ON A FAMILY OF COHOMOLOGICAL DEGREES

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1. Cohomological degrees

Definition

- $(R, \mathfrak{m}, k)$: a Noetherian local ring, $|k| = \infty$.
- Mod_R: the category of finitely generated R-modules.

Definition 1 (Doering-Gunston-Vasconcelos 1998)

A function

$$\mathrm{Deg} : \mathrm{Mod}_R \to \mathbb{R}_{\geq 0},$$

is a cohomological degree (extended degree) if

(a) $\mathrm{Deg}(M) = \mathrm{Deg}(M/H^0_{\mathfrak{m}}(M)) + \ell(H^0_{\mathfrak{m}}(M))$;

(b) (Bertini) If $\mathrm{depth}(M) > 0$ and $h \in \mathfrak{m} \setminus \mathfrak{m}^2$ then

$$\mathrm{Deg}(M) \geq \mathrm{Deg}(M/hM);$$

(c) (Calibration) If M is Cohen-Macaulay then $\mathrm{Deg}(M) = e(M)$.

Multiplicities of Cohen-Macaulay rings

- If M is Cohen-Macaulay then $e(M)$ gives bounds for several important invariants:

$$\mu_R(M) \leq e(M).$$

Theorem 2 (Sally, ...)

Let R be a Cohen-Macaulay ring and I be an ideal such that R/I is Cohen-Macaulay. Then

$$\mu(I) \leq e(R) + (g-1)e(R/I),$$

where $g = \mathrm{height}(I)$ and $r = \mathrm{depth}(R/I)$.

Cohomological degree replaces the multiplicity

- If M is not Cohen-Macaulay then a cohomological degree Deg gives rise to bounds for $\mu(M)$, $\mathrm{reg}(M)$ (graded case), $P_{\mathfrak{m},M}(n)$, $\beta_i(M)$, $\mu_i(M)$, etc.

Theorem 3 (Doering-Gunston-Vasconcelos 1998)

$$\beta_i^R(M) \leq \beta_i^R(k)\,\mathrm{Deg}(M),$$

$$\mu_i^R(M) \leq \mu_i^R(k)\,\mathrm{Deg}(M),$$

In particular, $\mu(M) \leq \mathrm{Deg}(M)$.

Theorem 4 (Doering-Gunston-Vasconcelos 1998)

Let R *be a Cohen-Macaulay ring and* I *be an ideal. Then*

$$\mu(I) \leq e(R) + (g-1)e(R/I) + (n-r+1)(\mathrm{Deg}(R/I) - e(R/I)),$$

where $n = \dim(R)$, $g = \mathrm{height}(I)$ *and* $r = \mathrm{depth}(R/I)$.

Key in applications

- A cohomological degree plays a role of a measure for the complexity of algebraic structure.
- In application, to obtain a bound for certain invariant, one reduce to a low dimensional case by using the Bertini's rule.

Examples of cohomological degrees

- Homological degree hdeg (Vasconcelos 1998): Let R be a quotient of a Gorenstein local ring S of dimension s.

$$\mathrm{hdeg}(M) = e(M) + \sum_{i=0}^{d-1} \binom{d-1}{i} \mathrm{hdeg}(\mathrm{Ext}_S^{s-d+j}(M,S)),$$

  where $d = \dim M$.
- Extreme cohomological degree bdeg (Gunston 1998[1]):

$$\mathrm{bdeg}(M) = \inf\{\mathrm{Deg}(M) : \text{ for all coh. degrees } \mathrm{Deg}\}.$$

- Unmixed degree udeg (N.T. Cuong-Quy 2016).

[1] T. Gunston, Cohomological degrees, Dilworth numbers and linear resolution, Ph.D. Thesis, Rutgers University, 1998.

Our aim: to construct a family of cohomological degrees and to study it relation with the $\mathrm{hdeg}, \mathrm{bdeg}, \mathrm{udeg}$.

Let Deg be a cohomological degree and fix a finitely generated R-module M.

- $\dim M = 0$: $\mathrm{Deg}(M) = \ell(M)$;
- $\dim M = 1$: $\mathrm{Deg}(M) = e(M) + \ell(H^0_{\mathfrak{m}}(M))$;
- $\dim M \geq 2$: $\mathrm{Deg}(M)$ can attain infinitely many values.

