Homological invariants of edge ideals

Somayeh Moradi

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In 1975 a new approach of commutative algebra appeared by a work of Richard Stanley, who was the first one which used in a symmetric way concepts and technique from commutative algebra to study simplicial complexes using Stanley-Reisner rings.

Since then, the study of squarefree monomial ideals from both algebraic and combinatorial point of view is one of the most exciting topics in commutative algebra.
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Since then, the study of squarefree monomial ideals from both algebraic and combinatorial point of view is one of the most exciting topics in commutative algebra.

The starting point, is to use a finite simple graph to construct a monomial ideal, usually called the edge ideal, and to study the properties of this monomial ideal using the properties of the graph, and vice versa (Edge ideal was first introduced by Villarreal in 1990).
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Main Problems in Combinatorial Commutative Algebra

1. To study edge ideals and describe their algebraic invariants using the combinatorial invariants of graph or hypergraph

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2. Finding minimal free resolution of monomial ideals
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Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over field $k$ and $I$ a squarefree monomial ideal. The **Stanley-Reisner simplicial complex associated to $I$ on the vertex set $V = \{x_i : x_i \notin I\}$** is defined as:

$$\Delta_I = \{\{x_{i_1}, \ldots, x_{i_k}\} : i_1 < \cdots < i_k, x_{i_1} \cdots x_{i_k} \notin I\}$$

For a simplicial complex $\Delta$ with vertex set $\{x_1, \ldots, x_n\}$, the **Stanley-Reisner ideal of $\Delta$ is defined as**:

$$I_\Delta = (x_{i_1} \cdots x_{i_k} : i_1 < \cdots < i_k, \{x_{i_1}, \ldots, x_{i_k}\} \notin \Delta)$$
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For a squarefree monomial ideal $I$:

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Stanley-Reisner ring associated to $\Delta$: $k[\Delta] = R/I_\Delta$
Stanley-Reisner rings

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Stanley-Reisner ring associated to $\Delta$:

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Let $\mathcal{X}$ be a finite set and $\mathcal{E} = \{E_1, \ldots, E_s\}$ a finite collection of non-empty subsets of $\mathcal{X}$. The pair $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ is called a hypergraph. The elements of $\mathcal{X}$ and $\mathcal{E}$, respectively, are called the vertices and the edges of the hypergraph.

A hypergraph is called simple if:

- $|E_i| \geq 2$ for all $i = 1, \ldots, s$
- $E_j \subseteq E_i$ only if $i = j$. 
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A hypergraph $\mathcal{H}$ is called $d$-uniform if $|E_i| = d$ for any $E_i \in \mathcal{E}(\mathcal{H})$. 
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For a hypergraph $\mathcal{H}$ with vertex set $\{x_1, \ldots, x_n\}$ the edge ideal of $\mathcal{H}$ in the polynomial ring $R = k[x_1, \ldots, x_n]$ is defined as:

$$I(\mathcal{H}) = (x^E : E \in \mathcal{E}(\mathcal{H})),$$

where $x^E = \prod_{x_i \in E} x_i$

For a hypergraph $\mathcal{H}$, the independence complex of $\mathcal{H}$ is defined as:

$$\Delta_{\mathcal{H}} = \{F \subseteq \mathcal{X}(\mathcal{H}) : E_i \notin F, \forall E_i \in \mathcal{E}(\mathcal{H})\}$$
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For a hypergraph $\mathcal{H}$, $I(\mathcal{H}) = I_{\Delta_\mathcal{H}}$.
For a simplicial complex \( \Delta \), let \( \mathcal{H} \) be a hypergraph whose edge set is the minimal non-faces of \( \Delta \). Then \( \Delta = \Delta_{\mathcal{H}} \) and so \( I_\Delta = I(\mathcal{H}) \).
A subset $C \subseteq X$ is called a vertex cover of $H$ if it intersects all the edges of $H$.

Let $C_1, \ldots, C_n$ be the vertex covers of $H$. Then

$$I(H) = \bigcap_{i=1}^{n} P_{C_i}$$
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A hypergraph $\mathcal{H}$ is called **Cohen-Macaulay** if $k[\Delta_{\mathcal{H}}]$ is Cohen-Macaulay.
A hypergraph $\mathcal{H}$ is called \textbf{unmixed} if all the minimal vertex covers of $\mathcal{H}$ are of the same cardinality.

