$$

Remark 1.5. TFAE (The Following Are Equivalent) for $W \subseteq \operatorname{Spec} R$.
(1) W is specialization-closed.
(2) W is a union of closed subsets of $\operatorname{Spec} R$.

Definition 1.6. Let $M \in \operatorname{mod} R$, $\mathcal{X} \subseteq \operatorname{mod} R$ and $W \subseteq \operatorname{Spec} R$.
(1) $\operatorname{Supp} M = \{ \mathfrak{p} \in \operatorname{Spec} R \mid M_{\mathfrak{p}} \neq 0 \}$.
(2) $\operatorname{Supp} \mathcal{X} = \bigcup_{M \in \mathcal{X}} \operatorname{Supp} M$.
(3) $\operatorname{Supp}^{-1} W = \{ M \in \operatorname{mod} R \mid \operatorname{Supp} M \subseteq W \}$.

Example 1.7. (1) $\{ M \in \operatorname{mod} R \mid M \text{ has finite length} \} = \operatorname{Supp}^{-1} \operatorname{Max} R$.
(2) $\{ M \in \operatorname{mod} R \mid M \text{ is a torsion module} \} = \operatorname{Supp}^{-1}\{ \text{Prime ideals containing nonzerodivisors} \}$.
(3) $\{ M \in \operatorname{mod} R \mid I^n M = 0 \text{ for some } n \geq 0 \} = \operatorname{Supp}^{-1} V(I)$.

Proof of Theorem 1.1. Fix $\begin{cases} \mathcal{X} \subseteq \operatorname{mod} R & \text{Serre,} \\ W \subseteq \operatorname{Spec} R & \text{specialization-closed.} \end{cases}$
Want to show:
(1) $\operatorname{Supp} \mathcal{X}$ is specialization-closed,
(2) $\operatorname{Supp}^{-1} W$ is Serre,
(3) $\operatorname{Supp} \operatorname{Supp}^{-1} W = W$,
(4) $\operatorname{Supp}^{-1} \operatorname{Supp} \mathcal{X} = \mathcal{X}$.

(1) $\operatorname{Supp} \mathcal{X} = \bigcup_{M \in \mathcal{X}} \operatorname{Supp} M$ and $\operatorname{Supp} M$ is closed.
(2) $0 \to L \to M \to N \to 0 \ \Rightarrow\ \operatorname{Supp} M = \operatorname{Supp} L \cup \operatorname{Supp} N$.

$$\begin{aligned} M \in \operatorname{Supp}^{-1} W &\Rightarrow \operatorname{Supp} M \subseteq W \\ &\Rightarrow \operatorname{Supp} L \subseteq W \text{ and } \operatorname{Supp} N \subseteq W \\ &\Rightarrow L, N \in \operatorname{Supp}^{-1} W. \end{aligned}$$

(3)($\subseteq$) $\mathfrak{p} \in \operatorname{Supp} \operatorname{Supp}^{-1} W \ \Rightarrow\ \mathfrak{p} \in \operatorname{Supp} M\ (\exists\, M \in \operatorname{Supp}^{-1} W) \ \Rightarrow\ \mathfrak{p} \in W$.
($\supseteq$) $\mathfrak{p} \in W \ \Rightarrow\ \mathfrak{p} \in \operatorname{Supp} R/\mathfrak{p} = V(\mathfrak{p}) \subseteq W \ \Rightarrow\ \mathfrak{p} \in \operatorname{Supp} \operatorname{Supp}^{-1} W$.
(4)($\supseteq$) $M \in \mathcal{X} \ \Rightarrow\ \operatorname{Supp} M \subseteq \operatorname{Supp} \mathcal{X} \ \Rightarrow\ M \in \operatorname{Supp}^{-1} \operatorname{Supp} \mathcal{X}$.

(⊆)

$$\begin{aligned}
& M \in \operatorname{Supp}^{-1} \operatorname{Supp} \mathcal{X} \\
\Rightarrow\ & \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} = \operatorname{Min} M \subseteq \operatorname{Supp} M \subseteq \operatorname{Supp} \mathcal{X} = \bigcup_{N \in \mathcal{X}} \operatorname{Supp} N \\
\Rightarrow\ & \mathfrak{p}_i \in \operatorname{Supp} N_i \ (\exists\, N_i \in \mathcal{X}),\ N := N_1 \oplus \cdots \oplus N_n \\
\Rightarrow\ & \operatorname{Supp} M = V(\mathfrak{p}_1) \cup \cdots \cup V(\mathfrak{p}_n) \subseteq \operatorname{Supp} N \\
\Rightarrow\ & M \in \operatorname{Serre} N \subseteq \mathcal{X} \text{ by Proposition 1.8 below.}
\end{aligned}$$

□

Proposition 1.8 (Essential part of Theorem 1.1)**.** Let $M, N \in \operatorname{mod} R$ with $\operatorname{Supp} M \subseteq \operatorname{Supp} N$. Then

$$M \in \operatorname{Serre} N,$$

where $\operatorname{Serre} N$ denotes the smallest Serre subcategory containing N.
In other words, M can be built from N by taking submodules, quotient modules and extensions finitely many times.