2. A Cohen-Macaulay obstruction

p-Standard system of parameters

- Suppose that R is a quotient of a Cohen-Macaulay local ring.
- $\mathfrak{a}(M) = \operatorname{Ann}_R H^0_{\mathfrak{m}}(M) \ldots \operatorname{Ann}_R H^{d-1}_{\mathfrak{m}}(M)$, $d = \dim M$. Then

$$\dim R/\mathfrak{a}(M) < \dim M.$$

- M admits a system of parameters which satisfies:

Definition 5 (N.T. Cuong 1990)

A system of parameters $x_1, \ldots, x_d$ of M is p-standard if

$$x_i \in \mathfrak{a}(M/(x_{i+1}, \ldots, x_d)M),\ i = 1, \ldots, d.$$

Almost p-standard system of parameters

Definition 6 (C-Nam 2015)

A system of parameters $x_1, \ldots, x_d$ of M is almost p-standard if

$$\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) = \sum_{i=0}^{d} \lambda_i n_1 \ldots n_i,$$

for given integer numbers $\lambda_0, \ldots, \lambda_d$ and for all $n_1, \ldots, n_d > 0$.

- (N.T. Cuong 1993) p-standard s.o.p $\Rightarrow$ almost p-standard s.o.p.
- $x_1, \ldots, x_d$ is an almost p-standard s.o.p of $M \Rightarrow x_1^{n_1}, x_2^{n_2}, \ldots, x_d^{n_d}$ is a p-standard s.o.p of M, for all $n_i \geq i$.
- If $x_1, \ldots, x_d$ is an almost s.o.p then

$$\lambda_i = e(x_1, \ldots, x_i; (0 : x_{i+1})_{M/(x_{i+2}, \ldots, x_d)}).$$

Almost p-standard system of parameters...

- An almost p-standard s.o.p is a strong d-sequence.
- (almost) p-Standard s.o.p.s are generalization of standard s.o.p of generalized Cohen-Macaulay modules.
- We may write

$$\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) = \sum_{j=0}^{r} \lambda_{d_j} n_1 \ldots n_{d_j},$$

where $\lambda_{d_j} \neq 0$ and $d_0 < d_1 < \ldots < d_r$. Then

  - $d_r = d = \dim M = \dim \operatorname{Supp}(M)$;
  - $d_{r-1} = \dim \operatorname{nCM}(M)$;
  - If one permutes $x_{d_j+1}, \ldots, x_{d_{j+1}}$ then the resulting s.o.p is also almost p-standard.

Existence

Theorem 7 (Kawasaki 2002, N.T. Cuong-C 2017)

Let R be a Noetherian local ring. TFAE

- *R is universally catenary and all formal fibers of R are CM;*
- *R is a quotient of a CM local ring;*
- *Any quasi-unmixed quotient of R has a CM Rees algebra;*
- *R admits a (almost) p-standard s.o.p;*
- *All finitely generated R-modules admit a (almost) p-standard s.o.p;*
- *A faithful finitely generated R-module admits a p-standard s.o.p;*

Corollary 8

A generalized Cohen-Macaulay local ring is a quotient of a Cohen-Macaulay ring.

Some subquotient modules

Theorem 9 (N.T. Cuong-Quy, C-Nam 2015)

Let $x_1, \ldots, x_d$ be a p-standard s.o.p of M and $0 \leq i < d$. Then the subquotient

$$(x_{i+2}^{n_{i+2}}, \ldots, x_d^{n_d})M : x_{i+1}^{n_{i+1}} / (x_{i+2}^{n_{i+2}}, \ldots, x_d^{n_d})M$$

is independent of the choice of the s.o.p and of the exponents $n_{i+1}, \ldots, n_d \gg 0$, up to an isomorphism.

Some more subquotient modules

Theorem 10 (C-Nam 2015)

Let $x_1, \ldots, x_d$ be a p-standard s.o.p of M and $0 \leq i < j \leq d$.

(a) *The subquotient*

$$(x_{i+2}^{n_{i+2}}, \ldots, x_j^{n_j})M : x_{i+1}/(x_{i+2}^{n_{i+2}}, \ldots, x_j^{n_j})M$$

is independent of the choice of the s.o.p and of the exponents $n_{i+2}, \ldots, n_d > 1$. This module is denoted by U_M^{ij}.

(b) *There is an injective homomorphism $\varphi : U_M^{i,j-1} \to U_M^{ij}$ such that $\mathrm{Im}(\varphi)$ is a direct summand of U_M^{ij}. We denote $\mathrm{Coker}(\varphi)$ by $\overline{U}_M^{ij}$. So there is a direct sum decomposition*

$$U_M^{ij} \simeq \overline{\mathrm{U}}_M^{ij} \oplus \overline{\mathrm{U}}_M^{i,j-1} \oplus \cdots \oplus \overline{\mathrm{U}}_M^{i,i+2} \oplus U_M^{i,i+1}.$$

A Cohen-Macaulay obstruction

Remark 11

- $\dim(U_M^{ij}) \leq i$, $0 \leq i < j \leq d$.
- $U_M^{i,i+1} = D_i$: the unique maximal submodule with $\dim(D_i) \leq i$.