A hypergraph $\mathcal{H}$ is called \textbf{Cohen-Macaulay} if $k[\Delta_{\mathcal{H}}]$ is Cohen-Macaulay.
Let $I \subseteq S = k[x_1, \ldots, x_n]$, with $G(I) = \{u_1, \ldots, u_m\}$, where $u_i = \prod_{j=1}^{n} x_j^{a_{ij}}$. For each $j$, let $a_j = \max\{a_{ij} : 1 \leq i \leq m\}$ and $T$ be the polynomial ring over $k$ in the variables $x_{11}, \ldots, x_{1a_1}, \ldots, x_{21}, \ldots, x_{2a_2}, \ldots, x_{n1}, \ldots, x_{na_n}$.

The ideal $J \subseteq T$ with generating set $G(J) = \{v_1, \ldots, v_m\}$, where

$$v_i = \prod_{j=1}^{n} \prod_{k=1}^{a_{ij}} x_{jk}$$

is called the polarization of $I$. 
Preliminaries

Regularity of edge ideals
Projective dimension of edge ideals
Alexander dual concepts
Vertex cover ideals
Persistence property for monomial ideals

Polarization

Let $I \subseteq S$ be a monomial ideal and $J \subseteq T$ its polarization. Then

- $\beta_{i,j}(I) = \beta_{i,j}(J)$ for all $i$ and $j$.
- $H_{S/I}(t) = (1 - t)^{\delta} H_{T/J}(t)$, where $\delta = \dim(T) - \dim(S)$.
- $ht(I) = ht(J)$.
- $pd(S/I) = pd(T/J)$ and $\reg(S/I) = \reg(T/J)$.
- $S/I$ is Cohen-Macaulay if and only if $T/J$ is Cohen-Macaulay.

⇓

Many questions in monomial ideals can be reduced to squarefree monomial ideals.
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Many questions in monomial ideals can be reduced to squarefree monomial ideals.
Let $\Delta$ be a simplicial complex and $F \in \Delta$. Then

$$\text{del}_{\Delta}(F) = \{ G \in \Delta : G \cap F = \emptyset \}$$

For a face $F \in \Delta$, link of $F$ is defined as:

$$\text{lk}(F) = \{ G \in \Delta : G \cap F = \emptyset, \ G \cup F \in \Delta \}$$
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Let $\Delta$ be a simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$. Then $\Delta$ is **vertex decomposable** if either:

- The only facet of $\Delta$ is $\{x_1, \ldots, x_n\}$, or $\Delta = \emptyset$.
- There exists a vertex $x \in V$ such that $\text{del}_\Delta(x)$ and $\text{lk}_\Delta(x)$ are vertex decomposable, and such that every facet of $\text{del}_\Delta(x)$ is a facet of $\Delta$.

The vertex $x$ is called a **shedding vertex**.
Let $\Delta$ be a simplicial complex and $F \in \Delta$. Then

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The vertex $x$ is called a shedding vertex.
Regularity of edge ideal of graphs

Let $G$ be a simple graph. Two edges $uv$ and $xy$ are called **3-disjoint** if the induced subgraph of $G$ on $\{x, y, u, v\}$ has only two edges.

The maximum number of pairwise 3-disjoint edges in $G$ is denoted by $c(G)$. 
[Zheng (2003)] For a tree graph $G$, $\text{reg}(R/I(G)) = c(G)$.

A graph $G$ is called chordal if any cycle of length $n \geq 4$ has a chord.
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[Khosh-Ahang, Moradi (2012)] Let $G$ be a $C_5$-free vertex decomposable graph. Then $\text{reg}(R/I(G)) = c(G)$.

