Betti Diagrams

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Notation

Throughout this talk S is a commutative ring (with identity) of the following types

- Noetherian local rings; or
- Homogeneous K-algebras, K a field.

M is a finitely generated S-module and I an ideal of S.

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$$0 \rightarrow S^{\beta_n^S} \xrightarrow{f_n} S^{\beta_{n-1}^S} \rightarrow \ldots \rightarrow S^{\beta_1^S} \xrightarrow{f_1} S^{\beta_0^S} \xrightarrow{f_0} \mathbf{M} \rightarrow 0$$

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Graded Betti numbers

$$0 \to \oplus_{j\in\mathbb{Z}} S(-j)^{\beta_{nj}^S} \to \cdots \to \oplus_{j\in\mathbb{Z}} S(-j)^{\beta_{1j}^S} \to \oplus_{j\in\mathbb{Z}} S(-j)^{\beta_{0j}^S} \to \mathbf{M} \to 0$$

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$\beta_{ij}^S = \dim_K \mathrm{Tor}_i^S(M, K)_j$.

Castelnuovo-Mumford regularity

Let K be a field and $S = K[x_1, \cdots, x_r]$.

1. Regularity in terms of minimal graded free resolution

The regularity of a finitely generated graded S-module M is the regularity of a minimal graded free resolution, $\mathbb{F}$, of M.

$$\mathbb{F} : \cdots \to \bigoplus_j S(-a_{i,j}) \to \bigoplus_j S(-a_{i-1,j}) \to \cdots ,$$

Then $\operatorname{reg}(\mathbb{F}) = \max\{a_{i,j} - i\}$.

Castelnuovo-Mumford regularity

1. Regularity in terms of minimal graded free resolution

2. Regularity in terms of Tor

$$\begin{aligned}\mathrm{reg}(M) &= \max_{i,j}\{j - i : \mathrm{Tor}_i(M, K)_j \neq 0\},\\ &= \max_{i,j}\{j - i; \beta_{i\,j}(M) \neq 0\}.\end{aligned}$$

Then $\mathrm{reg}(\mathbb{F}) = \max\{a_{i,j} - i\}$.

1. Regularity in terms of minimal graded free resolution

2. Regularity in terms of Tor

3. Regularity in terms of Local Cohomology

Let

$$a_i(M) = \max\{t; \mathrm{H}^i_{\mathfrak{m}}(M)_t \neq 0\}, 0 \leq i \leq n,$$

where $\mathrm{H}^i_{\mathfrak{m}}(M)$ is the ith local cohomology module with the support in $\mathfrak{m}$ (with the convention $\max \emptyset = -\infty$). Then the regularity is the number

$$\mathrm{reg}(M) = \max\{a_i(M) + i; 0 \leq i \leq n\}.$$

Castelnuovo-Mumford regularity

1. Regularity in terms of minimal graded free resolution

2. Regularity in terms of Tor

3. Regularity in terms of Local Cohomology

4. Regularity in terms of Linear Resolutions

$$\operatorname{reg}(M) = \min\{c : M_c \text{ has linear resolution}\},$$

where naturally $M_c = \bigoplus_{i \geq c} M_i$.

Some easy consequences:

1. $\operatorname{reg}(I) = \operatorname{reg}(S/I) + 1$ for a graded ideal I of S,

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2. For an Artinian graded S-module M, we have $\mathrm{H}^0_{\mathfrak{m}}(M) = M$, and hence

$$\mathrm{reg}(M) = \max\{j; M_j \neq 0\}.$$

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3. If I is generated by a regular sequence of forms of degrees $d_1, \cdots, d_r$,

$$\mathrm{reg}(S/I) = d_1 + \cdots + d_r - r.$$

Example

$S := K[x_1, x_2, x_3]$. *The graded free resolution of* $M = S/(x_1^2, x_2^3)$ *is*

$$0 \longrightarrow \overbrace{S(-5)}^{2} \xrightarrow{d_2} \overbrace{S(-2) \oplus S(-3)}^{1} \xrightarrow{d_1} \overbrace{S}^{0} \xrightarrow{d_0} M \longrightarrow 0,$$

where

$$d_0 : 1 \mapsto \overline{1},$$
$$d_1 : (1,0) \mapsto (-x_1^2), (0,1) \mapsto (-x_2^3),$$
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$$\mathrm{reg}(M) = \max\{0-0, 3-1, 5-2\} = 3$$

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$$0 \longrightarrow \overbrace{S(-7)^{15}}^{2} \longrightarrow \overbrace{S(-6)^{35}}^{1} \longrightarrow \overbrace{S(-5)^{21}}^{0} \longrightarrow \mathfrak{m}^5 \longrightarrow 0, \quad (*)$$

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$\mathfrak{m}^d$ has a d-linear resolution for all $d \geq 1$

For $i \geq 1$, the i-th free module in the resolution of $\mathfrak{m}^d$, if nonzero, is a direct sum of copies of $S(-d-i)$ and so $\operatorname{reg}(\mathfrak{m}^d) = d$.

$\mathrm{reg}(M) = 8$ and $\mathsf{proj.dim}(M) = 5$, then?