Proof. Put $\mathcal{X} = \operatorname{Serre} N$.

$$\begin{aligned}
\mathfrak{p} \in \operatorname{Supp} M \ &\Rightarrow\ \mathfrak{p} \in \operatorname{Supp} N \\
&\Rightarrow\ \exists\, \mathfrak{q} \in \operatorname{Min} N \subseteq \operatorname{Ass} N \text{ s.t. } \mathfrak{q} \subseteq \mathfrak{p} \\
&\Rightarrow\ 0 \to R/\mathfrak{q} \to N,\ N \in \mathcal{X} \\
&\Rightarrow\ R/\mathfrak{q} \in \mathcal{X},\ R/\mathfrak{q} \to R/\mathfrak{p} \to 0 \\
&\Rightarrow\ R/\mathfrak{p} \in \mathcal{X}.
\end{aligned}$$

$\exists\, 0 = M_0 \subsetneq \cdots \subsetneq M_n = M$ s.t. $M_i/M_{i-1} \cong R/\mathfrak{p}_i,\ \mathfrak{p}_i \in \operatorname{Supp} M$.
$0 \to M_{i-1} \to M_i \to R/\mathfrak{p}_i \to 0,\ R/\mathfrak{p}_i \in \mathcal{X} \ \rightsquigarrow\ M \in \mathcal{X}$. □

Remark 1.9. Let R be a commutative ring. Several extensions of Theorem 1.1:

(1) Gabriel (1962)
Localizing subcategories of $\operatorname{Mod} R$ if R is Noetherian
(2) Hovey (2000)
Wide subcategories of $\operatorname{mod} R$ if R is a quotient ring of a coherent regular ring by a f.g. ideal
(3) Garkusha-Prest (2008)
Serre subcategories of $\operatorname{mod} R$ if R is coherent
Torsion classes of finite type of $\operatorname{Mod} R$
(4) Krause (2008)
Wide subcategories of $\operatorname{Mod} R$ closed under (arbitrary) direct sums if R is Noetherian
(5) Stanley-Wang (2011)
Torsion classes and narrow subcategories of $\operatorname{mod} R$ if R is Noetherian

Theorem 1.10 (T (2008))**.** Let R be a commutative Noetherian ring.

$$\begin{array}{c}\{\,\text{Subcategories of } \operatorname{mod} R \text{ closed under submodules and extensions}\,\}\\ {\scriptstyle\mathrm{Ass}}\big\downarrow \quad \cong \quad \big\uparrow{\scriptstyle\mathrm{Ass}^{-1}}\\ \{\,\text{Subsets of } \operatorname{Spec} R\,\}.\end{array}$$

Moreover, restricting this to Serre subcategories, one can recover Theorem 1.1.

Definition 1.11. Let $M \in \operatorname{mod} R$, $\mathcal{X} \subseteq \operatorname{mod} R$ and $W \subseteq \operatorname{Spec} R$.

(1) $\operatorname{Ass} M = \{\, \mathfrak{p} \in \operatorname{Spec} R \mid \exists\, R/\mathfrak{p} \hookrightarrow M \,\}$.
(2) $\operatorname{Ass} \mathcal{X} = \bigcup_{M \in \mathcal{X}} \operatorname{Ass} M$.
(3) $\operatorname{Ass}^{-1} W = \{\, M \in \operatorname{mod} R \mid \operatorname{Ass} M \subseteq W \,\}$.

Proposition 1.12 (Essential part of Theorem 1.10)**.** Let $M, N \in \operatorname{mod} R$ with $\operatorname{Ass} M \subseteq \operatorname{Ass} N$. Then

$$M \in \operatorname{subext} N,$$

where $\operatorname{subext} N$ denotes the smallest subcategory containing N and closed under submodules and extensions.

Proof. Set $\mathcal{X} := \operatorname{subext} N$ and $\operatorname{Ass} M =: \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$.

$$\begin{cases} 0 = M_1 \cap \cdots \cap M_n \ (\exists\, M_i \subseteq M \quad \mathfrak{p}_i\text{-primary}), \\ M = M/M_1 \cap \cdots \cap M_n \hookrightarrow M/M_1 \oplus \cdots \oplus M/M_n, \\ \operatorname{Ass} M/M_i = \{\mathfrak{p}_i\} \subseteq \operatorname{Ass} M \subseteq \operatorname{Ass} N \end{cases}$$

$\rightsquigarrow$ May assume $\operatorname{Ass} M = \{\mathfrak{p}\}$.
Suppose $M \notin \mathcal{X}$.
Write $\operatorname{Hom}_R(M, R/\mathfrak{p}) = \langle f_1, \dots, f_m \rangle$.

$$\exists\, 0 \to M' \to M \xrightarrow{\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}} (R/\mathfrak{p})^{\oplus m}.$$

$(R/\mathfrak{p})^{\oplus m} \in \mathcal{X},\ M \notin \mathcal{X} \ \Rightarrow\ M' \notin \mathcal{X},\ \operatorname{Ass} M' = \{\mathfrak{p}\}$.
Write $\operatorname{Hom}_R(M', R/\mathfrak{p}) = \langle f'_1, \dots, f'_{m'} \rangle$.

$$\exists\, 0 \to M'' \to M' \xrightarrow{\begin{pmatrix} f'_1 \\ \vdots \\ f'_{m'} \end{pmatrix}} (R/\mathfrak{p})^{\oplus m'}.$$

$(R/\mathfrak{p})^{\oplus m'} \in \mathcal{X},\ M' \notin \mathcal{X} \ \Rightarrow\ M'' \notin \mathcal{X},\ \operatorname{Ass} M'' = \{\mathfrak{p}\}$.
$0 \neq \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, \kappa(\mathfrak{p})) = \langle (f_1)_\mathfrak{p}, \dots, (f_m)_\mathfrak{p} \rangle \ \Rightarrow\ M_\mathfrak{p} \supsetneq M'_\mathfrak{p}$
$\Rightarrow\ M_\mathfrak{p} \supsetneq M'_\mathfrak{p} \supsetneq M''_\mathfrak{p} \supsetneq \cdots$ $R_\mathfrak{p}$-modules of finite length $\rightsquigarrow$ contradiction. $\square$

2. Resolving subcategories of module categories

Definition 2.1 (Auslander-Bridger (1969))**.** $\mathcal{X} \subseteq \operatorname{mod} R$ is resolving if it satisfies

(R1) $\mathcal{X}$ contains $\operatorname{proj} R := \{\text{ f.g. projective } R\text{-modules }\}$.
(R2) $\mathcal{X}$ is closed under direct summands:
$M \in \mathcal{X},\ N \lessdot M \ \Rightarrow\ N \in \mathcal{X}$.
(R3) $\mathcal{X}$ is closed under extensions:
$0 \to L \to M \to N \to 0,\ L, N \in \mathcal{X} \ \Rightarrow\ M \in \mathcal{X}$.
(R4) $\mathcal{X}$ is closed under kernels of epimorphisms:
$0 \to L \to M \to N \to 0,\ M, N \in \mathcal{X} \ \Rightarrow\ L \in \mathcal{X}$.

