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Relative homological algebra in categories of representations of quivers

A general overview of the Theory of Covers and Envelopes

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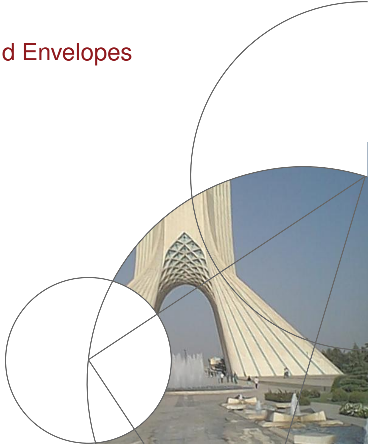

- R.Baer, Abelian groups which are direct summands of every containing group, *Bull. Amer. Math. Soc.* **46** (1940), 800-806.

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- H. Bass, Finitistic dimension and a homological generalization of semiprimary rings, *Trans. Amer. Math. Soc.* **95** (1960), 466-488.

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General definition of covers and envelopes with respect to a certain class $\mathcal{F}$:
E. Enochs, Injective and flat covers, envelopes and resolvents, *Israel J. Math.* **39** (1981), 190-209.

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Definition

An $\mathcal{F}$-precover of M is a morphism $\varphi : F \to M$ such that $F \in \mathcal{F}$ and every diagram

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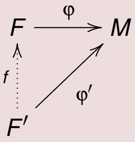

can be completed commutatively, with $F' \in \mathcal{F}$.
If $\varphi \circ f = \varphi \Rightarrow f$ is an automorphism, φ is an $\mathcal{F}$-cover.
$\mathcal{F}$-(pre)envelopes are defined in a dual manner.
$\mathcal{F}$: the class of flat modules, $\mathcal{F}$-cover=flat cover.
$\mathcal{E}$: the class of injective modules, $\mathcal{E}$-envelope= injective envelope.

Flat cover conjecture: “Every module over an associative ring has a flat cover”.

It was known to be true:

- For modules over a left perfect ring.
- For modules over a Prüfer domain, torsion-free=flat.

E. Enochs, Torsion free covering modules, *Proc. Amer. Math. Soc.* **14** (1963), 884-889.

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Before of the later resolution to the conjecture, the most significant advance was obtained by:

J. Xu, Flat covers of modules, Lecture Notes in Math. 1634, Springer-Verlag 1996.

Positive solution to the conjecture:

L. Bican, R. El Bashir and E. Enochs, All Modules have flat covers, *Bull. London Math. Soc.* **33(4)** (2001), 385-390.

Enochs' proof:

L. Salce, Cotorsion theories for abelian groups, Symposia Mathematica, Vol. 23 (1979), 11-32.

P.C. Eklof and J. Trlifaj, How to make Ext vanish, *Bull. London Math. Soc.* **33(1)** (2001), 41-51.

S.T. Aldrich, E. Enochs, J.R. García Rozas and L. Oyonarte, Covers and envelopes in Grothendieck categories. Flat covers of complexes with applications, *J. Algebra* **243(2)** (2001), 615-630.

E. Enochs, L. Oyonarte and B. Torrecillas, Flat covers and flat representations of quivers, *Comm. Algebra*, **32(4)**(2004), 1319-1338.

J.R. García Rozas, J.A. López Ramos and B. Torrecillas, On the existence of flat covers in *R*-gr, *Comm. Algebra*, **29(8)** (2001), 3341-3349.

Categories without enough projectives: sheaves of O_X-modules over a topological space, quasi-coherent sheaves on a scheme.

E. Enochs and L. Oyonarte, Flat covers and cotorsion envelopes of sheaves, *Proc. Amer. Math. Soc.* **130(5)** (2001), 1285-1292.

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Quasi-coherent sheaves?

- Existence theorems of covers in categories without enough projectives.

- $\mathfrak{Qco}(X)$ is locally κ-presentable.