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Thus we have the following ($\mathrm{proj.dim}(M) + 1 = 6$) conditions on M and its syzygies:

$$\begin{aligned}
&M \text{ is generated in degrees } \leq 8,\\
&\Omega_1(M) \text{ is generated in degrees } \leq 9,\\
&\Omega_2(M) \text{ is generated in degrees } \leq 10,\\
&\cdots,\\
&\Omega_5(M) \text{ is generated in degrees } \leq 8 + 5 = 13.
\end{aligned}$$

Hilbert Syzygy Theorem

Any finitely generated graded S-module M has a finite graded free resolution

$$0 \to F_m \xrightarrow{\phi_m} F_{m-1} \to \ldots \to F_1 \xrightarrow{\phi_1} F_0$$

Moreover, we may take $m \leq r + 1$, the number of variables in S.

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Koszul complexes

$$\mathbf{K}(x_0) : 0 \longrightarrow S(-1) \xrightarrow{(x_0)} S$$

$$\mathbf{K}(x_0, x_1) : 0 \longrightarrow S(-2) \xrightarrow{\begin{pmatrix} x_1 \\ -x_0 \end{pmatrix}} S^2(-1) \xrightarrow{(x_0\ x_1)} S$$

$$\mathbf{K}(x_0, x_1, x_2) : 0 \longrightarrow S(-3) \xrightarrow{\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}} S^3(-2) \xrightarrow{\begin{pmatrix} 0 & x_2 & -x_1 \\ -x_2 & 0 & x_0 \\ x_1 & -x_0 & 0 \end{pmatrix}} S^3(-1) \xrightarrow{(x_0\ x_1\ x_2)} S.$$

Minimal Free Resolution

A complex of graded S-modules

$$\cdots \to F_i \xrightarrow{\delta_i} F_{i-1} \to \ldots$$

is called minimal if for each i the image of δ_i is contained in $\mathfrak{m}F_{i-1}$.

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Informally, we may say that a complex of free modules is minimal if its differential is represented by matrices with entries in the maximal ideal.

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If $\mathbf{F} : \cdots \to F_1 \to F_0$ is the minimal free resolution of a finitely generated graded S-module M and K denotes the residue field $S/\mathfrak{m}$, then any minimal set of homogeneous generators of F_i contains precisely $\dim_K \mathrm{Tor}_i^S(K, M)_j$ generators of degree j.

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If M is a finitely generated graded S-module then the projective dimension of M is equal to the length of the minimal free resolution.

Describing Resolutions: Betti Diagrams

Suppose that **F** is a free complex

$$\mathbf{F} : 0 \to \mathbb{F}_s \to \cdots \to F_m \to \cdots \to F_0$$

where $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$, that is, F_i requires $\beta_{i,j}$ minimal generators of degree j. The Betti diagram of **F** has the form

	0	1	$\cdots$	s
i	$\beta_{0,i}$	$\beta_{1,i+1}$	$\cdots$	$\beta_{s,i+s}$
$i+1$	$\beta_{0,i+1}$	$\beta_{1,i+2}$	$\cdots$	$\beta_{s,i+s+1}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
j	$\beta_{0,j}$	$\beta_{1,j+1}$	$\cdots$	$\beta_{s,j+s}$

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The Betti diagram consists of a table with $s+1$ columns, labeled $0, 1, \cdots, s$, corresponding to the free modules $F_0, \cdots, F_s$. It has rows labeled with consecutive integers corresponding to degrees.

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Why $\beta_{i,i+j}$ rather than $\beta_{i,j}$?

Let $\{\beta_{i,j}\}$ be the graded Betti numbers of a finitely generated S-module. If for a given i there is d such that $\beta_{i,j} = 0$ for all $j < d$, then $\beta_{i+1,j+1} = 0$ for all $j < d$.

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The entry in the j-th row of the i-th column is $\beta_{i,i+j}$ rather than $\beta_{i,j}$. In fact if the i-th column of the Betti diagram has zeros above the j-th row, then the $(i+1)$-st column also has zeros above the j-th row. This allows a more compact display of Betti numbers than if we had written $\beta_{i,j}$ in the i-th column and j-th row.

Some Applications

1. Regularity of powers of ideals

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2. Invariants similar to regularity

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3. Resolution of path ideals of cycles

1. Regularity of powers of ideals

Chandler (1997)

If $\dim S/I \leq 1$,

$$\mathrm{reg}(I^k) \leq k\mathrm{reg}(I).$$

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Eisenbud-Huneke-Ulrich (2007)

If $\dim \mathrm{Tor}_1^R(M, N) \leq 1$, then for any q,

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Kodiyalam (2000), Cutkosky-Herzog-Trung (1999)

$$\mathrm{reg}(I^k) = kb + r, \quad \forall k \gg 0$$

with $\mathrm{indeg}(I) \leq b \leq b_0(I)$.

Counter Examples

(The first) Terai

An example of Reisner $\longleftrightarrow$ Triangulation of the real projective plane $\mathbb{P}^2$.
Let $S := K[x_1, \cdots, x_6]$ one has

$$J = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6).$$

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J is characteristic dependant

J is a square-free monomial ideal whose Betti numbers, regularity and projective dimension depend on the characteristic of the base field.

- If $\mathrm{char}(K) \neq 2$, S/J is Cohen-Macaulay. $\mathrm{reg}(J) = 3$ and $\mathrm{reg}(J^2) = 7$ $(> 2 \times 3)$.
- If $\mathrm{char}(K) = 2$, S/J is not Cohen-Macaulay. J itself has no linear resolution.

Natural Question:

How it goes on for the regularity of powers of J?

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Answer by [CHT]

By the help of [CHT] we are able to write

$$\mathrm{reg}(J^k) = 3k + b(J),\ \forall k \geq c(J).$$

But what are $b(J)$ and $c(J)$?