Proposition 12

Let $d = \dim M$. *TFAE*

(a) M *is Cohen-Macaulay;*

(b) $U_M^{id} = 0$ *for all* $0 \leq i < d$*;*

(c) $U_M^{ij} = 0$ *for all* $0 \leq i < j \leq d$*;*

(d) $\overline{\mathrm{U}}_M^{ij} = 0$ *for all* $0 \leq i < j \leq d$*;*

- So the subquotients U_M^{ij} and $\overline{\mathrm{U}}_M^{ij}$'s are new Cohen-Macaulay obstruction of M.
- Question: Should U_M^{ij} and $\overline{\mathrm{U}}_M^{ij}$ come from some spectral sequence?

Unmixed degree

- Let $i \geq 0$. For a f.g. R-module N with $\dim N \leq i$, we denote

$$e(N)_i = \begin{cases} e(N) & \text{if } \dim N = i, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 13 (N.T. Cuong-Quy 2016)

Define the unmixed degree by

$$\operatorname{udeg}(M) := \sum_{i=0}^{d} e(U_M^{id})_i,$$

where $d = \dim M$. Then udeg is a cohomological degree.

3. The extreme cohomological degree bdeg

Definition 14 (Gunston 1998)

The extreme cohomological degree is a function $\operatorname{bdeg} : \operatorname{Mod}_R \to \mathbb{R}_{\geq 0}$ defined by

$$\operatorname{bdeg}(M) = \inf\{\operatorname{Deg}(M) : \operatorname{Deg} \text{ is a cohomological degree}\}.$$

Theorem 15 (Gunston 1998)

(a) bdeg *is a cohomological degree.*

(b) *For each finitely generated R-module M and a generic hyperplane section h,*

$$\operatorname{bdeg}(M) = \operatorname{bdeg}(M/hM) + \ell(H^0_{\mathfrak{m}}(M)) - \ell(H^0_{\mathfrak{m}}(M)/hH^0_{\mathfrak{m}}(M)).$$

- The last property enables us to compute the bdeg by induction.
- Up to now, no precise formulas for the bdeg seemingly exist.

The arithmetic degree

- $U_M^{i,i+1}$ is the maximal submodule of M of dimension $\leq i$.
- The arithmetic degree of M is defined by

$$\operatorname{arith.deg}(M) := e(M) + \sum_{i=0}^{d-1} e(U_M^{i,i+1})_i.$$

- Let $H_M = \{h \in \mathfrak{m} \setminus \mathfrak{m}^2 : \exists h^t, x_2, \ldots, x_d$ a p-standard sop, $t \gg 0\}$.

Theorem 16

We have

$$\operatorname{bdeg}(M) = \operatorname{arith.deg}(M) + t(M),$$

where let $d = \dim(M)$, we may choose $h_1, \ldots, h_{d-1} \in \mathfrak{m}$ generically such that $h_i \in H_{M/(h_1,\ldots,h_{i-1})M}$ for $i = 1, \ldots, d-1$, and

$$t(M) = \sum_{i=1}^{d-1} \ell\left((0 : h_i)_{\overline{\mathrm{U}}^{02}_{M/(h_1,\ldots,h_{i-1})M}}\right).$$

The invariant $t(M)$

Corollary 17

Let $d = \dim(M)$.

(a) $t(M) = \operatorname{bdeg}(M) - \operatorname{arith.deg}(M) \le \operatorname{Deg}(M) - \operatorname{arith.deg}(M)$ *for any cohomological degree* Deg*;*

(b) $t(M) = t(M/H^0_{\mathfrak{m}}(M))$*;*

(c) *For lower dimension modules, we have*

$$t(M) = \begin{cases} 0 & \textit{if } \dim(M) = 0, 1; \\ \ell((0:h)_{U_M^{02}}) & \textit{if } \dim(M) = 2, \ \textit{for all } h \in H_M. \end{cases}$$

(d) *If* M *is Buchsbaum then*

$$t(M) + \ell(H^0_{\mathfrak{m}}(M)) = I(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H^i_{\mathfrak{m}}(M)).$$

4. A family of cohomological degrees

From now on, R is a quotient of a Cohen-Macaulay ring, $\dim(R) = n$.