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Regularity of edge ideal of graphs

**Question 1:** Does the equality $\text{reg}(R/I(G)) = c(G)$ hold for any vertex decomposable graph?

**Question 2:** For which families of graphs does the equality $\text{reg}(R/I(G)) = c(G)$ hold?
Regularity of edge ideal of graphs

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**Question 2:** For which families of graphs does the equality $\text{reg}(R/I(G)) = c(G)$ hold?
Some bounds for the regularity of edge ideals

[Ha, Van Tuyl (2007)] For a graph \( G \), \( \text{reg}(R/I(G)) \leq a(G) \), where \( a(G) \) is the matching number of \( G \).

[Kiani, Moradi (2010)] Let \( G \) be a shellable graph. Then \( \text{reg}(R/I(G)) \leq n(G) \).

\[
n(G) = \max\{|V(H)| : H \in \mathcal{S}(G), H \cup W(H) \in \mathcal{S}(G)\}
\]

, where \( \mathcal{S}(G) \) is the set of all induced subgraphs of \( G \).
Some bounds for the regularity of edge ideals

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$$n(G) = \max\{|V(H)| : H \in S(G), H \cup W(H) \in S(G)|$$

, where $S(G)$ is the set of all induced subgraphs of $G$. 
[Kiani, Moradi (2010)] Let $G$ be a vertex decomposable graph. Then $\text{reg}(R/I(G)) \leq \min\{a'(G), n(G)\}$.

$a'(G)$ : the maximum number of vertex disjoint paths of length at most two in $G$ such that paths of lengths one are pairwise 3-disjoint in $G$. 
Let $G$ be a graph such that $G^c$ has no triangle, then $\text{reg}(R/I(G)) \leq 2$. In addition if $G^c$ is not chordal, then $\text{reg}(R/I(G)) = 2$. 

[Kiani, Moradi (2010)]
The graph $B$ with vertex set $V(B) = \{z, w_1, \ldots, w_d\}$ and edge set $E(B) = \{\{z, w_i\} \mid 1 \leq i \leq d\}$ is called a bouquet. The vertex $z$ is called the root of $B$, the vertices $w_i$ flowers of $B$ and the edges $\{z, w_i\}$ the stems of $B$. 
A subgraph of $G$ which is a bouquet is called a bouquet of $G$.

A set of bouquets $\mathcal{B} = \{B_1, \ldots, B_n\}$ is called strongly disjoint in $G$ if

(i) $V(B_i) \cap V(B_j) = \emptyset$ for all $i \neq j$,

(ii) we can choose a stem $e_i$ from each bouquet $B_i \in \mathcal{B}$ such that $\{e_1, \ldots, e_n\}$ is pairwise 3-disjoint in $G$. 
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$\{e_1, \ldots, e_n\}$ is pairwise 3-disjoint in $G$. 
Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a set of bouquets of $G$.

$$F(\mathcal{B}) = \{w \in V(G) \mid w \text{ is a flower of some bouquet in } \mathcal{B}\}$$

$$R(\mathcal{B}) = \{z \in V(G) \mid z \text{ is a root of some bouquet in } \mathcal{B}\}$$
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$$d_G := \max\{|F(\mathcal{B})| \mid \mathcal{B} \text{ is a strongly disjoint set of bouquets of } G\}$$

$\mathcal{B}$ is called a bouquet of type $(|F(\mathcal{B})|, |R(\mathcal{B})|)$. 
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Homological invariants of edge ideals
Projective dimension of edge ideal of graphs

In the following graph $d(G) = 4$. 
A set of bouquets $\mathcal{B} = \{B_1, \ldots, B_n\}$ is called semi strongly disjoint in $G$ if

(i) $V(B_i) \cap V(B_j) = \emptyset$ for all $i \neq j$, and

(ii) $R(\mathcal{B})$ is an independent set of $G$.

$d'_G := \max\{|F(\mathcal{B})| : \mathcal{B} \text{ is a semi-strongly disjoint set of bouquets of } G\}$. 

$d'(G) = 5$
A set of bouquets \( \mathcal{B} = \{B_1, \ldots, B_n\} \) is called semi strongly disjoint in \( G \) if

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\[ d'_G := \max\{|F(\mathcal{B})| : \mathcal{B} \text{ is a semi-strongly disjoint set of bouquets of } G\}. \]
Kimura (2011)

(i) Suppose that $G$ is a chordal graph. Then $\beta_{i,i+j}(R/I(G)) \neq 0$ if and only if there exists a subset $W$ of $V$ such that the induced subgraph $G_W$ contains a strongly disjoint set of bouquets of type $(i,j)$.