For a given $\mathcal{F}$ in $\mathcal{A}$,

$$\mathcal{F}^{\perp} = \{C \in \mathrm{Ob}(\mathcal{A}) : \mathrm{Ext}^1(F, C) = 0, \ \forall F \in \mathcal{F}\}.$$

Analogously ${}^{\perp}\mathcal{F}$ will denote

$${}^{\perp}\mathcal{F} = \{G \in \mathrm{Ob}(\mathcal{A}) : \mathrm{Ext}^1(G, D) = 0, \ \forall D \in \mathcal{F}\}.$$

The pair $(\mathcal{F}, \mathcal{F}^{\perp})$ is *cogenerated by a set T* if:

$$C \in \mathcal{F}^{\perp} \quad \Leftrightarrow \quad \mathrm{Ext}^1(F, C) = 0 \ \ \forall F \in T.$$

Definition

$(\mathcal{F}, \mathcal{C})$ is a cotorsion theory if
$\mathcal{F}^{\perp} = \mathcal{C}$ and ${}^{\perp}\mathcal{C} = \mathcal{F}$.

1. The pair of classes

$$(\mathcal{P}roj, R\text{-Mod}) \quad \text{and} \quad (R\text{-Mod}, \mathcal{I}nj)$$

   are cotorsion theories, $\mathcal{P}roj$ the class of projective modules and $\mathcal{I}nj$ the class of injective modules. $(R\text{-Mod}, \mathcal{I}nj)$ is cogenerated by the set $\{R/I : \ I \leq_R R\}$.
2. The pair $(\mathcal{F}, \mathcal{C})$ composed by flat modules and cotorsion modules (flat cotorsion theory).

Theorem

Let $\mathcal{A}$ be a Grothendieck category and $\mathcal{F}$ a class of objects of $\mathcal{A}$ closed under direct sums, extensions and well ordered direct limits. $(\mathcal{F}, \mathcal{F}^{\perp})$ cogenerated by a set $\Rightarrow \forall M \in Ob(\mathcal{A})$ there exists an $\mathcal{F}$-cover and an $\mathcal{F}^{\perp}$-envelope.

Proof: Every object in a Grothendieck category is small.

Lemma

There exists an ordinal number λ such that $\forall \lambda' \geq \lambda$ and for every well ordered inductive system $(E_\alpha, f_{\beta\alpha})$, $\alpha < \lambda'$ of injective objects of $\mathcal{C}$, $\varinjlim_{\alpha<\lambda'} E_\alpha$ is also injective $\forall \lambda' \geq \lambda$.

Proposition

For every object $M \in \mathrm{Ob}(\mathcal{C})$ there exists an ordinal number λ such that $\forall \lambda' \geq \lambda$

$$\mathrm{Ext}^n(M, \varinjlim_{\alpha<\lambda'} M_\alpha) \cong \varinjlim_{\alpha<\lambda'} \mathrm{Ext}^n(M, M_\alpha),$$

for every w. o. inductive system $(M_\alpha, f_{\beta\alpha})$, $\alpha < \lambda'$ in $\mathcal{C}$.

There is another proof without using homological methods, by means of "The Small Object Argument"
D.G. Quillen, *Homotopical algebra*, *Lecture Notes in Mathematics*, Vol. 43, Springer-Verlag, 1967.

by using a more general version which appears in

M. Hovey, *Model categories*, *Mathematical Surveys and Monographs*, Vol. 63, American Mathematical Society, Providence, RI, 1998.

So in any case everything reduces to prove that the pair $(\mathcal{F}, \mathcal{F}^{\perp})$ is cogenerated by a set.

Lemma (Eklof)

Let $\mathcal{A}$ be an abelian category with direct limits and A, C objects of $\mathcal{A}$. Let us suppose that

- $A = \cup_{\alpha<\lambda} A_\alpha$ *for an ordinal number* λ
- $\mathrm{Ext}^1(A_0, C) = 0$ *and* $\mathrm{Ext}^1(A_{\alpha+1}/A_\alpha, C) = 0$ *for every* $\alpha < \lambda$.

Then $\mathrm{Ext}^1(A, C) = 0$.