Theorem 18

Let $\Lambda = \{\lambda_{ij} \in \mathbb{R} : 0 \leq i \leq j \leq n\}$ *such that*

$$\lambda_{01} = 1, \lambda_{0j} \leq \lambda_{0,j+1}, \text{ and } \lambda_{ij} \leq \lambda_{i+1,j+1} \text{ for } 0 \leq i < j < n.$$

Define a function $\operatorname{Deg}_\Lambda : \operatorname{Mod}_R \to \mathbb{R}$ *by*

$$\operatorname{Deg}_\Lambda(M) = e(M) + \sum_{0 \leq i < j \leq \dim M} \lambda_{ij} e(\overline{U}_M^{ij})_i.$$

Then $\operatorname{Deg}_\Lambda$ *is a cohomological degree.*

Corollary 19

The function $\operatorname{Deg}_\Lambda(M) + t(M)$ *is also a cohomological degree.*

Proof

(a) $\mathrm{Deg}_\Lambda(M) = \mathrm{Deg}_\Lambda(M/H^0_{\mathfrak{m}}(M)) + \ell(H^0_{\mathfrak{m}}(M))$;

(b) (Bertini) $\mathrm{depth}(M) > 0$, h generic, $\mathrm{Deg}_\Lambda(M) \geq \mathrm{Deg}_\Lambda(M/hM)$

(c) (Calibration) If M is Cohen-Macaulay then $\mathrm{Deg}_\Lambda(M) = e(M)$.

Proposition 20

Let $h \in H_M$ and $1 \leq i < j \leq d$. There are exact sequences

$$0 \to U^{ij}_M/hU^{ij}_M \to U^{i-1,j-1}_{M/hM} \to L \to 0,$$

$$0 \to \overline{U}^{ij}_M/h\overline{U}^{ij}_M \to \overline{U}^{i-1,j-1}_{M/hM} \to N \to 0,$$

where $\dim(L), \dim(N) \leq 0$ such that

$$(0:h)_{U^{i-1,j}_M} \simeq L \oplus (0:h)_{U^{ij}_M},$$

$$\ell(N) = \begin{cases} \ell((0:h)_{\overline{U}^{i-1,j}_M}) - \ell((0:h)_{\overline{U}^{ij}_M}) & \text{if } j > i+1, \\ \ell((0:h)_{\overline{U}^{i-1,i+1}_M}) & \text{if } j = i+1. \end{cases}$$

A refinement

Theorem 21

Let $\lambda_{ijk} \in \mathbb{R}$, $0 \leq i \leq j \leq k \leq n$, *be such that*

$\lambda_{01k} = 1, \lambda_{0jk} \leq \lambda_{0,j+1,k},$ *and* $\lambda_{ijk} \leq \lambda_{i+1,j+1,k+1}$ *for* $0 \leq i < j \leq k < n.$

The following function is a cohomological degree

$$\mathrm{Deg}_{(\lambda_{ijd})}(M) = e(M) + \sum_{0 \leq i < j \leq d} \lambda_{ijd} e\left(\overline{\mathrm{U}}_M^{ij}\right)_i .$$

Remark 22

Let $\mathcal{L}(R)$ be the family of cohomological degree constructed in Theorem 21. Then $\mathcal{L}(R)$ is a convex set and

$$\dim(\mathcal{L}(R)) = \binom{n+2}{3} - n.$$

Examples

Example 23

(a) Given numbers $1 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n$. Let $\lambda_{ij} = \lambda_i$ for $i = 0, 1, \ldots, n$ and $j = i + 1, \ldots, n$. We get a cohomological degree

$$\mathrm{Deg}(M) = e(M) + \lambda_{d-1} e(U_M^{d-1,d})_{d-1} + \ldots + \lambda_1 e(U_M^{1d})_1 + \ell(U_M^{0d}),$$

where $\dim(M) = d$.

(b) If $\lambda_0 = \ldots = \lambda_n = 1$ then one gets the unmixed degree udeg.

(c) Let $\lambda_{ij} = \binom{j-1}{i}$, $0 \leq i < j$. By Theorem 9,

$$\mathrm{Deg}_b(M) := e(M) + \sum_{0 \leq i < j \leq \dim M} \binom{j-1}{i} e(\overline{U}_M^{ij})_i.$$

5. Relation with hdeg

The homological degree hdeg

- Suppose R is a quotient of a Gorenstein local ring S, $\dim(R) = \dim(S)$.
- $M \in \mathrm{Mod}_R$, $\dim(M) = d = \dim(R)$.
- $K_M^i := \mathrm{Ext}_S^{d-i}(M, S)$: the i-th module of deficiency of M, for $i = 0, 1, \ldots, d$.
- K_M^d: the canonical module of M.
- Def:

$$\mathrm{hdeg}(M) = e(M) + \sum_{i=0}^{d-1} \binom{d-1}{i} \mathrm{hdeg}(K_M^i).$$

Question 24

What are relations between hdeg and Deg_Λ, or more precisely, with Deg_b?