(ii) When $G$ is a forest, the graded Betti number $\beta_{i,i+j}(R/I(G))$ coincides with the number of subsets $W$ of $V$ with the same condition as in (i).
For a hypergraph $\mathcal{H}$, big height of $\mathcal{H}$ is equal to:

$$\text{bight}(I(\mathcal{H})) = \max\{|C_i| : C_i \text{ is a minimal vertex cover of } \mathcal{H}\}$$

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[Kimura (2011)] For a chordal graph $G$,

$$\text{pd}(R/I(G)) = d(G) = d'(G) = \text{bight}(I(G)).$$
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2. [Morey, Villarreal (2011)] Let $\Delta$ be a sequentially Cohen-Macaulay simplicial complex. Then 
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Projective dimension of edge ideal of graphs

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   $$\text{pd}(R/I_\Delta) = \text{bight}(I_\Delta).$$
[Khosh-Ahang, Moradi (2012)] For a $C_5$-free vertex decomposable graph $G$, $\text{pd}(R/I(G)) = d'(G) = \text{bight}(I(G))$.

A $d$-tree is a chordal graph defined inductively as follows:

(i) $K_{d+1}$ is a $d$-tree.
(ii) If $H$ is a $d$-tree, then so is $G = H \cup_{K_d} K_{d+1}$.
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[Kiani, Moradi (2010)] Let $G$ be a graph such that $G^c$ is a $d$-tree. Then $\text{pd}(R/I(G)) = \max_{v \in V(G)} \{\deg_G(v)\}$. 
Projective dimension of edge ideal of graphs

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Question: For which families of graphs, there are descriptions of $\text{pd}(\mathbb{R}/I(G))$ in terms of information from $G$?
For a $C_5$-free vertex decomposable graph $G$, $\text{pd}(R/I(G)) = d'(G) = \text{bight}(I(G))$.

A $d$-tree is a chordal graph defined inductively as follows:

(i) $K_{d+1}$ is a $d$-tree.

(ii) If $H$ is a $d$-tree, then so is $G = H \cup_{K_d} K_{d+1}$.

Let $G$ be a graph such that $G^c$ is a $d$-tree. Then $\text{pd}(R/I(G)) = \max_{v \in V(G)} \{\deg_G(v)\}$.

Question: For which families of graphs, there are descriptions of $\text{pd}(R/I(G))$ in terms of information from $G$?
Roth, Van Tuyl (2006)

Let $G$ be a graph with no minimal cycle of length 4. Let $k_{i+2}(G)$ denote the number of $(i + 2)$-cliques in $G$. Then, for any $i \geq 0$,

$$\beta_{i,i+2}(I(G)) = \sum_{u \in V(G)} \binom{\deg(u)}{i+1} - k_{i+2}(G).$$
Roth, Van Tuyl (2006)

Let $G$ be a forest. Then, for any $i \geq 1$,

$$\beta_{i,i+2}(I(G)) = \sum_{u \in V(G)} \binom{\deg(u)}{i+1}.$$

Katzman (2004)

For any $i \geq 1$, $\beta_{i,2(i+1)}(I(G))$ is equal to the number of induced subgraphs of $G$ consisting of exactly $i + 1$, 3-disjoint edges.
Roth, Van Tuyl (2006)

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Question: For which families of graphs $\beta_{i,j}(R/I(G))$ is independent of the ground field and can be described in terms of properties of $G$?
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Question: For which families of graphs $\beta_{i,j}(R/I(G))$ is independent of the ground field and can be described in terms of properties of $G$?
Alexander dual ideal