Proposition

Let $F \in \mathcal{F}$ and $x \in F$. Suppose there exists a cardinal $\aleph$ and $S \subseteq F$ such that.

- $x \in S$, $|S| \leq \aleph$
- S, $F/S \in \mathcal{F}$.

Then the pair $(\mathcal{F}, \mathcal{F}^\perp)$ is cogenerated by a set.

Representations of quivers

A quiver Q is a directed graph.

A path p of Q is a sequence of arrows. If $t(p) = i(q)$ we get the path qp.

$P(Q)_v$, (left) path tree associated to Q with root in v: is a quiver whose vertices are the paths p of Q begining in v and the arrows the pairs (p, ap).

A representation by modules of Q is a functor $X : Q \to R$-Mod. A morphism between X and Y is a natural transformation.

$(Q, R\text{-Mod})$ is the family of representations by modules of a quiver Q, it is a Grothendieck category with enough projectives.

RQ (path algebra of Q) is the free R-module whose base are the paths p of Q, and

$$q \cdot p = \begin{cases} qp & \text{if } t(p) = i(q) \\ 0 & \text{in other case} \end{cases}$$

If Q has a finite number of vertices, RQ has identity

$$1 = v_1 + \cdots + v_n,$$

if not is a ring with local units.
The categories $(Q, R\text{-Mod})$ and RQ-Mod are equivalent.

Quasi-coherent $\mathfrak{R}$-modules

$Q = (V, E)$ is a quiver.
$\mathfrak{R}$ is a representation of Q in the category of rings, that is,
for each $v \in V$ we have a ring $\mathfrak{R}(v)$ and
for an arrow $a : v \to w \in E$ a homomorphism of rings

$$\mathfrak{R}(a) : \mathfrak{R}(v) \to \mathfrak{R}(w).$$

An $\mathfrak{R}$-module M is an $\mathfrak{R}(v)$-module $M(v)$ for $v \in V$
and an $\mathfrak{R}(v)$-morphism $M(a) : M(v) \to M(w)$ for an arrow $a : v \to w$.

M is quasi-coherent if for each edge a the morphism

$$\mathfrak{R}(w) \otimes_{\mathfrak{R}(v)} M(v) \to M(w)$$

is an $\mathfrak{R}(w)$-isomorphism.

The category of quasi-coherent $\mathfrak{R}$-modules is cocomplete with exact direct limits and abelian if $\mathfrak{R}(w)$ is a flat $\mathfrak{R}(v)$-module for $v \to w$.

Consider quasi-coherent sheaves over $(X, \mathcal{O}_X)$.
If $\mathcal{U}$ are the affine opens $U \subseteq X$, a quasi-coherent sheaf over $(X, \mathcal{O}_X)$ (or a quasi-coherent $\mathcal{O}_X$-module) is uniquely determined by

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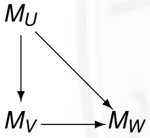
ii) The compatibility condition,
if $W \subseteq V \subseteq U$, $(W, V, U \in \mathcal{U})$, then



is commutative.

Proposition

For M a right $\mathfrak{R}$-module and N a left $\mathfrak{R}$-module, the tensor product $M \otimes_{\mathfrak{R}} N$ is the $\mathbb{Z}$-module ($\mathbb{Z}(v) = \mathbb{Z}, \forall v \in V$ and $\mathbb{Z}(a) = id_{\mathbb{Z}}$ for all $a \in E$) such that

$$M \otimes_{\mathfrak{R}} N(v) = M(v) \otimes_{\mathfrak{R}(v)} N(v),$$

with $M \otimes_{\mathfrak{R}} N(a)$ the obvious map.

- $_{\mathfrak{R}}F$ is flat $\overset{def.}{\Leftrightarrow} - \otimes_{\mathfrak{R}} F$ is exact.
- $\mathcal{F}$ will denote the class of all flat quasi-coherent $\mathfrak{R}$-modules.