Remark 2.2. (1) (R3) implies
$\mathcal{X}$ is closed under (finite) direct sums:
$M, N \in \mathcal{X} \ \Rightarrow\ M \oplus N \in \mathcal{X}$.
(2) (R1) can be replaced with
(R1') $R \in \mathcal{X}$.
(3) (R4) can be replaced with
(R4') $\mathcal{X}$ is closed under syzygies:
$M \in \mathcal{X} \ \Rightarrow\ \Omega M \in \mathcal{X}$.

Example 2.3. The following are resolving.

(1) $\operatorname{mod} R$
(2) $\operatorname{proj} R$
(3) $\mathrm{CM}(R) := \{\text{ MCM } R\text{-modules }\}$ if R is a CM local ring
(4) (Auslander-Bridger (1969))
$\mathrm{G}(R) := \{\text{ Totally reflexive } R\text{-modules }\}$
(5) For a fixed module $E \in \operatorname{mod} R$:
$\{\, M \in \operatorname{mod} R \mid \operatorname{Tor}^R_{>0}(M, E) = 0 \,\}$
$\{\, M \in \operatorname{mod} R \mid \operatorname{Ext}_R^{>0}(M, E) = 0 \,\}$
(6) For a fixed ideal $I \subseteq R$:
$\{\, M \in \operatorname{mod} R \mid \operatorname{grade}(I, M) \geq \operatorname{grade}(I, R) \,\}$
(7) If R is local:
$\{\, M \in \operatorname{mod} R \mid \sup_{i \geq 0}\{\beta_i^R(M)\} < \infty \,\}$
$\{\, M \in \operatorname{mod} R \mid \operatorname{cx}_R M < \infty \,\}$
$\{\, M \in \operatorname{mod} R \mid \mathrm{CI}_*\mathrm{dim}_R M \leq 0 \,\}$

Remark 2.4 (Auslander-Bridger (1969))**.** Let $\mathcal{X} \subseteq \operatorname{mod} R$ be resolving. If

$$0 \to N \to X_{n-1} \to \cdots \to X_0 \to M \to 0,$$
$$0 \to N' \to X'_{n-1} \to \cdots \to X'_0 \to M \to 0$$

are exact sequences in $\operatorname{mod} R$ with $X_i, X'_i \in \mathcal{X}\ \forall i$, then

$$N \in \mathcal{X} \ \Leftrightarrow\ N' \in \mathcal{X}.$$

Definition 2.5. Let $\mathcal{X} \subseteq \operatorname{mod} R$.

(1) A homomorphism $\phi : X \to M$ in $\operatorname{mod} R$ with $X \in \mathcal{X}$ is a right $\mathcal{X}$-approximation (or an $\mathcal{X}$-precover) of M if

$$\operatorname{Hom}_R(X', \phi) : \operatorname{Hom}_R(X', X) \to \operatorname{Hom}_R(X', M)$$

is surjective.

$$\begin{array}{ccc} X & \xrightarrow{\phi} & M \\ & & \Big\uparrow{\scriptstyle \forall \phi'} \\ & & X' \end{array}$$

(2) $\mathcal{X}$ is contravariantly finite (or precovering) if every $M \in \operatorname{mod} R$ has a right $\mathcal{X}$-approximation.

Example 2.6. The following are contravariantly finite.

(1) $\operatorname{proj} R$
(2) $\operatorname{mod} R$
(3) (Auslander-Bridger (1969))
$\{ M \in \operatorname{mod} R \mid \operatorname{Ext}^1_R(M, R) = 0 \}$
(4) (Auslander-Reiten (1992, 1994))
When R is Artinian and $n \geq 0$:
$\{ M \in \operatorname{mod} R \mid \exists\, 0 \to M \to P^0 \to \cdots \to P^{n-1},\ P^i \in \operatorname{proj} R\ \forall i \}$
$\{ M \in \operatorname{mod} R \mid \exists\, I_{n-1} \to \cdots \to I_0 \to M \to 0,\ I_i \in \operatorname{inj} R\ \forall i \}$
(5) (Iyama (2003))
When $R \to S$ module-finite:
$\{ M \in \operatorname{mod} R \mid M \text{ is a f.g. } S\text{-module} \}$
(6) (Auslander-Smalø (1980))
A subcategory closed under direct sums and having only finite many indecomposable modules

Proof. Let $\mathcal{X}$ be such a subcategory. Set

$$\{ \text{indecomposable modules in } \mathcal{X} \} =: \{X_1, \dots, X_n\}.$$

Then $X := X_1 \oplus \cdots \oplus X_n \in \mathcal{X}$. Write $\operatorname{Hom}_R(X, M) = \langle f_1, \dots, f_m \rangle$. Then

$$\phi : X^{\oplus m} \to M,\ \phi \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \sum_{i=1}^{m} f_i(x_i)$$

is a right $\mathcal{X}$-approximation of M. □

(7) (Auslander-Buchweitz (1989))
$\mathrm{CM}(R)$ if R is CM local with canonical module

Proof. Every $M \in \operatorname{mod} R$ has a CM approximation:

$$0 \to Y \to X \xrightarrow{\phi} M \to 0$$

s.t. $X \in \mathrm{CM}(R)$, $\operatorname{id}_R Y < \infty \rightsquigarrow \phi$ is a right $\mathrm{CM}(R)$-approximation. □

Remark 2.7. TFAE for $\mathcal{X} \subseteq \operatorname{mod} R$.