For any simplicial complex $\Delta$ with vertex set $X$, the Alexander dual simplicial complex $\Delta^\vee$ to $\Delta$ is defined as follows:

$$\Delta^\vee = \{ F \subseteq X; X \setminus F \notin \Delta \}$$

For a squarefree monomial ideal $I = (x_{1,1}x_{1,2} \cdots x_{1,k_1}, \ldots, x_{n,1}x_{n,2} \cdots x_{n,k_n})$, Alexander dual ideal of $I$ is defined as:

$$I^\vee = (x_{1,1}, x_{1,2}, \ldots, x_{1,k_1}) \cap \cdots \cap (x_{n,1}, x_{n,2}, \ldots, x_{n,k_n})$$

$$I_{\Delta^\vee} = (I_{\Delta})^\vee = (x_F^c : F \text{ is a facet of } \Delta)$$
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For a graph $G$, the ideal $I(G)^\vee$ is called the vertex cover ideal of $G$. 
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$$\Delta^\vee = \{ F \subseteq X; X \setminus F \notin \Delta \}$$

For a squarefree monomial ideal $I = (x_1,1x_1,2 \cdots x_1,k_1, \ldots, x_n,1x_n,2 \cdots x_n,k_n)$, Alexander dual ideal of $I$ is defined as:

$$I^\vee = (x_1,1, x_1,2, \cdots , x_1,k_1) \cap \cdots \cap (x_n,1, x_n,2, \cdots , x_n,k_n)$$

$$I_{\Delta^\vee} = (I_{\Delta})^\vee = (x^F \cap F \text{ is a facet of } \Delta)$$

For a graph $G$, the ideal $I(G)^\vee$ is called the vertex cover ideal of $G$. 
Terai (1999)

For a simplicial complex $\Delta$, $\text{pd}(I_{\Delta}) = \text{reg}(R/I_{\Delta^\vee})$.

Eagon-Reiner (1998)

Let $\Delta$ be a simplicial complex. Then

$k[\Delta]$ is Cohen–Macaulay $\iff I_{\Delta^\vee}$ has linear resolution
Alexander dual ideal

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Eagon-Reiner (1998)

Let $\Delta$ be a simplicial complex. Then

$k[\Delta]$ is Cohen–Macaulay $\Leftrightarrow I_{\Delta^\vee}$ has linear resolution
Alexander dual ideal

A simplicial complex $\Delta$ is called shellable, if the facets of $\Delta$ can be ordered $F_1 < \cdots < F_n$, such that for any $i < j$, there exists $v \in F_j \setminus F_i$ and $k < j$ such that $F_j \setminus F_k = \{v\}$.

A monomial ideal $I = (f_1, \ldots, f_m)$ has linear quotients, if there exists an order $f_1 < \cdots < f_m$ on the generators of $I$ such that the colon ideal $(f_1, \ldots, f_{i-1}) : f_i$ is generated by a subset of variables for all $2 \leq i \leq m$. 
A simplicial complex $\Delta$ is called **shellable**, if the facets of $\Delta$ can be ordered $F_1 < \cdots < F_n$, such that for any $i < j$, there exists $v \in F_j \setminus F_i$ and $k < j$ such that $F_j \setminus F_k = \{v\}$.

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Let $\Delta$ be a simplicial complex and $I = I_\Delta$. Then

$\Delta$ is shellable $\iff I^\vee$ has linear quotients.
Alexander dual ideal

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Let $\Delta$ be a simplicial complex and $I = I_\Delta$. Then

$\Delta$ is shellable $\iff I^\vee$ has linear quotients
For a monomial ideal $I \subseteq R$ and $d \geq 1$, let $I_{(d)}$ be the ideal generated by all monomials of degree $d$ in $I$. The ideal $I$ is called componentwise linear if for each $d$, $I_{(d)}$ has a linear resolution.

Herzog-Hibi (1999)

$k[\Delta]$ is sequentially Cohen-Macaulay if and only if $I_\Delta^\vee$ is componentwise linear.
For a monomial ideal $I \subset R$ and $d \geq 1$, let $I_{(d)}$ be the ideal generated by all monomials of degree $d$ in $I$. The ideal $I$ is called **componentwise linear** if for each $d$, $I_{(d)}$ has a linear resolution.

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$k[\Delta]$ is sequentially Cohen-Macaulay if and only if $I_{\Delta^\vee}$ is componentwise linear.

**Question**: What is the dual concept for vertex decomposability?
For a monomial ideal $I \subset R$ and $d \geq 1$, let $I_{(d)}$ be the ideal generated by all monomials of degree $d$ in $I$. The ideal $I$ is called componentwise linear if for each $d$, $I_{(d)}$ has a linear resolution.

