On local rings with zero-dimensional formal fibers

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1. Introduction

- If $\alpha(R) = 0$ then R is quite close to its completion $\hat{R}$ geometrically. In special situation, we may expect to understand R by studying $\hat{R}$.
- Examples of such rings are from

Theorem

Let $(R, \mathfrak{m})$ be a complete one-dimensional local ring. Let X be a curve over R, x be a closed point in the special fiber X_k. Suppose for all components X_i of X through x, $\dim_x(X_i) > 1$. Then

$$\alpha(\mathcal{O}_{X,x}) = 0.$$

- In applications, R is the ring of p-adic integers in a p-adic field.

2. Characterization

Theorem

Let R be a universally catenary local domain. TFAE

(a) $\alpha(R) = 0$;

(b) *For any ideal $I \subset \widehat{R}$,* $\dim \widehat{R}/I = \dim R/I \cap R$;

(c) *For any parameter element $x \in \widehat{R}$, there is $y \in \widehat{R}$ such that $xy \in R$ which is a parameter element;*

(d) $\pi^{-1}(0) = \operatorname{Assh}(\widehat{R})$.

Theorem

Let R be a local domain with $\alpha(R) = 0$. TFAE

(a) *R is universally catenary;*

(b) *R is a homomorphic image of a Cohen-Macaulay ring;*

(c) $\dim R/\operatorname{Ann}_R H^i_{\mathfrak{m}}(R) < i$ *for* $i = 0, 1, \ldots, \dim(R) - 1$.

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Going up property

- $\mathcal{R}^0 := \{\text{local rings with trivial formal fibers}\}$;
- $\mathcal{R}^0_{uc} := \{R \in \mathcal{R}^0 : R \text{ is universally catenary}\}$;
- $\mathcal{R}^0_{gu} := \{\text{local rings with going-up property}\}$;
- $\mathcal{R}^0_{int} := \{\text{local rings } R \text{ such that } \hat{R} \text{ is integral over } R\}$.

Theorem (Zöschinger 2010)

$$\mathcal{R}^0_{gu} \subseteq \mathcal{R}^0_{uc}.$$

- In general,

$$\mathcal{R}^0_{int} \subset \mathcal{R}^0_{gu} \subset \mathcal{R}^0_{uc} \subset \mathcal{R}^0.$$

Lichstenbaum-Hartshorne Vanishing

Theorem

Let R be a universally catenary local ring with $\alpha(R) = 0$. TFAE

- *R satisfies the going-up theorem;*
- *For any ideal $I \subset R$ and any quotient S of R, $H_I^{\dim S}(S) = 0$ if and only if $\dim R/(\mathfrak{p} + I) > 0$ for all $\mathfrak{p} \in \operatorname{Assh}(S)$.*

3. Weierstrass preparation theorem

Definition

Let A be a commutative ring. A Weierstrass extension of A is a ring extension $A \subseteq B$ such that for any $b \in B$,

$$b = ua,$$

for some $a \in A$ and an invertible element $u \in B^{\times}$.

Examples of Weierstrass extensions.

- Let M be a complex smooth algebraic curve and $a \in M$. Then $\mathcal{O}_{M,a} \subset \mathcal{O}_{M,a}^{an}$ is a Weierstrass extension (classical WPT in one variable).
- $k[X] \subset k[[X]]$ and $k[[X]][Y] \subset k[[X, Y]]$.
- For a domain A, a localization $A \to S^{-1}A$ is Weierstrass.

- In fact, each result giving a new Weierstrass extension could be regarded as a new Weierstrass preparation theorem.

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- ▶ In fact, each result giving a new Weierstrass extension could be regarded as a new Weierstrass preparation theorem.

u-invariant

- k is a field, $\mathrm{char}(k) \neq 2$. A quadratic form $\lambda_1 x_1^2 + \ldots + \lambda_n x_n^2$ is anisotropic if it has no non-zero roots.
- u-invariant:

$$u(k) = max\{n : \exists\, \lambda_1 x_1^2 + \ldots + \lambda_n x_n^2 \text{ anisotropic}\}.$$

- $u(\mathbb{C}) = 1$; $u(\mathbb{R}) = \infty$ and $u(\mathbb{F}_p) = 2$.
- $u(k) = 4$ if k is a p-adic field (Springer's theorem).