(1) $\mathcal{X}$ is contravariantly finite.

(2) $\mathrm{Hom}_R(-, M) \in \mathrm{Mod}\,\mathcal{X}$ is f.g. for all $M \in \mathrm{mod}\,R$.
Here, $\mathrm{Mod}\,\mathcal{X}$ denotes the functor category of $\mathcal{X}$.

Theorem 2.8 (Auslander-Reiten (1991))**.** Let R be an Artin algebra with $\mathrm{gldim}\,R < \infty$. Then

$$\{\text{ Contravariantly finite resolving subcategories of } \mathrm{mod}\,R\,\} \quad \Big\downarrow \ \cong\ \Big\uparrow (-)^{\perp} \quad \{\text{ Basic cotilting } R\text{-modules }\}/\cong .$$

Theorem 2.9 (T (2011))**.** Let R be a Gorenstein complete local ring. Then

$$\{\text{ Contravariantly finite resolving subcategories of } \mathrm{mod}\,R\,\} \;\; \| \;\; \{\,\mathrm{proj}\,R,\ \mathrm{CM}(R),\ \mathrm{mod}\,R\,\}.$$

In particular, all resolving subcategories other than these three contain infinitely many indecomposable modules.

More generally:

Theorem 2.10 (T (2011))**.** Let R be complete local. Let $\mathcal{X} \subsetneq \mathrm{mod}\,R$ be contravariantly finite resolving. Assume $\exists\, G \in \mathcal{X}$ s.t. $\mathrm{Ext}_R^{\gg 0}(G, R) = 0$ and $\mathrm{pd}_R\, G = \infty$. Then R is CM, and $\mathcal{X} = \mathrm{CM}(R)$.

Proof of Theorem 2.9. Let $\mathcal{X} \subseteq \mathrm{mod}\,R$ be contravariantly finite resolving.
(1) When $\mathrm{pd}\,G = \infty$ for some $G \in \mathcal{X}$:

$$R \text{ is Gorenstein} \Rightarrow \mathrm{Ext}^{>\dim R}(G, R) = 0$$

$$\Rightarrow \begin{cases} \mathcal{X} = \mathrm{mod}\,R, \\ \mathcal{X} = \mathrm{CM}(R) \end{cases} \quad \text{by Theorem 2.10.}$$

(2) When $\mathrm{pd}\,X < \infty$ for all $X \in \mathcal{X}$:
May assume $\mathcal{X} \neq \mathrm{mod}\,R \;\Rightarrow\; \exists\, M \in \mathrm{mod}\,R$ s.t. $M \notin \mathcal{X}$.
$\mathcal{X}$ is contravariantly finite $\Rightarrow$ M has a right $\mathcal{X}$-approximation $\phi : X \to M$.

$$\begin{array}{ccc} X & \xrightarrow{\phi} & M \\ & & \big\uparrow {\scriptstyle \exists\,\mathrm{surj}} \\ & & R^{\oplus} \end{array}$$

$\rightsquigarrow$ ϕ is surjective.

$$\exists\, 0 \to Y \to X \xrightarrow{\phi} M \to 0.$$

By Wakamatsu's lemma: $\mathrm{Ext}^1(N, Y) = 0$ for all $N \in \mathcal{X} \;\rightsquigarrow\; \mathrm{Ext}^{>0}(N, Y) = 0$.

$$\begin{aligned} N \in \mathcal{X} \;&\Rightarrow\; \mathrm{pd}\,N < \infty \\ &\Rightarrow\; 0 = \sup\{\, i \mid \mathrm{Ext}^i(N, Y) \neq 0\,\} = \mathrm{depth}\,R - \mathrm{depth}\,N = \mathrm{pd}\,N \\ &\Rightarrow\; N \in \mathrm{proj}\,R. \end{aligned}$$

Thus $\mathcal{X} = \mathrm{proj}\,R$. □

Corollary 2.11 (Christensen-Piepmeyer-Striuli-T (2008))**.** Let R be a complete local ring with an algebraically closed coefficient field of characteristic 0 and with a nonfree totally reflexive module. TFAE:

(1) R is a simple singularity.
(2) There are only finitely many indecomposable totally reflexive R-modules.

Proof. (2) $\Rightarrow$ (1)
By Example 2.3(4) and Example 2.6(6), $\mathrm{G}(R)$ is contravariantly finite resolving.

$$\begin{aligned}\exists\, G \in \mathrm{G}(R) \text{ nonfree } &\Rightarrow \ \operatorname{pd} G = \infty,\ \operatorname{Ext}^{>0}(G,R) = 0 \\ &\Rightarrow \ \text{By Theorem 2.10, either holds:}\end{aligned}$$

(a) $\mathrm{G}(R) = \operatorname{mod} R$,
(b) R is CM and $\mathrm{G}(R) = \mathrm{CM}(R)$.

In either case, R is Gorenstein.
By Buchweitz-Greuel-Herzog-Knörrer-Schreyer, R is a simple singularity. □

3. Thick subcategories of triangulated categories

Definition 3.1. Let $\mathcal{T}$ be a triangulated category. $\mathcal{X} \subseteq \mathcal{T}$ is thick if

(T1) $\mathcal{X}$ is closed under direct summands:
$M \in \mathcal{X},\ N \lessdot M$ in $\mathcal{T}\ \Rightarrow\ N \in \mathcal{X}$.
(T2) $\mathcal{X}$ is closed under exact triangles:
$L \to M \to N \to \Sigma L$ an exact triangle,
$(2 \text{ of } L, M, N) \in \mathcal{X}\ \Rightarrow\ (\text{3rd}) \in \mathcal{X}$.