**Herzog-Hibi (1999)**

$k[\Delta]$ is sequentially Cohen-Macaulay if and only if $I_{\Delta^\vee}$ is componentwise linear.

**Question:** What is the dual concept for vertex decomposability?
A monomial ideal \( I \) of \( R \) is called **vertex splittable** if it can be obtained by the following recursive procedure:

(i) If \( u \) is a monomial and \( I = (u) \), then \( I \) is a vertex splittable ideal.

(ii) If there is a variable \( x \in X \) and vertex splittable ideals \( I_1 \) and \( I_2 \) of \( k[X \setminus \{x\}] \) so that \( I = xI_1 + I_2 \) and \( I_2 \subseteq I_1 \), then \( I \) is a vertex splittable ideal.

With the above notations if \( I = xI_1 + I_2 \) is a vertex splittable ideal, then \( xI_1 + I_2 \) is called a **vertex splitting** for \( I \).
Khosh-Ahang, Moradi

A simplicial complex $\Delta$ is vertex decomposable if and only if $I_{\Delta^\vee}$ is a vertex splittable ideal.

Any vertex splittable ideal has linear quotients.
A simplicial complex $\Delta$ is vertex decomposable if and only if $I_{\Delta^v}$ is a vertex splittable ideal.

Any vertex splittable ideal has linear quotients.

Let $I$ be a vertex splittable ideal generated by monomials in the same degrees. Then $I$ has linear resolution.
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Any vertex splittable ideal has linear quotients.

Let $I$ be a vertex splittable ideal generated by monomials in the same degrees. Then $I$ has linear resolution.
**Definition**

Let $I$, $J$ and $K$ be monomial ideals such that $G(I)$ is the disjoint union of $G(J)$ and $G(K)$. Then $I = J + K$ is a **Betti splitting** if

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)$$

for all $i \in \mathbb{N}$ and (multi)degrees $j$.

**Khosh-Ahang, Moradi**

Let $I = xI_1 + I_2$ be a vertex splitting for the monomial ideal $I$. Then $I = xI_1 + I_2$ is a Betti splitting.
Betti splitting

Definition

Let \( I, J \) and \( K \) be monomial ideals such that \( \mathcal{G}(I) \) is the disjoint union of \( \mathcal{G}(J) \) and \( \mathcal{G}(K) \). Then \( I = J + K \) is a Betti splitting if

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Khosh-Ahang, Moradi

Let \( I = xI_1 + I_2 \) be a vertex splitting for the monomial ideal \( I \).
Then \( I = xI_1 + I_2 \) is a Betti splitting.
For a vertex splittable ideal $I$ with vertex splitting $I = xI_1 + I_2$, the graded Betti numbers of $I$ can be computed by the following recursive formula

$$
\beta_{i,j}(I) = \beta_{i,j-1}(I_1) + \beta_{i,j}(I_2) + \beta_{i-1,j-1}(I_2).
$$

Let $\Delta$ be a vertex decomposable simplicial complex, $x$ a shedding vertex of $\Delta$, $\Delta_1 = \text{del}_\Delta(x)$ and $\Delta_2 = \text{lk}_\Delta(x)$. Then

$$
\beta_{i,j}(I_{\Delta^\vee}) = \beta_{i,j-1}(I_{\Delta_1^\vee}) + \beta_{i,j}(I_{\Delta_2^\vee}) + \beta_{i-1,j-1}(I_{\Delta_2^\vee}).
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Corollary

Let $\Delta$ be a vertex decomposable simplicial complex, $x$ a shedding vertex of $\Delta$ and $\Delta_1 = \text{del}_\Delta(x)$ and $\Delta_2 = \text{lk}_\Delta(x)$. Then

$$\text{pd}(R/I_\Delta) = \max\{\text{pd}(R/I_{\Delta_1}) + 1, \text{pd}(R/I_{\Delta_2})\},$$

$$\text{reg}(R/I_\Delta) = \max\{\text{reg}(R/I_{\Delta_1}), \text{reg}(R/I_{\Delta_2}) + 1\}.$$
Let $G$ be a vertex decomposable simplicial complex, $v \in V(G)$ be a shedding vertex of $G$, $G' = G \setminus \{v\}$, $G'' = G \setminus N_G[v]$ and $\deg_G(v) = t$. Then

$$\beta_{i,j}(I(G)^\vee) = \beta_{i,j-1}(I(G')^\vee) + \beta_{i,j-t}(I(G'')^\vee) + \beta_{i-1,j-t-1}(I(G'')^\vee).$$