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[HHZ]

Let $I \subseteq K[x_1, \cdots, x_n] := S$ be an equigenerated graded ideal. Let m be the number of generators of I and let $T := S[t_1, \cdots, t_m]$, and let $\mathcal{R}(I) = T/P$ be the Rees algebra associated to I.
If for some term order $<$ on T, P has a Gröbner basis $\mathcal{G}$ whose elements are at most linear in the variables $x_1, \cdots, x_n$, that is $\deg_x(f) \leq 1$ for all $f \in \mathcal{G}$, then each power of I has a linear resolution.

Rees ring of I

$I = (f_1, \ldots, f_m)$ be a graded ideal of $S = K[x_1, \ldots, x_r]$ generated in a single degree, say d.

$$\mathcal{R}(I) = \bigoplus_{j \geq 0} I^j t^j = S[f_1 t, \ldots, f_m t] \subseteq S[t],$$

The standard bigraded structure of $\mathcal{R}(I)$

$$\mathcal{R}(I)_{(k,j)} = (I^k)_{kd+j} \qquad (1)$$

But..,

$\mathrm{in}(P)$ has at least 3 elements with $deg_x > 1$, no matter if we take initial ideal w.r.t term ordering $\underline{x} > \underline{t}$ or $\underline{t} > \underline{x}$ in either Lex or DegRevLex order:

	$\underline{x} > \underline{t}$	$\underline{t} > \underline{x}$
DegRevLex	(1,2):2,(2,2):2	(1,2):2,(2,2):1
Lex	(1,2):2,(2,2):1	(1,2):2,(2,2):1

Table: Count of elements of $\mathrm{in}(P)$ with $deg_x > 1$ for J

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For example; $(\overbrace{1}^{\text{t-degree}}, \overbrace{2}^{\text{x-degree}})$ and so forth.

So, what do we do?

We prove that J^k has linear resolution $\forall k \neq 2$

$$b(J) = 0 \quad \text{and } c(J) = 3.$$

That is,

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Our criterion

$Q \subseteq S = K[x_1, \cdots, x_r]$ a graded ideal generated by m polynomials all of the same degree d,
$I = \operatorname{in}(g(P))$ for some linear bi-transformation $g \in \mathrm{GL}_r(K) \times \mathrm{GL}_m(K)$.
Write $I = G + B$ where G is generated by elements of $\deg_x \leq 1$;
B is generated by elements of $\deg_x > 1$.
If $I_{(k,j)} = G_{(k,j)}$ for all $k \geq k_0$ and for all $j \in \mathbb{Z}$, then Q^k has linear resolution for all $k \geq k_0$. In other words, $\operatorname{reg}(Q^k) = kd$ for all $k \geq k_0$.

The Algorithms and the coding in CoCoA

1. Algorithm for calculating P

Data: an equigenerated ideal I of S

Result: The associated ideal of Rees ring I, i.e., P

begin

$R \longleftarrow k[x_1, \ldots, x_r, t_1, \cdots, t_m, u]$

$I \longleftarrow IR$

$G \longleftarrow \text{Gens}(I)$

$P \longleftarrow \texttt{Elim}(u, \text{Ideal}([t[i] - u * G[i] \mid i = 1, \cdots \texttt{Len}(G)]))$

return P

end

2. Algorithm for calculating Good and Bad parts of $\mathrm{in}(P)$

Data: a monomial ideal U of T

Result: The Good and Bad parts of U, i.e., G and B

```
begin
    G ⟵ Ideal(m ∈ Gens(U) | deg_x(m) ≤ 1)
    B ⟵ Ideal(m ∈ Gens(U) | deg_x(m) > 1)
    return G, B
end
```

3. Algorithm for calculating x-degree and t-degree

Data: a term p of T, and a fixed term order $<$ on T

Result: $\deg_x(p), \deg_t(p)$

begin

$M \longleftarrow \texttt{Len}(\underline{\mathrm{x}}) = r, N \longleftarrow \texttt{Len}(\underline{\mathrm{t}}) = m, L \longleftarrow \texttt{Log}(p)$

/*Log, $F := x^3y^2z^5 + x^2y + xz^4$ then Log(F) $= [3, 2, 5]$.*/

if $\underline{\mathrm{x}} < \underline{\mathrm{t}}$ **then**

$$\deg_x(p) = \sum_{i=1}^{M} L[i],\ \deg_t(p) = \sum_{i=M+1}^{M+N} L[i]$$

else

$$\deg_x(p) = \sum_{i=N+1}^{N+M} L[i],\ \deg_t(p) = \sum_{i=1}^{N} L[i]$$

return $\deg_x(p), \deg_t(p)$

end

4. Algorithm for calculating maximum t-degree of a subset of T

Data: a subset X of T

Result: $\max \deg_t(X)$

```
begin
    MaxTDeg ⟵ 0
    foreach x in X do
        if deg_t(x) > MaxTDeg then
            MaxTDeg := deg_t(x)
        endif
    end
    return MaxTDeg
end
```

5. Algorithm for the condition $x - gen = 1$.

Data: a monomial ideal U of T

Result: $N := | \{ b \in B \mid \mathrm{Ideal}(b)(\underline{t})^{M+1-deg_t(b)} \not\subseteq G \} |$.

begin

    $Counter \longleftarrow 0$,   `/*see Algorithm 2*/`

    $B \longleftarrow$ `BadParts`$(U), G \longleftarrow$ `GoodParts`(U) `/*see Algorithm 4*/`