Theorem (u-invariant)

Let F be a function field in one variable over a p-adic field. Then $u(F) = 8$.

- Parimala-Suresh (Ann. Math. 2010): cohomological method.
- Harbater-Hartmann-Krashen (Invent. Math. 2009): patching method.
- Leep (Crelle' 2012): p-adic analytic method.

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WPT and u-invariant

- For a domain A, K_A is the field of fractions.
- Assume the following
  - A, B are domains, $A \subseteq B$ is a Weierstrass extension.
  - $I \subset A$ an ideal such that B is henselian relative to IB.
  - $A/I \simeq B/IB$ with 2 invertible.

Lemma (Hensel)

For a unit $b \in B^\times$, there are $a \in A, \alpha \in B^\times$ such that $b = a\alpha^2$.

Proof.

- $A/I \simeq B/IB \Rightarrow a \in A^\times$ such that $a \equiv b \pmod{IB}$.
- Let $F(X) = aX^2 - b \in B[X] \Rightarrow F'(1) = 2a$ is invertible in B/IB.
- $F(1) = a - b \equiv 0 \pmod{IB}$.
- B is henselian relative to $IB \Rightarrow \exists! \alpha \in B$ such that $F(\alpha) = a\alpha^2 - b = 0$ and $\alpha \equiv 1 \pmod{IB}$.

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Theorem

$$u(K_B) \leq u(K_A).$$

Proof.

$$\begin{aligned}
&\lambda_1 x_1^2 + \ldots + \lambda_n x_n^2, \quad (\lambda_i \in B)\\
\Longrightarrow\ & a_1 b_i x_1^2 + \ldots + a_n b_n x_n^2, \quad (a_i \in A, b_i \in B^\times)\\
\Longrightarrow\ & c_1 \alpha_1^2 x_1^2 + \ldots + c_n \alpha_n^2 x_n^2, \quad (c_i \in A, \alpha_i \in B^\times\\
\Longrightarrow\ & c_1(\alpha_1 x_1)^2 + \ldots + c_n(\alpha_n x_n)^2
\end{aligned}$$

□

Remark

1. In many cases, K_B is henselian or even complete, and hence there are effective ways to compute $u(K_B)$. We then get a lower bound for $u(K_A)$.
2. The theorem works for higher forms.
3. The theorem works at the level of rings.
4. We are interested in the case $A \subseteq B$ is a flat extension.

4. Flat Weierstrass extension

- $A \subseteq B$ are Noetherian domains, B is local.
- For a prime ideal P of B, denote $k_B(P) = B_P/PB_P$, the residue field.

Theorem

Assume B/A is flat. Then B/A is Weierstrass if and only if any prime ideal $P \in \operatorname{Spec}(B)$ satisfies

(a) $P = (A \cap P)B$;

(b) *Either B_P is a principal ideal domain or the natural inclusion*

$$k_A(A \cap P) \hookrightarrow k_B(P),$$

is an isomorphism.

Flat Weierstrass extension (continued)

Corollary

Let $A \subseteq B$ be a flat, Weierstrass extension, where A, B are Noetherian local rings. Then either

1. *B is a field, or*
2. *B is a DVR, or*
3. *A and B are analytically isomorphic, i. e., $\hat{A} \simeq \hat{B}$.*

Remark

- Condition (a) is equivalent to
  - (a1) The dimensions of the fibers are zero, and
  - (a2) For $\mathfrak{p} \in \operatorname{Spec}(A)$, either $\mathfrak{p}B = B$ or $\mathfrak{p}B$ is a prime ideal.
- $\mathbb{R}[[X]] \to \mathbb{C}[[X]]$ is faithfully flat and Weierstrass which is not an isomorphism.

- $(R, \mathfrak{m}_R)$: a complete Noetherian local domain of $\dim R = 1$, with $K = \mathrm{Frac}(R)$, $k = R/\mathfrak{m}_R$.
- $\mathfrak{X}$: a connected projective normal curve over R with closed fiber X and function field F.
- $U \subset X$ is a subset contained in an affine set.
- A: ring of rational functions in F which are regular on U.
- $\widehat{A}$ is the $\mathfrak{m}_R A$-adic completion of the ring A.