C. Jensen and H. Lenzing. Model-theoretic algebra with particular emphasis on fields, rings, modules, Algebra, Logic and Applications, Vol. 2, Gordon and Breach, 1989.

Proposition

$Q=(V,E)$ is a quiver, and M a quasi-coherent R-module over Q.
Let κ be an infinite cardinal such that $\kappa \geq |\mathfrak{R}(v)|\ \forall v \in V$ and $\kappa \geq |E|, |V|$.
Let $X_v \subseteq M(v)$ be subsets with $|X_v| \leq \kappa\ \forall v \in V$.
There exits a quasi-coherent submodule $M' \subseteq M$ with
i) $M'(v) \subseteq M(v)$ pure, $\forall v \in V$
ii) $X_v \subseteq M'(v)$, $\forall v \in V$ and
iii) $|M'| \leq \kappa$.

B. Conrad. Grothendieck duality and base change, Lecture Notes in Mathematics, Vol. 1750, Springer-Verlag, 2000.

Corollary (Gabber)

The category of quasi-coherent $\mathfrak{R}$-modules is locally κ-presentable.

Theorem

Every quasi-coherent $\mathfrak{R}$-module has a flat cover and a cotorsion envelope.

Corollary

For a given scheme $(X, \mathcal{O}_X)$, every quasi-coherent sheaf on $\mathcal{O}_X$ admits a flat cover and a cotorsion envelope.

Gorenstein categories

$\mathcal{A}$ Grothendieck category.

Definition

$X \in Ob(\mathcal{A})$

$$pd\, X \leq n \Leftrightarrow \mathrm{Ext}^i(X, -) = 0 \text{ for } i \geq n+1.$$

$FPD(\mathcal{A}) = sup\{pd\, X : \ \forall X,\ pd\, X < \infty\}$
$FID(\mathcal{A}) = sup\{id\, X : \ \forall X,\ id\, X < \infty\}$.

Definition

We will say that $\mathcal{A}$ is a Gorenstein category if the following hold:
1) For any object L of $\mathcal{A}$, $pd\, L < \infty$ if and only if $id\, L < \infty$.
2) $FPD(\mathcal{A}) < \infty$ and $FID(\mathcal{A}) < \infty$.
3) $\mathcal{A}$ has a generator L such that $pd\, L < \infty$.

Definition

E is Gorenstein injective if there exists an exact complex

$$\cdots \to E_{-1} \to E_0 \to E_1 \to \cdots$$

such that $E = \mathrm{Ker}\,(E_0 \to E_1)$ and it is $\mathrm{Hom}(U,-)$-exact for all injective U.
Dually we define Gorenstein projective objects.

Definition

$Gpd(X) = n$ if the first syzygy of X that is Gorenstein projective is the n-th one and $Gpd(X) = \infty$ if there is no such syzygy.
$glGpd(\mathcal{A}) = sup\{Gpd(X) : \ X \in Ob(\mathcal{A})\}$
Then also define $Gid(Y)$ and $glGid(\mathcal{A})$.

$\mathcal{L} = \{X \in Ob(\mathcal{A}) : \ id\, X < \infty\}$

Theorem

If $(\mathcal{A}, \mathcal{L})$ is a Gorenstein category then

1. *$(\mathcal{L}, \mathcal{L}^{\perp})$ is a cotorsion theory.*
2. *$\mathcal{L}^{\perp}$ is the class of Gorenstein injective objects of $\mathcal{A}$.*
3. *For Each $M \in Ob(\mathcal{A})$ has a special $\mathcal{L}$-precover and a special $\mathcal{L}^{\perp}$-preenvelope (so $(\mathcal{L}, \mathcal{L}^{\perp})$ is a complete and hereditary cotorsion theory).*

If $FID(\mathcal{A}) = n$ then $Gid(Y) \leq n$ for all objects Y of $\mathcal{A}$.