    $M \longleftarrow$ `MaxTDeg`(B)

    foreach b *in* B **do**

        $P_b \longleftarrow M + 1 - deg_t(b)$

        $W_b \longleftarrow \mathrm{Ideal}(b)(\underline{t}^{P_b})$

        if $W_b \not\subseteq G$ **then**

            Counter=Counter+1

        endif

    end

    return *Counter*

end

$N := \mid \{ b \in B \mid \mathrm{Ideal}(b)(\underline{t})^{M+1-deg_t(b)} \not\subseteq G \} \mid .$

If fortunately $N = 0$, we are done and from our criterion we deduce the linear resolution of I^k for $k > N$. Otherwise having N in hand, we suggest the following two approaches; even most of the time, we use a combination of the two:

1. Change order
2. Switch to a sparse upper triangular bi-change of coordinates

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If fortunately $N = 0$, we are done and from our criterion we deduce the linear resolution of I^k for $k > N$. Otherwise having N in hand, we suggest the following two approaches; even most of the time, we use a combination of the two:

1. Change order
2. Switch to a sparse upper triangular bi-change of coordinates

case 1

If $\mid N \mid$ is large enough or more precisely when $\frac{|N|}{|B|}$ is almost 1, we are advised to do the change ordering. That is, if the large powers of P are more concentrating on t's than x's, it is a good idea to choose the term order $\underline{t} > \underline{x}$.

$N := \mid \{ b \in B \mid \text{Ideal}(b)(\underline{t})^{M+1-deg_t(b)} \nsubseteq G \} \mid .$

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case 2

Start with $g(P)$ instead of P, where g is a bi-homogenous isomorphism on $K[\underline{x}, \underline{t}]$. Hence we suggest to use the following algorithm to generate **a Sparse Random Upper Triangular bi-change of coordinates**.

$$
\begin{aligned}
P_{J_1} = Ideal(&-t[2]x[3] + t[1]x[4], -t[3]x[2] + t[1]x[5], -t[7]x[1] + t[2]x[5],\\
&-t[9]x[1] + t[3]x[4], -t[9]x[2] + t[7]x[3], -t[6]x[1] + t[1]x[6],\\
&-t[4]x[2] + t[2]x[6], -t[10]x[1] + t[4]x[3], -t[10]x[2] + t[6]x[4],\\
&-t[8]x[1] + t[5]x[2], -t[5]x[3] + t[3]x[6], -t[8]x[3] + t[6]x[5],\\
&-t[5]x[4] + t[4]x[5], -t[8]x[4] + t[7]x[6], -t[10]x[5] + t[9]x[6],\\
&t[2]t[6]t[9] - t[1]t[7]t[10], -t[1]t[4]t[9] + t[2]t[3]t[10], t[3]t[6]t[7]\\
&-t[1]t[8]t[9], t[3]t[4]t[7] - t[2]t[5]t[9], -t[1]t[5]t[7] + t[2]t[3]t[8],\\
&t[4]t[6]t[7] - t[2]t[8]t[10], t[5]t[6]t[9] - t[3]t[8]t[10], t[3]t[4]t[6]\\
&-t[1]t[5]t[10], t[4]t[8]t[9] - t[5]t[7]t[10], t[2]t[5]t[6] - t[1]t[4]t[8],\\
&t[2]^2t[3]t[6] - t[1]^2t[4]t[7], t[2]t[3]^2t[6] - t[1]^2t[5]t[9], -t[1]t[4]^2t[7]\\
&+t[2]^2t[5]t[10], t[2]t[3]t[6]^2 - t[1]^2t[8]t[10], -t[1]t[4]t[7]^2+\\
&t[2]^2t[8]t[9], -t[1]t[5]t[9]^2 + t[3]^2t[7]t[10], -t[2]t[8]t[9]^2+\\
&t[3]t[7]^2t[10], -t[4]t[6]t[9]^2 + t[3]t[7]t[10]^2, -t[3]^2t[4]t[8]+\\
&t[1]t[5]^2t[9], -t[5]t[6]t[7]^2 + t[2]t[8]^2t[9], -t[4]^2t[6]t[9]+
\end{aligned}
$$

A Sparse Random Lower Triangular Matrix

$$\begin{cases} x_1 \longmapsto c_{1\,1}x_1 \\ x_2 \longmapsto c_{2\,1}x_1 + c_{2\,2}x_2 \\ \dots \\ x_n \longmapsto c_{n\,1}x_1 + c_{n\,2}x_2 + \cdots + c_{n\,n}x_n \\ \dots \\ t_1 \longmapsto d_{1\,1}t_1 \\ t_2 \longmapsto d_{2\,1}t_1 + d_{2\,2}t_2 \\ \dots \\ t_m \longmapsto d_{m\,1}t_1 + d_{m\,2}t_2 + \cdots + d_{m\,m}t_m \end{cases}$$

where $c_{i\,j}, d_{i\,j} \in \{-1, 0, 1\}$ are randomly chosen.