Remark 3.2. (1) (T2) means: $\mathcal{X}$ is a triangulated subcategory of $\mathcal{T}$.
(2) Verdier (1977): "épaisse subcategory"
Localization of triangulated categories (Verdier quotient)

Classification of thick subcategories

Stable homotopy theory

- Ravenel (1984)
- Devinatz-Hopkins-Smith (1988)
- Hopkins-Smith (1998) (Telescope Conjecture)

Thick subcategories of the category of compact objects in the p-local stable homotopy category

Ring theory

- Hopkins (1987)
- Neeman (1992)

Notation 3.3. Denote:
$\mathrm{D^b}(R)$ = the bounded derived category of $\operatorname{mod} R$,
$\mathrm{D_{perf}}(R) = \{\text{Perfect complexes}\} \subseteq \mathrm{D^b}(R)$,
perfect complex = bounded complex of f.g. projective R-modules.

Theorem 3.4 (Hopkins-Neeman)**.**

$$\begin{array}{c}\{\,\text{Thick subcategories of } \mathrm{D}_{\mathrm{perf}}(R)\,\}\\ {\scriptstyle\mathrm{Supp}}\Big\downarrow \;\cong\; \Big\uparrow{\scriptstyle\mathrm{Supp}^{-1}}\\ \{\,\text{Specialization closed subsets of } \operatorname{Spec} R\,\}.\end{array}$$

For $\mathcal{X} \subseteq \mathrm{D}_{\mathrm{perf}}(R)$ and $W \subseteq \operatorname{Spec} R$:

(1) $\operatorname{Supp} \mathcal{X} = \bigcup_{M \in \mathcal{X}} \operatorname{Supp} M = \{\, \mathfrak{p} \in \operatorname{Spec} R \mid \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R M \ncong 0 \,\}$,
where $\operatorname{Supp} M = \operatorname{Supp} \mathrm{H}(M)$, $\mathrm{H}(M) = \bigoplus_{i \in \mathbb{Z}} \mathrm{H}^i(M) \in \operatorname{mod} R$,

(2) $\operatorname{Supp}^{-1} W = \{\, M \in \mathrm{D}_{\mathrm{perf}}(R) \mid \operatorname{Supp} M \subseteq W \,\}$.

- Thomason (1997)
  Extension to quasi-compact quasi-separated schemes
- Avramov-Buchweitz-Christensen-Iyengar-Piepmeyer (2010)
  Thick subcategories of the derived category of perfect differential modules

Modular representation theory

- Benson-Carlson-Rickard (1997)
  Thick subcategories of the stable category of finite dimensional representations of a finite p-group
- Friedlander-Pevtsova (2007)
  Extension to finite group schemes
- Benson-Iyengar-Krause (2012)
  Extension to thick subcategories of $\mathrm{D}^{\mathrm{b}}(kG)$

Definition 3.5. Let R be CM.

$$\underline{\mathrm{CM}}(R) = \frac{\mathrm{CM}(R)}{\operatorname{proj} R} \quad \underline{\text{CM stable category}} \text{ of } R$$

$$\begin{cases} \mathrm{Ob}(\underline{\mathrm{CM}}(R)) = \mathrm{Ob}(\mathrm{CM}(R)) = \{\,\text{MCM } R\text{-modules}\,\}, \\ \mathrm{Hom}_{\underline{\mathrm{CM}}(R)}(M,N) = \underline{\mathrm{Hom}}_R(M,N) = \dfrac{\mathrm{Hom}_R(M,N)}{\{\, f \mid f \text{ factors through a projective module}\,\}}. \end{cases}$$

Fact 3.6 (Buchweitz (1987), Happel (1988))**.** $\underline{\mathrm{CM}}(R)$ is triangulated if R is Gorenstein.

$\forall\, M \in \underline{\mathrm{CM}}(R)$, $\exists\, 0 \to M \to R^{\oplus} \to N \to 0$.
$\Sigma M := N \;\rightsquigarrow\; \Sigma : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R)$ an automorphism with a quasi-inverse Ω.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & R^{\oplus} & \longrightarrow & \Sigma M & \longrightarrow & 0 \\ & & {\scriptstyle f}\big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & N & \longrightarrow & C & \longrightarrow & \Sigma M & \longrightarrow & 0 \end{array}$$

$\rightsquigarrow \quad M \xrightarrow{f} N \to C \to \Sigma M$ an exact triangle in $\underline{\mathrm{CM}}(R)$, $\operatorname{cone}(f) := C$.

Remark 3.7. Let R be a d-dimensional CM local ring with canonical module ω and having an isolated singularity.

(1) Auslander-Reiten duality:

$$\underline{\mathrm{Hom}}_R(M,N) \cong \mathrm{Ext}^1_R(N,\tau M)^\vee$$

for $M,N \in \underline{\mathrm{CM}}(R)$, where $\tau M = \mathrm{Hom}_R(\Omega^d \mathrm{Tr} M, \omega)$ is the Auslander-Reiten translation.

(2) Calabi-Yau property ($\rightsquigarrow$ Cluster theory):
If R is Gorenstein, then

$$\underline{\mathrm{Hom}}_R(M,N) \cong \underline{\mathrm{Hom}}_R(N,\Sigma^{d-1}M)^\vee,$$

i.e., $\underline{\mathrm{CM}}(R)$ is $(d-1)$-Calabi-Yau.

Definition 3.8. Let R be a (commutative) Noetherian ring.

$$\mathrm{D_{sg}}(R) = \frac{\mathrm{D^b}(R)}{\mathrm{D_{perf}}(R)} \quad \text{the \underline{singularity category} of } R$$

$$\begin{cases} \mathrm{Ob}(\mathrm{D_{sg}}(R)) = \mathrm{Ob}(\mathrm{D^b}(R)) = \{\,\text{Bounded complexes of f.g. } R\text{-modules}\,\}, \\ \mathrm{Hom}_{\mathrm{D_{sg}}(R)}(M,N) = \{\,(M \xleftarrow{s} X \xrightarrow{f} N) \mid s,f \in \mathrm{Mor}\,\mathrm{D^b}(R),\ \mathrm{cone}(s) \in \mathrm{D_{perf}}(R)\,\}. \end{cases}$$

Remark 3.9. Orlov: Mirror symmetry

Fact 3.10 (Buchweitz (1987))**.** $\mathrm{D_{sg}}(R) \cong \underline{\mathrm{CM}}(R)$ if R is Gorenstein.