Sequentially (generalized) Cohen-Macaulay modules

Definition 25 (N. T. Cuong-Schenzel-N. V. Trung)

M is called generalized Cohen-Macaulay if $\ell(H^i_{\mathfrak{m}}(M)) < \infty$, for all $0 \leq i \leq d$, where $d = \dim(M)$. Set $\mathrm{I}(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H^i_{\mathfrak{m}}(M))$.

Definition 26 (Stanley, Schenzel, N.T. Cuong-Nhan)

M is sequentially (generalized) Cohen-Macaulay if D_{i+1}/D_i is (generalized) Cohen-Macaulay for $i = 0, 1, \ldots, \dim(M) - 1$, where D_i is the unique maximal submodule of smaller dimension of D_{i+1} for $i = t-1, \ldots, 0$, where $D_t = M$.

Theorem 27 (Stanley, Schenzel, N. T. Cuong-Nhan)

M is sequentially (generalized) Cohen-Macaulay if and only if K^i_M is either (generalized) Cohen-Macaulay of dimension i or zero (finite length) for $i = 0, 1, \ldots, \dim(M) - 1$.

hdeg and Deg_b on seq. gCM modules

Proposition 28

Let M be a sequentially generalized CM module with the dimension filtration $D_0 = H^0_{\mathfrak{m}}(M) \subset D_1 \subset \cdots \subset D_t = M$ and $d_j = \dim(D_j)$, $j = 0, 1, \ldots, t$, then

$$\mathrm{hdeg}(M) = e(M) + \sum_{j=0}^{t-1} \binom{d-1}{d_j} e(D_j) + \sum_{i=1}^{d-1} \binom{d-1}{i} I(K^i_M),$$

$$\mathrm{Deg}_b(M) = e(M) + \sum_{j=0}^{t-1} \binom{d-1}{d_j} e(D_j) + c,$$

where

$$c = \sum_{i=0}^{t-1} \sum_{j=0}^{d_{i+1}-1} \left(\binom{d_{i+1}-1}{j} - \binom{d_i-1}{j}\right)\ell(H^j_{\mathfrak{m}}(M/D_i).$$

hdeg and Deg_b

Corollary 29

Let M be a finitely generated R-module. We have

$$\mathrm{hdeg}(M) = \mathrm{Deg}_b(M),$$

in the following cases
(a) M is a sequentially Cohen-Macaulay module;
(b) M is a generalized Cohen-Macaulay module;
(c) $\dim(M) = 2$.

Proposition 30

Let $\dim M = 2$. Then

(a) $\mathrm{bdeg}(M) = e(M) + e(U_M^{12}) + \ell(U_M^{01}) + t(M)$;

(b) $\mathrm{hdeg}(M) = \mathrm{udeg}(M) = e(M) + e(U_M^{12}) + e(U_M^{02})$.

Vasconcelos's question

Question 31 (Vasconcelos)

Let R be a CM ring, denote by $\mathcal{N}$ the set of all rational numbers

$$\frac{\operatorname{hdeg}(M) - \operatorname{hdeg}(M/hM)}{e(M)},$$

where $M \neq 0 \in \operatorname{Mod}_R$ and $h \in \mathfrak{m} \setminus \mathfrak{m}^2$ generic. Is $\mathcal{N}$ finite or bounded?

Example 32

Let $R = k[[X, Y]]$ for a field k. Let $\mathfrak{m} = (X, Y)$ and $L_t = \mathfrak{m}^{t+1}$ for $t \geq 0$.

$$\operatorname{hdeg}(L_t) = e(L_t) + \ell(R/\mathfrak{m}^{t+1}) = 1 + \binom{t+2}{2},$$

$$\operatorname{hdeg}(L_t/hL_t) = 1 + \ell(\mathfrak{m}^t/\mathfrak{m}^{t+1}) = t + 2,$$

$$\frac{\operatorname{hdeg}(L_t) - \operatorname{hdeg}(L_t/hL_t)}{e(L_t)} = \binom{t+1}{2}.$$

Thank you for your attention!