6. Algorithm for generating such matrix.

begin

$DS \longleftarrow 5$

for $i \leftarrow 1$ **to** r **do**

$$X_i \longleftarrow x_i + \sum_{j=1}^{i-1}\Big(\prod_{k=1}^{DS} \mathtt{Rand}(-1,1) x_j\Big)$$

end

for $i \leftarrow 1$ **to** m **do**

$$T_i \longleftarrow t_i + \sum_{j=1}^{i-1}\Big(\prod_{k=1}^{DS} \mathtt{Rand}(-1,1) t_j\Big)$$

end

if $Ideal(X_1, \cdots, X_r, T_1, \cdots, T_m) = Ideal(\underline{\mathrm{x}}, \underline{\mathrm{t}})$ **then**

$g := x_1 \mapsto X_1, \cdots, x_r \mapsto X_r, t_1 \mapsto T_1, \cdots, t_m \mapsto T_m$ **return** g

else

Generate again

endif

7. Algorithm for searching for a desired g.

```
/*MainFnc(); the function in algorithm 5*/
/*CalcP(); the function in algorithm 1*/
/*Randgen(); the function in algorithm 6*/
```

Data: I an equigenerated ideal I of S

Result: A bi-transformation g for which our criterion works

begin
- $P \longleftarrow \texttt{CalcP}(I)$
- $C \longleftarrow \texttt{MainFnc}(\mathrm{in}(P))$
- **repeat**
  - $g \longleftarrow \texttt{Randgen}()$
  - $C \longleftarrow \texttt{MainFnc}(\mathrm{in}(g(P)))$
- **until** $C = 0$

end

The second example, Conca (2006)

J_1, the ideal of 3-minors of a 4×4 symmetric matrix of linear forms in 6 variables, that is, 3-minors of

$$\begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & 0 & x_4 & x_5 \\ x_2 & x_4 & 0 & x_6 \\ x_3 & x_5 & x_6 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & 0 & x_4 & x_5 \\ x_2 & x_4 & 0 & x_6 \\ x_3 & x_5 & x_6 & 0 \end{bmatrix}$$

As an ideal of $S = \mathbb{Q}[x_1, \cdots, x_6]$ one has:
$J_1 := (2x_1x_2x_4, 2x_1x_3x_5, 2x_2x_3x_6, 2x_4x_5x_6, x_1x_3x_4 + x_1x_2x_5 - x_1^2x_6, x_3x_4x_6 + x_2x_5x_6 - x_1x_6^2, -x_2x_3x_4 + x_2^2x_5 - x_1x_2x_6, -x_3^2x_4 + x_2x_3x_5 + x_1x_3x_6, -x_3x_4^2 + x_2x_4x_5 + x_1x_4x_6, -x_3x_4x_5 + x_2x_5^2 - x_1x_5x_6)$.

Again [HHZ] fails

Check: $\mathrm{in}(P_1)$, where P_1 is the associated ideal to Rees ring of J_1, has at least 9 elements with $deg_x > 1$, no matter if we take initial ideal w.r.t term ordering $\underline{\mathrm{x}} > \underline{\mathrm{t}}$ or $\underline{\mathrm{t}} > \underline{\mathrm{x}}$ in Lex or DegRevLex order:

	$\underline{\mathrm{x}} > \underline{\mathrm{t}}$	$\underline{\mathrm{t}} > \underline{\mathrm{x}}$
DegRevLex	(1,2):6,(2,2):5,(1,3):1,(4,2):1	(1,2):6,(2,2):3,(1,3):1
Lex	(1,2):6,(2,2):3	(1,2):6,(2,2):5

Table: Count of elements of $\mathrm{in}(P_1)$ with $deg_x > 1$ for J_1.

J, J_1 resemble in many sense

The same behavior of regularity of powers

$$\mathrm{reg}(J^k) = 3k, \quad \forall k \neq 2.$$

$$\mathrm{reg}(J_1^k) = 3k, \quad \forall k \neq 2.$$

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The same behavior of Hilbert series of powers

J and J_1 and their powers have the same Hilbert series (HS for short) respectively:

$$\mathrm{HS}(S/J^k) = \mathrm{HS}(S/J_1^k), \quad \forall k.$$

Linear bi-transformation $g \in \mathrm{GL}_6(K) \times \mathrm{GL}_{10}(K)$ for J

$g := g_1 \times g_2 \in \mathrm{GL}_6(\mathbb{Q}) \times \mathrm{GL}_{10}(\mathbb{Q})$

$$
\begin{aligned}
g_1 : \mathbb{Q}[\underline{x}] &\longrightarrow \mathbb{Q}[\underline{x}] \\
x_4 &\longmapsto x_1 + x_4, \\
x_6 &\longmapsto x_3 + x_6, \\
x_i &\longmapsto x_i, \; i \neq 4, 6
\end{aligned}
$$

$$
\begin{aligned}
g_2 : \mathbb{Q}[\underline{t}] &\longrightarrow \mathbb{Q}[\underline{t}] \\
t_i &\longmapsto t_i, \forall i
\end{aligned}
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$$g_2 : \mathbb{Q}[\underline{t}] \longrightarrow \mathbb{Q}[\underline{t}]$$

G, B

Then $\mid G \mid = 98$, $B = (t_7 x_3^2, t_4 t_6 x_5^2)$.

$$I_{(k,\star)} = G_{(k,\star)}, \text{ for } k > 2 \Longleftrightarrow \begin{cases} (t_7 x_3^2)(t_1, \cdots, t_{10})^2 \subseteq G, \\ (t_4 t_6 x_5^2)(t_1, \cdots, t_{10}) \subseteq G, \end{cases}$$

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Then $\mid G \mid = 144$, $B = (t_{10}x_2x_3, t_2t_4x_5^2)$.

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An example of Sturmfels

Sturmfels constructed a 2-dimensional Cohen-Macaulay ideal I generated by 8 square-free monomials in 6 variables such that $\mathrm{reg}(I) = 3$ but $reg(I^2) = 7$ for any base field K.