Proposition

For any prime ideal $\mathfrak{p} \neq 0$ of A, the canonical embedding $A/\mathfrak{p} \hookrightarrow \hat{A}/\mathfrak{p}\widehat{A}$ is an isomorphism. In particular, $\mathfrak{p}\widehat{A}$ is a prime ideal.
Conversely, for a non-minimal prime ideal Q of $\widehat{A}$, if we denote $\mathfrak{q} = Q \cap A$, then $Q = \mathfrak{q}\widehat{A}$ and $k_A(\mathfrak{q}) \simeq k_{\hat{A}}(Q)$.

A Weierstrass preparation theorem

Theorem

Let

- *U be a subset of the closed fiber X;*
- *$P_1, \ldots, P_t \in \operatorname{Spec} \widehat{A}$ be generic points of the irreducible components of $\operatorname{Spec} \widehat{A}$;*
- *Q be a maximal ideal of $\widehat{A}$.*

For each $P_i \subseteq Q$, we have a Weierstrass extension

$$A \subset (\widehat{A}/P_i)_Q.$$

A Weierstrass preparation theorem (continued)

Corollary (Harbater-Hartmann-Krashen)

Let U be either

(a) *A set consisting of one closed point; or*

(b) *An open subset of an irreducible component of X which does not intersect other components.*

For each prime ideal Q of $\widehat{A}$, the extension $A \subset (\widehat{A})_Q$ is Weierstrass.

A Weierstrass preparation theorem (continued)

Theorem (Harbater-Hartmann-Krashen)

Let $x \in X$ be a closed point of the closed fiber. Assume

- *For each irreducible component X_i of X and $x \in X_i$, the ring $\mathcal{O}_{X_i,x}$ is analytically unramified and unibranch.*

Then the $\mathfrak{m}_x$-adic completion $R_x \subset \widetilde{R_x}$ is Weierstrass.

It is the case if x is a non-singular point of the component X_i.

6. Henselization and completion

- $(A, \mathfrak{m}_A) \subseteq (B, \mathfrak{m}_B)$ is a faithfully flat extension consisting of Noetherian local domains.
- B is henselian.
- A^h: the Henselization of A.
- $\hat{A}$: the $\mathfrak{m}_A$-adic completion of A.

Proposition

Suppose B/A is Weierstrass. Then

(a) *There are faithfully flat embeddings $A \hookrightarrow A^h \hookrightarrow B$;*

(b) *If B is not a PID, then there are faithfully flat embeddings*

$$A^h \hookrightarrow B \hookrightarrow \hat{A}.$$

(c) *The Henselization $A \subseteq A^h$ is Weierstrass.*

Henselization

A local ring R is unibranch if the reduction R_{red} is a domain and the integral closure of R in its field of fractions is a local ring.

Proposition

Let A be a Noetherian local domain. The Henselization $A \subseteq A^h$ is Weierstrass if and only if

(a) *For $\mathfrak{p} \in \mathrm{Spec}(A)$, $A/\mathfrak{p}$ is unibranch, and*

(b) *For $\mathfrak{p} \in \mathrm{Spec}(A)$, if $A_{\mathfrak{p}}$ is neither a field nor a DVR, then*

$$k_A(\mathfrak{p}) \simeq k_{A^h}(\mathfrak{p}A^h).$$

Completion

Corollary

Let A be a Noetherian local domain of dimension one. Then $A \subseteq \hat{A}$ is Weierstrass if and only if A is analytically unramified and unibranch.

Example (Matsumura)

Let X be a projective variety over a field k and $x \in X$. The dimension of the generic fiber of $\mathcal{O}_{X,x} \hookrightarrow \widehat{\mathcal{O}}_{X,x}$ is $\operatorname{codim}_x(X) - 1$. So if the completion map is Weierstrass, we get $\operatorname{codim}_x(X) = 1$.

Higher dimension WPT?

Question

How to formulate a higher dimensional WPT which is applicable to study u-invariants?

Proposition

Let $(A, \mathfrak{m}_A)$ be a Noetherian local ring. Let P be a prime ideal which is maximal in the generic fiber of the extension $A \hookrightarrow \hat{A}$. If in addition $\dim \hat{A}/P = 1$, then $A \hookrightarrow \hat{A}/P$ as a Weierstrass extension.

Thank you for your attention