Problem. Classify the thick subcategories of $\underline{\mathrm{CM}}(R)$ when R is Gorenstein!

Theorem 3.11 (T (2010, 2011))**.** (1) Let R be a local hypersurface. Set

$$\begin{aligned}
A &= \{\,\text{Specialization-closed subsets of } \mathrm{Sing}\,R\,\},\\
B &= \{\,\text{Thick subcategories of } \underline{\mathrm{CM}}(R)\,\},\\
C &= \{\,\text{Thick subcategories of } \mathrm{CM}(R) \text{ containing } R\,\},\\
D &= \{\,\text{Thick subcategories of } \mathrm{mod}\,R \text{ containing } R\,\},\\
E &= \{\,\text{Thick subcategories of } \mathrm{D^b}(R) \text{ containing } R\,\},\\
F &= \{\,\text{Resolving subcategories of } \mathrm{mod}\,R \text{ contained in } \mathrm{CM}(R)\,\}.
\end{aligned}$$

(2) Let $(R,\mathfrak{m},k)$ be d-dimensional Gorenstein singular (= nonregular) local ring, locally a hypersurface on the punctured spectrum. Set

$$\begin{aligned}
A &= \{\,\text{Nonempty specialization-closed subsets of } \mathrm{Sing}\,R\,\},\\
B &= \{\,\text{Thick subcategories of } \underline{\mathrm{CM}}(R) \text{ containing } \Omega^d k\,\},\\
C &= \{\,\text{Thick subcategories of } \mathrm{CM}(R) \text{ containing } R, \Omega^d k\,\},\\
D &= \{\,\text{Thick subcategories of } \mathrm{mod}\,R \text{ containing } R, \Omega^d k\,\},\\
E &= \{\,\text{Thick subcategories of } \mathrm{D^b}(R) \text{ containing } R, k\,\},\\
F &= \{\,\text{Resolving subcategories of } \mathrm{mod}\,R \text{ contained in } \mathrm{CM}(R) \text{ containing } \Omega^d k\,\}.
\end{aligned}$$

In each case, $A \cong B \cong C \cong D \cong E \cong F$ holds. More precisely:

$$\begin{array}{ccccc} & & E & & \\ & & \uparrow{\scriptstyle \text{thick}} & & \\ & & D & & \\ & & \uparrow{\scriptstyle \text{thick}} & & \\ A & \xleftarrow{\text{NF}} & C & = & F \\ & & \uparrow{\scriptstyle \text{can}} & & \\ & & B & & \end{array}$$

Remark 3.12. Recently, Stevenson extended $A \cong B$ in (1) to a Noetherian ring which is locally a hypersurface. More recently, he and Krause extended $C \cong D \cong E$ in (1) to an exact category with enough projective objects.

Definition 3.13. (1) $\operatorname{Sing} R = \{ \mathfrak{p} \in \operatorname{Spec} R \mid R_\mathfrak{p} \text{ is singular} \}$.
(2) A local ring R is a hypersurface if $\widehat{R} \cong S/(f)$ for some complete RLR S and $f \in S$.
(3) For $\mathcal{X} \subseteq \operatorname{mod} R$, $\operatorname{NF}(\mathcal{X}) = \bigcup_{M \in \mathcal{X}} \operatorname{NF}(M)$, where

$$\operatorname{NF}(M) = \{ \mathfrak{p} \in \operatorname{Spec} R \mid M_\mathfrak{p} \text{ is nonfree over } R_\mathfrak{p} \}.$$

(4) For $\mathcal{X} \subseteq \mathrm{D^b}(R)$, $\operatorname{IPD}(\mathcal{X}) = \bigcup_{M \in \mathcal{X}} \operatorname{IPD}(M)$, where

$$\operatorname{IPD}(M) = \{ \mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} = \infty \}.$$

(5) For $\mathcal{X} \subseteq \underline{\mathrm{CM}}(R)$, $\underline{\operatorname{Supp}}\, \mathcal{X} = \bigcup_{M \in \mathcal{X}} \underline{\operatorname{Supp}}\, M$, where

$$\underline{\operatorname{Supp}}\, M = \{ \mathfrak{p} \in \operatorname{Spec} R \mid M_\mathfrak{p} \ncong 0 \text{ in } \underline{\mathrm{CM}}(R_\mathfrak{p}) \}.$$

(6) $\mathcal{X} \subseteq \operatorname{mod} R$ is thick if
  (a) $\mathcal{X}$ is closed under direct summands,
  (b) $\mathcal{X}$ is closed under short exact sequences:
    $0 \to L \to M \to N \to 0$ an exact sequence of f.g. modules,
    $(2 \text{ of } L, M, N) \in \mathcal{X} \ \Rightarrow\ (3\text{rd}) \in \mathcal{X}$.
(7) $\mathcal{X} \subseteq \mathrm{CM}(R)$ is thick if
  (a) $\mathcal{X}$ is closed under direct summands,
  (b) $\mathcal{X}$ is closed under short exact sequences:
    $0 \to L \to M \to N \to 0$ an exact sequence of MCM mods,
    $(2 \text{ of } L, M, N) \in \mathcal{X} \ \Rightarrow\ (3\text{rd}) \in \mathcal{X}$.