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There exists at least 9 elements of x-degree > 1. So again HHZ fails.

Question

Can we find a suitable upper triangular bi-change of $\underline{\mathrm{x}}$ and $\underline{\mathrm{t}}$ that fulfils the requirements of our criterion.

After 122, 000 times of tests the answer was "No"!

We believe that powers of J_2 have non-linear resolution.

$$\begin{aligned}
\operatorname{reg}(J_2) &= 3, \\
\operatorname{reg}(J_2^2) &= 7 = 3 * 2 + 1, \\
\operatorname{reg}(J_2^3) &= 10 = 3 * 3 + 1, \\
\operatorname{reg}(J_2^4) &= 13 = 3 * 4 + 1, \\
\operatorname{reg}(J_2^5) &= 16 = 3 * 5 + 1, \text{ and} \\
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It attracts our interests to the following question:

Question

Is it true that $\operatorname{reg}(J_2^k) = 3k + 1, \ \forall k \geq 2$?

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S is a polynomial ring over a field K.

2. Invariants similar to regularity

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$|I|$ denotes the total degree of I.

Objective

1. To find the **regularity** of monomial ideals that satisfy some conditions on their **primary representation**.

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1. To find the **regularity** of monomial ideals that satisfy some conditions on their **primary representation**.
2. To ensure which associated primes of I still belong to associated primes of its powers

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- In general there is no guaranty for having the implication $\mathfrak{p} \in \operatorname{ass}_S I \Longrightarrow \mathfrak{p} \in \operatorname{ass}_S I^n$ for all $n \geq 1$.
- We want to know if $\mathfrak{p} \in \operatorname{ass}_S I$ and $\operatorname{ht}\mathfrak{p} = \lambda(I)$, then $\mathfrak{p} \in \operatorname{ass}_S I^n$ for all $n \geq 1$?

[Herzog-Hibi, Thm. 1.3.1]

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- In other words, each Q_i is of the form $(x_{i_1}^{a_1}, x_{i_2}^{a_2}, \ldots, x_{i_t}^{a_t})$ which is $\mathfrak{p}_i$-primary, where $\mathfrak{p}_i = (x_{i_1}, \ldots, x_{i_t})$.

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- Thus $\operatorname{ass}_S I = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$.
- Finally an irredundant presentation of this form is unique.

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- Then $I = (x, z) \cap (y, z)$ and so $(x, y, z) \notin \operatorname{ass}_S I$.
- $I^2 = (x^2y^2, xyz, z^2) = (x^2, xyz, z^2) \cap (y^2, xyz, z^2) = \cdots = (x, z^2) \cap (x^2, z) \cap (y, z^2) \cap (y^2, z) \cap (x^2, y, z^2) \cap (x, y^2, z^2)$.

MHC

- $I = \cap_{i=1}^{k} Q_i$, where each Q_j is generated by pure powers of the variables. Thus Q_j is a $\mathfrak{p}_j$-primary ideal, where $\mathfrak{p}_j = (x_{i_1}, \cdots, x_{i_c})$ for some positive integer c.

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- One now can note that $\operatorname{ht}(Q_j) = \operatorname{ht}(x_{i_1}, \cdots, x_{i_c}) = c$ and indeed $c \leq \lambda(I)$. Hence **ht** $(\mathfrak{p}) \leq \lambda(I)$ for all $\mathfrak{p} \in \operatorname{ass}_S I$.

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- One now can note that $\operatorname{ht}(Q_j) = \operatorname{ht}(x_{i_1}, \cdots, x_{i_c}) = c$ and indeed $c \leq \lambda(I)$. Hence $\mathbf{ht}(\mathfrak{p}) \leq \lambda(I)$ for all $\mathfrak{p} \in \operatorname{ass}_S I$.
- It now makes perfect sense to see whenever the equality holds in fact.

MHC

- We say that I satisfies the *maximal height condition* for $\operatorname{ass}_S I$ (*MHC* for short), if there exists a prime ideal $\mathfrak{p} \in \operatorname{ass}_S I$ with $\mathbf{ht}\,(\mathfrak{p}) = \lambda(I)$.

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- We show that if I satisfies the MHC, then $\operatorname{reg}(S/I) = m(I) = \max\{|Q_j| - \operatorname{ht}(Q_j) \mid j = 1, \ldots, k\}$.

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- We show that if I satisfies the MHC, then $\mathrm{reg}(S/I) = m(I) = \max\{|Q_j| - \mathrm{ht}\,(Q_j) \mid j = 1, \ldots, k\}$.
- That is, regularity of such ideals is given by $\max\{|Q| - ht(Q)\}$, where Q appears in the irredundant pure primary representaion of I.

MHC in terms of depth

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- In order to see this note that

$$I \text{ satisfies the MHC} \iff (x_1, \cdots, x_t) \in \mathrm{Ass}_S\, S/I \iff (x_1, \cdots, x_t)S_{(x_1,\cdots,x_t)} \in \mathrm{Ass}_S\, (S/I)_{(x_1,\cdots,x_t)} \iff \mathrm{depth}((S/I)_{(x_1,\cdots,x_t)}) = 0.$$

Veronese ideals

- The **square-free Veronese** ideal of degree d in the variables $x_{i_1}, \cdots, x_{i_t}$ is the ideal of S which is generated by all square-free monomials in $x_{i_1}, \cdots, x_{i_t}$ of degree d.