Francisco, Ha, Van Tuyl (2009)

Let $G$ be a Cohen-Macaulay bipartite graph, $x, y \in V(G)$ be adjacent vertices with $\deg_G(x) = 1$ such that $G' = G \setminus N_G[x]$ and $G'' = G \setminus N_G[y]$ are Cohen-Macaulay and $\deg_G(y) = t$. Then

$$\beta_i(I(G)^\vee) = \beta_i(I(G')^\vee) + \beta_i(I(G'')^\vee) + \beta_{i-1}(I(G'')^\vee).$$
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$$\beta_i(I(G)^\vee) = \beta_i(I(G')^\vee) + \beta_i(I(G'')^\vee) + \beta_{i-1}(I(G'')^\vee).$$
Let $G$ be a sequentially Cohen-Macaulay bipartite graph, $x, y \in V(G)$ be adjacent vertices with $\deg_G(x) = 1$ such that $G' = G \setminus N_G[x]$ and $G'' = G \setminus N_G[y]$ are sequentially Cohen-Macaulay and $\deg_G(y) = t$. Then

$$\beta_{i,j}(I(G)^\wedge) = \beta_{i,j-1}(I(G')^\wedge) + \beta_{i,j-t}(I(G'')^\wedge) + \beta_{i-1,j-t-1}(I(G'')^\wedge).$$

Let $G$ be a chordal graph with simplicial vertex $x$ and $y \in N_G(x)$ with $\deg_G(y) = t$. Let $G' = G \setminus \{y\}$ and $G'' = G \setminus N_G[y]$. Then

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Let $G$ be a sequentially Cohen-Macaulay bipartite graph, $x, y \in V(G)$ be adjacent vertices with $\deg_G(x) = 1$ such that $G' = G \setminus N_G[x]$ and $G'' = G \setminus N_G[y]$ are sequentially Cohen-Macaulay and $\deg_G(y) = t$. Then

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$$\beta_{i,j}(I(G)^\vee) = \beta_{i,j-1}(I(G')^\vee) + \beta_{i,j-t}(I(G'')^\vee) + \beta_{i-1,j-t-1}(I(G'')^\vee).$$
Francisco, Ha, Van Tuyl (2006)

Let $e = uv$ be an edge of the graph $G$. $I(G) = (uv) + I(G \setminus e)$ is a splitting if and only if $N(u) \subseteq N[v]$ or $N(v) \subseteq N[u]$. 
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Francisco, Ha, Van Tuyl (2006)

Let $v$ be a vertex of the graph $G$ such that $d = \deg(v) > 0$ and $G \setminus \{v\}$ is not the graph of isolated vertices and $N(v) = \{v_1, \ldots, v_d\}$. Then $I(G) = (vv_1, \ldots, vv_d) + I(G \setminus \{v\})$ is a splitting.
Let $e = uv$ be an edge of the graph $G$. $I(G) = (uv) + I(G \setminus e)$ is a splitting if and only if $N(u) \subseteq N[v]$ or $N(v) \subseteq N[u]$.

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Question: Are there splittings of edge ideal and vertex cover ideal for another families of graphs?
Francisco, Ha, Van Tuyl (2006)

*Let* *e* = *uv* *be an edge of the graph* *G*. *I*(*G*) = (*uv*) + *I*(*G* \ *e*) *is a splitting if and only if* *N*(*u*) ⊆ *N*[v] *or* *N*(*v*) ⊆ *N*[u].