Corollary 3.14. Let R be a local hypersurface with an isolated singularity.
(1) $\{ \text{Resolving subcategories contained in } \mathrm{CM}(R) \} = \{ \operatorname{proj} R,\ \mathrm{CM}(R) \}$.
(2) $\{ \text{Thick subcategories of } \underline{\mathrm{CM}}(R) \} = \{ 0,\ \underline{\mathrm{CM}}(R) \}$.
(3) $\{ \text{Thick subcategories of } \mathrm{D^b}(R) \text{ containing } R \} = \{ \mathrm{D_{perf}}(R),\ \mathrm{D^b}(R) \}$.

Example 3.15. Let k be a field.
(1) Let $R = k[[x, y]]/(x^2)$.

$$\mathrm{CM}(R) = \operatorname{add}\{ R,\ (x),\ (x, y^n) \mid n \geq 1 \}.$$

Setting $\mathfrak{p} = (x)$, $\mathfrak{m} = (x, y)$, we have

$$\operatorname{Sing} R = \operatorname{Spec} R = \{ \mathfrak{p}, \mathfrak{m} \}.$$

Hence

$$\{ \text{Specialization-closed subsets of } \operatorname{Sing} R \} = \{ \emptyset, \{\mathfrak{m}\}, \operatorname{Sing} R \}.$$

$$\begin{cases} \mathrm{NF}^{-1}(\emptyset) = \operatorname{proj} R, \\ \mathrm{NF}^{-1}(\{\mathfrak{m}\}) = \{ \text{MCM modules locally free on the punctured spectrum} \} \\ \qquad\qquad\quad = \operatorname{add}\{ R, (x, y^n) \mid n \geq 1 \}, \\ \mathrm{NF}^{-1}(\operatorname{Sing} R) = \mathrm{CM}(R). \end{cases}$$

By Theorem 3.11(1),

$$\begin{aligned} &\{ \text{Resolving subcategories of } \operatorname{mod} R \text{ contained in } \mathrm{CM}(R) \} \\ =\ &\{ \operatorname{proj} R, \operatorname{add}\{ R, (x, y^n) \mid n \geq 1 \}, \mathrm{CM}(R) \}, \\ &\{ \text{Thick subcategories of } \underline{\mathrm{CM}}(R) \} \\ =\ &\{ 0, \underline{\operatorname{add}\{ (x, y^n) \mid n \geq 1\}}, \underline{\mathrm{CM}}(R) \}. \end{aligned}$$

(2) Let $R = k[[x, y, z]]/(x^2, yz)$.

$$\operatorname{Spec} R = \{ \mathfrak{p}, \mathfrak{q}, \mathfrak{m} \},$$

where $\mathfrak{p} = (x, y)$, $\mathfrak{q} = (x, z)$ and $\mathfrak{m} = (x, y, z)$.

$$\begin{cases} R_{\mathfrak{p}} \cong k[[x, z]]_{(x)}/(x^2), \\ R_{\mathfrak{q}} \cong k[[x, y]]_{(x)}/(x^2), \end{cases} \quad \text{and } \operatorname{Sing} R = \{ \mathfrak{p}, \mathfrak{q}, \mathfrak{m} \}.$$

$$\begin{aligned} &\{ \text{Nonempty specialization-closed subsets of } \operatorname{Sing} R \} \\ =\ &\{ V(\mathfrak{p}), V(\mathfrak{q}), V(\mathfrak{p}, \mathfrak{q}), V(\mathfrak{p}, \mathfrak{q}, \mathfrak{m}) \}. \end{aligned}$$

By Theorem 3.11(2),

$$\begin{aligned} &\#\{ \text{Thick subcategories of } \underline{\mathrm{CM}}(R) \text{ containing } \mathfrak{m} \} \\ =\ &\#\{ \text{Resolving subcategories of } \operatorname{mod} R \text{ contained in } \mathrm{CM}(R) \text{ containing } \mathfrak{m} \} \\ =\ &4. \end{aligned}$$

Corollary 3.16 (Huneke-Wiegand (1997))**.** Let R be a local hypersurface. Let $M, N \in \operatorname{mod} R$.

(1) $\operatorname{Tor}^R_{\gg 0}(M, N) = 0 \;\Leftrightarrow\; \operatorname{pd}_R M < \infty$ or $\operatorname{pd}_R N < \infty$
(2) $\operatorname{Ext}_R^{\gg 0}(M, N) = 0 \;\Leftrightarrow\; \operatorname{pd}_R M < \infty$ or $\operatorname{id}_R N < \infty$

Proof. (1)($\Rightarrow$)
$M \in \mathcal{X} := \{ X \in \operatorname{mod} R \mid \operatorname{Tor}^R_{\gg 0}(X, N) = 0 \}$.
$\mathcal{X}$ is a thick subcategory of $\operatorname{mod} R$ containing R.

By Theorem 3.11(1), $\mathcal{X} = \mathrm{IPD}^{-1}(\mathrm{IPD}(\mathcal{X}))$.

$$\begin{aligned}\operatorname{pd} M = \infty &\Rightarrow \mathfrak{m} \in \mathrm{IPD}(M) \subseteq \mathrm{IPD}(\mathcal{X})\\ &\Rightarrow \mathrm{IPD}(k) \subseteq \{\mathfrak{m}\} \subseteq \mathrm{IPD}(\mathcal{X})\\ &\Rightarrow k \in \mathrm{IPD}^{-1}(\mathrm{IPD}(\mathcal{X})) = \mathcal{X}\\ &\Rightarrow \operatorname{pd} N < \infty.\end{aligned}$$

□

Remark 3.17. Let $R = k[[x,y]]/(x^2,y^2)$, where k is a commutative ring. Then

$$\begin{cases}\mathrm{Tor}^R_{>0}(R/(x), R/(y)) = 0 = \mathrm{Ext}^{>0}_R(R/(x), R/(y)),\\ \operatorname{pd}_R R/(x) = \operatorname{pd}_R R/(y) = \operatorname{id}_R R/(y) = \infty.\end{cases}$$

4. Proof of Theorem 3.11(1)

Lemma 4.1. Let $\mathcal{T}$ be a triangulated category and $\mathcal{U} \subseteq \mathcal{T}$ a thick subcategory.

$$\begin{array}{c}\{\,\text{Thick subcategories of } \mathcal{T} \text{ containing } \mathcal{U}\,\}\\ \pi\Big\downarrow \quad \cong \quad \Big\uparrow\\ \{\,\text{Thick subcategories of } \mathcal{T}/\mathcal{U}\,\},\end{array}$$

where $\pi(\mathcal{X}) = \mathcal{X}/\mathcal{U}$.