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- Now let $2 \leq d < n$ and let $I = I_{n,d}$ be the square-free Veronese ideal of degree d in the variables $x_1, \cdots, x_n$.

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- Now let $2 \leq d < n$ and let $I = I_{n,d}$ be the square-free Veronese ideal of degree d in the variables $x_1, \cdots, x_n$.
- Since each power I^k is the ideal of Veronese type indexed by kd and $(k, k, \cdots, k)$, then $\mathrm{depth} S/I^k = \max\{0, s - k(s-d) - 1\}$.

Veronese ideals

- The **square-free Veronese** ideal of degree d in the variables $x_{i_1}, \cdots, x_{i_t}$ is the ideal of S which is generated by all square-free monomials in $x_{i_1}, \cdots, x_{i_t}$ of degree d.
- Now let $2 \leq d < n$ and let $I = I_{n,d}$ be the square-free Veronese ideal of degree d in the variables $x_1, \cdots, x_n$.
- Since each power I^k is the ideal of Veronese type indexed by kd and $(k, k, \cdots, k)$, then $\mathrm{depth} S/I^k = \max\{0, s - k(s-d) - 1\}$.
- Hence for $k >> 0$ we have $\mathrm{depth} S/I^k = 0$, thus I^k are satisfying MHC.

$m(I)$

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Assume that I is a monomial ideal of S. Define

$$m(I) = \max\{|Q_j| - \operatorname{ht}(Q_j) \mid j = 1, \ldots, k\}$$

Some elementary properties of $m(I)$

- Assume that I and J are monomial ideals and $u = x_1^{\alpha_1} \cdots x_s^{\alpha_s}$, where $\alpha_i \geq 0$ for each $1 \leq i \leq s$, is a monomial of S. Let $\Lambda(I) = \{x_1, \ldots, x_t\}$.

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- Then
  - *(i)* $(I \cap J)u = Iu \cap Ju$.
  - *(ii)* $m(I \cap J) \leq \max\{m(I), m(J)\}$.
  - *(iii)* $m(Iu) = \max\{m(I) + \deg(u), a_{t+1} - 1, \ldots, a_s - 1\}$.

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- Let $S = K[x, y, z]$ and $I = (xy, z) = (x, z) \cap (y, z)$. Then $\lambda(I) = 3$ and $(x, y, z) \notin \text{ass}_S I$. Thus I does not satisfy MHC.

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- $I^2 = (x^2y^2, xyz, z^2) = (x^2, xyz, z^2) \cap (y^2, xyz, z^2) = Q_1 \cap \cdots \cap Q_6$, where $Q_1 = (x, z^2)$, $Q_2 = (x^2, z)$, $Q_3 = (y, z^2)$, $Q_4 = (y^2, z)$, $Q_5 = (x^2, y, z^2)$ and $Q_6 = (x, y^2, z^2)$.

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- Note that for $i \in \{1, 2, 3, 4\}$, Q_i are of height 2 and $|Q_i| = 3$.

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- Note that for $i \in \{1, 2, 3, 4\}$, Q_i are of height 2 and $|Q_i| = 3$.
- For $i \in \{5, 6\}$, Q_i are of height 3 and $|Q_i| = 5$.

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- Note that for $i \in \{1, 2, 3, 4\}$, Q_i are of height 2 and $|Q_i| = 3$.
- For $i \in \{5, 6\}$, Q_i are of height 3 and $|Q_i| = 5$.
- Thus I^2 satisfies MHC and $m(I^2) = \max\{5 - 3, 3 - 2\} = 2$.

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- The primary decomposition of I^3 is, $I^3 = Q_1 \cap \cdots \cap Q_7$, where $Q_1 = (x, z^3)$, $Q_2 = (x^3, z)$, $Q_3 = (x^2, z^2)$, $Q_4 = (x^2, y, z^3)$, $Q_5 = (x^3, y, z^3)$, $Q_6 = (x^3, y^2, z^2)$ and $Q_7 = (x, y^3, z^3)$.

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- Hence I^3 satisfies the MHC.

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- Hence I^3 satisfies the MHC.
- Note that $|Q_i| = 4$ for $i \in \{1, 2, 3\}$, $|Q_4| = 6$ and $|Q_i| = 7$ for $i \in \{5, 6, 7\}$.

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- The primary decomposition of I^3 is, $I^3 = Q_1 \cap \cdots \cap Q_7$, where $Q_1 = (x, z^3)$, $Q_2 = (x^3, z)$, $Q_3 = (x^2, z^2)$, $Q_4 = (x^2, y, z^3)$, $Q_5 = (x^3, y, z^3)$, $Q_6 = (x^3, y^2, z^2)$ and $Q_7 = (x, y^3, z^3)$.
- Hence I^3 satisfies the MHC.
- Note that $|Q_i| = 4$ for $i \in \{1, 2, 3\}$, $|Q_4| = 6$ and $|Q_i| = 7$ for $i \in \{5, 6, 7\}$.
- Thus $m(I^3) = \max\{4 - 2, 6 - 3, 7 - 3\} = 4$.

An attractive question

If I is MHC, then so is I^n?

- Let I be a monomial ideal of S that satisfies the MHC. Then I^n satisfies the MHC for all $n \geq 1$?

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- Let I be a monomial ideal of S that satisfies the MHC. Then I^n satisfies the MHC for all $n \geq 1$?
- That is for a monomial ideal I, if $\mathfrak{p} \in \operatorname{ass}_S I$ and $\operatorname{ht} \mathfrak{p} = \lambda(I)$, then $\mathfrak{p} \in \operatorname{ass}_S I^n$ for all $n \geq 1$?