Theorem

If $(\mathcal{A},\mathcal{L})$ be a Gorenstein category of dimension at most n having enough projectives. Then for an object C of $\mathcal{A}$ the following are equivalent:
1) C is an n-th syzygy.
2) $C \in {}^{\perp}\mathcal{L}$.
3) C is Gorenstein projective.
As a consequence we get that glGpd$(\mathcal{A}) \leq n$ and that $({}^{\perp}\mathcal{L},\mathcal{L})$ is a complete hereditary cotorsion pair.

$\mathcal{A}$ Grothendieck category with enough projectives.

Theorem

Let $\mathcal{A}$ be a Grothendieck category with enough projectives. Then the following are equivalent:
1) $\mathcal{A}$ is Gorenstein.
2) $glGpd(\mathcal{A}) < \infty$ and $glGid(\mathcal{A}) < \infty$.
Moreover, if (1) (or (2)) holds we have

$$FID(\mathcal{A}) = FPD(\mathcal{A}) = glGpd(\mathcal{A}) = glGid(\mathcal{A}).$$

Gext^i **functors**

$\mathcal{A}$ Gorenstein category.

- For a given Y there exists a *Gorenstein injective resolution*, that is an exact sequence

$$0 \to Y \to G_0 \to G_1 \to \cdots$$

  such that $\mathrm{Hom}(-, G)$ leaves the sequence exact, for all Gorenstein injective G.
- We define right derived functors $\mathrm{Gext}^i(X, Y)$, $i \geq 0$ of Hom by using Gorenstein injective resolutions of Y.

$\mathcal{A}$ Gorenstein category with $glGid(\mathcal{A}) = n$.

Tate cohomology functors $\widehat{\mathrm{Ext}}^i(X, Y)$**,** $i \in \mathbb{Z}$

- The n-th cosyzygy of Y, G, is Gorenstein injective, so there exists

  $$\mathbf{E} : \cdots \to E^{-1} \to E^0 \to E^1 \to \cdots$$

  with $G = \mathrm{Ker}\,(E^0 \to E^1)$, such that

  $$\mathrm{Hom}(U, \mathbf{E})$$

  is exact, U injective.
- **E** is unique up to homotopy

$$\widehat{\mathrm{Ext}}^i(X, Y) \overset{def}{=} i\text{−th cohomology}$$

$$\text{groups of } \mathrm{Hom}(X, \mathbf{E})$$

L. Avramov and A. Martsinkovsky, *Absolute, relative and Tate cohomology of modules of finite Gorenstein dimension*, *Proc. London Math. Soc.* **85(3)** (2002), 393-440.
A. Iacob, *Generalized Tate Cohomology*, *Tsukuba J. Math.* **29(2)** (2005), 389-404.

Proposition

If $\mathcal{A}$ is a Gorenstein category of dimension at most n then for all objects X and Y of $\mathcal{A}$ there exist natural exact sequences

$$0 \to Gext^1(X,Y) \to \mathrm{Ext}^1(X,Y) \to \widehat{\mathrm{Ext}}^1(X,Y) \to Gext^2(X,Y) \to \cdots \to Gext^n(X,Y) \to \mathrm{Ext}^n(X,Y) \to \widehat{\mathrm{Ext}}^n(X,Y) \to 0.$$

$X \subseteq \mathbf{P^n}(A)$ closed subscheme.
X locally Gorenstein scheme (so $\mathfrak{R}(v)$ is Gorenstein ring, i.e., commutative noetherian and *id* $\mathfrak{R}(v) < \infty$, $\forall v$).

Theorem

$\mathfrak{Qco}(X)$ *is a Gorenstein category.*

Projective and injective model structure on Gorenstein categories

Theorem

(Hovey). If $(\mathcal{A}, \mathcal{L})$ is a Gorenstein category then there is a cofibrantly generated model structure on $\mathcal{A}$ with $\mathcal{L}$ the full subcategory of trivial objects and such that the fibrant objects are the Gorenstein injective objects.
If $\mathcal{A}$ has enough projectives then there is a cofibrantly generated model structure on $\mathcal{A}$ with $\mathcal{L}$ the trivial objects and such that the cofibrant objects are the Gorenstein projective objects.