Francisco, Ha, Van Tuyl (2006)

*Let* *v* *be a vertex of the graph* *G* *such that* *d* = *deg*(*v*) > 0 *and* *G* \ {v} *is not the graph of isolated vertices and* *N*(*v*) = {v₁,...,v₅}. *Then* *I*(*G*) = (vv₁,...,vv₅) + *I*(*G* \ {v}) *is a splitting.*

**Question:** Are there splittings of edge ideal and vertex cover ideal for another families of graphs?
Fröberg (1990)

For a graph $G$, the edge ideal $I(G)$ has linear resolution if and only if $G^c$ is a chordal graph.

Khosh-Ahang, Moradi

Let $G$ be a chordal graph. Then $I(G^c)$ is a vertex splittable ideal.
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Khosh-Ahang, Moradi

Let $G$ be a chordal graph. Then $I(G^c)$ is a vertex splittable ideal.

Corollary

For a graph $G$, the edge ideal $I(G)$ is vertex splittable if and only if $I(G)$ has linear resolution.
An ideal $I$ in a Noetherian ring $R$ has the \textit{persistence property} if
\[
\text{Ass}(I) \subseteq \text{Ass}(I^2) \subseteq \ldots \text{Ass}(I^k) \subseteq \ldots.
\]

\text{Martinez-Bernal, Morey, Villarreal (2011)}

For any graph $G$, the edge ideal $I(G)$ has the persistence property.
An ideal $I$ in a Noetherian ring $R$ has the persistence property if

$$\text{Ass}(I) \subseteq \text{Ass}(I^2) \subseteq \ldots \text{Ass}(I^k) \subseteq \ldots \ldots$$

**Martinez-Bernal, Morey, Villarreal (2011)**

For any graph $G$, the edge ideal $I(G)$ has the persistence property.
A graph $G$ is called **perfect** if for every induced subgraph $G_S$, with $S \subseteq V(G)$, we have $\chi(G_S) = \omega(G_S)$.

Francisco, Ha, Van Tuyl (2011)

If $G$ is a perfect graph, then $I(G)^\vee$ has the persistence property.
A graph $G$ is called perfect if for every induced subgraph $G_S$, with $S \subseteq V(G)$, we have $\chi(G_S) = \omega(G_S)$.

**Francisco, Ha, Van Tuyl (2011)**

If $G$ is a perfect graph, then $I(G)^\vee$ has the persistence property.

**Herzog, Rauf, Vladoiu (2011)**

Let $I$ be a polymatroidal ideal. Then $I$ has the persistence property.
A graph $G$ is called perfect if for every induced subgraph $G_S$, with $S \subseteq V(G)$, we have $\chi(G_S) = \omega(G_S)$.

**Francisco, Ha, Van Tuyl (2011)**
If $G$ is a perfect graph, then $I(G)^\vee$ has the persistence property.

**Herzog, Rauf, Vladoiu (2011)**
Let $I$ be a polymatroidal ideal. Then $I$ has the persistence property.
For an integer $t \geq 1$, the partial $t$-cover ideal of $G$ is the monomial ideal

$$J_t(G) = \bigcap_{x \in V(G)} \left( \bigcap_{\{x_i_1, \ldots, x_i_t\} \subseteq N(x)} (x, x_i_1, \ldots, x_i_t) \right).$$

Bhat, Biermann, Van Tuyl (2013)

Let $G$ be a tree. Then for any integer $t \geq 1$, the partial $t$-cover ideal $J_t(G)$ satisfies the persistence property.
For an integer $t \geq 1$, the partial $t$-cover ideal of $G$ is the monomial ideal
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Let $G$ be a tree. Then for any integer $t \geq 1$, the partial $t$-cover ideal $J_t(G)$ satisfies the persistence property.

Open question: Do all square-free monomial ideals have the persistence property?
Persistence property for monomial ideals

For an integer $t \geq 1$, the partial $t$-cover ideal of $G$ is the monomial ideal

$$J_t(G) = \bigcap_{x \in V(G)} \left( \bigcap_{\{x_{i_1}, \ldots, x_{i_t}\} \subseteq N(x)} (x, x_{i_1}, \ldots, x_{i_t}) \right).$$

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Let $G$ be a tree. Then for any integer $t \geq 1$, the partial $t$-cover ideal $J_t(G)$ satisfies the persistence property.

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REFERENCES

Thanks!