Castelnuovo-Mumford Regularity

Regularity of principal monomial ideals

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- One can easily see that for a principle monomial ideal $I = (x_1^{\alpha_1} \dots x_s^{\alpha_s})$ we have $\mathrm{reg}(I) = \sum_{i=1}^{s} \alpha_i = (\mathbf{m(I)+1})$.

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- Now the Auslander-Buchsbaum theorem implies that $\mathrm{pd}_S\, I = \mathrm{depth} S - \mathrm{depth} S/I = 0$, i.e., $0 \to S(-\sum_{i=1}^{s} \alpha_i) \to S$ is exact.

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Example

- Let $S = K[x, y, z]$ and $I = (x^2, xy, y^{10}, xz) = (x^2, y, z) \cap (x, y^{10})$.

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- Then I satisfies the MHC and $m(I) = \max\{4 - 3, 11 - 2\} = 9$.
- We also know that $\operatorname{reg}(S/I) = 9$.
- We show that this phenomena happens for all ideals that satisfy MHC.

Regularity of primary monomial ideals and monomial MHC

- Let I be a primary monomial ideal of S. Then $\mathrm{reg}(S/I) = m(I)$.

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- The following simple example shows that one can not remove the MHC assumption.

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- Let I be a monomial ideal of S that satisfies the MHC. Then $\mathrm{reg}(S/I) = m(I)$.
- The following simple example shows that one can not remove the MHC assumption.
- Let $I = (xy, xz^3, y^2) = (x, y^2) \cap (y, z^3)$. Then I does not satisfy the MHC and $m(I) = 2 < 3 = \mathrm{reg}(S/I)$.

Implementation

How do we find $m(I)$ in practice?

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- Let $I = (x_1^2x_2, x_1^2x_3^2, x_2^2, x_2x_3^2)$.
- Then $t = 3$ and $r = 4$.
- One can write I as in the following table:

2 0 0	0 1 0	0 0 0
2 0 0	0 0 0	0 0 2
0 0 0	0 2 0	0 0 0
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```
private int[] numtopath(int x){
  int y,z;
  int a[] = new int[r];
  a[0]=x/t^(r-1);
  for(j=1;j<=r-1;j++){
    y=(int)(a[j-1]*t^(r-j));
    x=x-y;
    z=t^(r-j-1);
    a[j]=x/z;
  }
  return a;
}
```

How do we find $m(I)$ in practice?

- Now consider the following three paths among all $t^r = 3^4 = 81$ possible paths.

(1)

2 0 0	0 1 0	0 0 0
2 0 0	0 0 0	0 0 2
0 0 0	0 2 0	0 0 0
0 0 0	0 1 0	0 0 2

(2)

2 0 0	0 1 0	0 0 0
2 0 0	0 0 0	0 0 2
0 0 0	0 2 0	0 0 0
0 0 0	**0 1 0**	0 0 2

(3)

2 0 0	0 1 0	0 0 0
2 0 0	0 0 0	0 0 2
0 0 0	**0 2 0**	0 0 0
0 0 0	0 1 0	**0 0 2**

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- Following the first node in the path; $b = (2, 0, 0)$.

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- Following the first node in the path; $b = (2, 0, 0)$.
- Continuing the second node, b is still $b = (2, 0, 0)$.
- Then following the third node; $b = (2, 2, 0)$. Finally we finish this path with $b = (2, 2, 3)$.

How do we find $m(I)$ in practice?

Note that two vectors of size t in each path will operate to each other by the following rule:

```
int [] e = new int[t];
for(i=0;i<t;i++){
  if(c[i]*d[i]==0)e[i]=Math.max(c[i],d[i]);
  else e[i]=Math.min(c[i],d[i]);
}
```

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- As it is easily seen, the value of $m(I)$ does not increase along other remaining paths.
- Hence $m(I) = 3 = \mathrm{reg}(S/I)$.

3. Resolution of path ideals of cycles

Path ideals

- K is a field and $R = K[x1, ..., xn]$ is a polynomial ring in n variables.

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- χ is a **simplicial complex** on vertex set $X = \{x1, ..., xn\}$ is a collection Δ of subsets of X satisfying $xi \in \Delta$ for all i and $F \in \Delta, G \subset F \Longrightarrow G \in \Delta$.

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- Two special cases that [Alilooee, Faridi] considered in their recent paper are when G is a cycle C_n, or a line graph L_n on vertices $\{x1, ..., xn\}$.
- $C_n =< x_1x_2, ..., x_{n-1}x_n, x_nx_1 >$, $L_n =< x_1x_2, ..., x_{n-1}x_n >$.

If I is the degree t path ideal of a cycle, then

$$\beta_{i,j}(R/I) = 0 \text{ for all } i \geq 1, j \geq ti.$$

The Betti numbers of path ideals of cycles

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Betti numbers of degree n

Let p, t, n, d be integers such that $n = (t+1)p + d$, where $p \geq 0, 0 \leq d \leq t$, and $2 \leq t \leq n$. If C_n is a cycle over n vertices, then

$$\beta_{i,n}(R/I_t(C_n)) = \begin{cases} t, & d = 0, i = 2(\frac{n}{t+1}) = 2p; \\ 1, & d \neq 0, i = 2(\frac{n-d}{t+1}) + 1 = 2p+1; \\ 0, & \text{o.w.} \end{cases}$$

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MANY THANKS FOR YOUR ATTENTION