In what follows, unless otherwise specified, let $(R,\mathfrak{m},k)$ be a local hypersurface with $\dim R = d$.

Proposition 4.2.

$$\begin{array}{c}E = \{\,\text{Thick subcategories of } \mathrm{D^b}(R) \text{ containing } R\,\}\\ \mathrm{IPD}\Big\downarrow \quad \cong \quad \Big\uparrow \mathrm{IPD}^{-1}\\ A = \{\,\text{Specialization-closed subsets of } \operatorname{Spec} R\,\}.\end{array}$$

Proof.

$$\begin{array}{c}
\{\,\text{Thick subcategories of } \mathrm{D^b}(R) \text{ containing } R\,\} \\
\| \\
\{\,\text{Thick subcategories of } \mathrm{D^b}(R) \text{ containing } \mathrm{D_{perf}}(R)\,\} \\
{\scriptstyle \text{Lemma 4.1}} \big\downarrow \cong \\
\{\,\text{Thick subcategories of } \mathrm{D_{sg}}(R)\,\} \\
{\scriptstyle \text{Fact 3.10}} \big\downarrow \cong \\
\{\,\text{Thick subcategories of } \underline{\mathrm{CM}}(R)\,\} \\
{\scriptstyle \text{nat}} \big\downarrow \cong \\
\{\,\text{Thick subcategories of } \mathrm{CM}(R) \text{ containing } R\,\} \\
{\scriptstyle \text{Proposition 4.3 below}} \big\downarrow \cong \\
\{\,\text{Specialization-closed subsets of } \operatorname{Spec} R\,\}
\end{array}$$

This sends $\mathcal{X}$ to $\mathrm{IPD}(\mathcal{X})$. □

Proposition 4.3.

$$\begin{array}{c}
C = \{\,\text{Thick subcategories of } \mathrm{CM}(R) \text{ containing } R\,\} \\
{\scriptstyle \mathrm{NF}} \big\downarrow \quad \cong \quad \big\uparrow {\scriptstyle \mathrm{NF}^{-1}} \\
A = \{\,\text{Specialization-closed subsets of } \operatorname{Spec} R\,\}.
\end{array}$$

Proposition 4.4 (Essential part of Proposition 4.3)**.** Let $M, N \in \mathrm{CM}(R)$ with $\mathrm{NF}(M) \subseteq \mathrm{NF}(N)$. Then

$$M \in \operatorname{res} N,$$

where $\operatorname{res} N$ denotes the smallest resolving subcategory containing N.

Lemma 4.5. (1) Let $(R, \mathfrak{m}, k)$ be a CM local ring with $\dim R = d$. Let $M \in \mathrm{CM}(R)$. If M is locally free on the punctured spectrum, then

$$M \in \operatorname{res} \Omega^d k.$$

(2) Let R be an Artinian hypersurface. Then

$$\{\,\text{Resolving subcategories of } \operatorname{mod} R\,\} = \{\,\operatorname{proj} R,\ \operatorname{mod} R\,\}.$$

(3) Let R be CM local, $\mathcal{X} \subseteq \mathrm{CM}(R)$ resolving, $M \in \mathrm{CM}(R)$ and $W \subseteq \operatorname{Spec} R$ nonempty finite. If $M_\mathfrak{p} \in \operatorname{add}_{R_\mathfrak{p}} \mathcal{X}_\mathfrak{p}$ for all $\mathfrak{p} \in W$, then

$$\exists\, 0 \to L \to N \to X \to 0 \text{ an exact sequence of MCM modules}$$

$$\text{s.t.} \begin{cases} X \in \mathcal{X}, \\ M \text{ is a direct summand of } N, \\ \mathrm{NF}(L) \subseteq \mathrm{NF}(M), \\ \mathrm{NF}(L) \cap W = \emptyset. \end{cases}$$

Proof of Proposition 4.4. Set $\mathcal{X} := \operatorname{res} N$. Induction on $n := \dim \operatorname{NF}(M)$.

(1) When $n \le 0$:

$\operatorname{NF}(M) \subseteq \{\mathfrak{m}\}$, i.e., M is locally free on the punctured spectrum
$\rightsquigarrow M \in \operatorname{res}(\Omega^d k) \subseteq \mathcal{X}$ by Lemma 4.5(1)(2).

(2) When $n \ge 1$:

$\mathfrak{p} \in \min \operatorname{NF}(M) \;\Rightarrow\; M_{\mathfrak{p}} \in \operatorname{add}_{R_{\mathfrak{p}}} \mathcal{X}_{\mathfrak{p}}$ by the induction hypothesis.
By Lemma 4.5(3),

$$\exists\, 0 \to L \to N \to X \to 0 \text{ s.t.} \begin{cases} X \in \mathcal{X}, \\ M \text{ is a direct summand of } N, \\ \operatorname{NF}(L) \subseteq \operatorname{NF}(M), \\ \operatorname{NF}(L) \cap \min \operatorname{NF}(M) = \emptyset \end{cases}$$

$\rightsquigarrow \dim \operatorname{NF}(L) < n$.

By the induction hypothesis $L \in \mathcal{X}$, and hence $M \in \mathcal{X}$. □

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